18.336 Pset 2 Solutions

Problem 1: Crank-Nicolson

We'll analyze it for a general α first. As usual, we look at a Fourier eigenmode: let $v_m^n = g^n e^{im\theta}$ and solve for the amplification factor $g(\theta, \Delta x, \Delta t)$. Then:

$$g = 1 - a\Delta t \left[\alpha g \frac{2i\sin\theta}{2\Delta x} + (1-\alpha)\frac{2i\sin\theta}{2\Delta x} \right],$$

and thus:

$$g = \frac{1 - ia\lambda(1-\alpha)\sin\theta}{1 + ia\lambda\alpha\sin\theta}$$

where $\lambda = \Delta t/\Delta x$. Therefore, $|g|^2 \le 1$ when:

$$1 + a^2\lambda^2(1-\alpha^2)\sin^2\theta \le 1 + a^2\lambda^2\alpha^2\sin^2\theta$$

and thus when $\alpha \ge 0.5$. In particular:

(a) For Crank-Nicolson ($\alpha = 0.5$), it is unconditionally stable and $|g| = 1$.

(b) It is also unconditionally stable for $0.5 < \alpha \le 1$, where it is a dissipative first-order implicit scheme.

Problem 2: Consistency and Stability

(a) We want to show that the following scheme is consistent with $u_t + au_x = 0$.

$$\frac{v_m^{n+1} - v_m^n}{\Delta t} + \frac{a}{2}\left(\frac{v_{m+1}^{n+1} - v_m^{n+1}}{\Delta x}\right) + \frac{a}{2}\left(\frac{v_m^n - v_{m-1}^n}{\Delta x}\right) = 0.$$

We plug in a smooth function $u(x,t)$ and expand around $u[m\Delta x, (n+\frac{1}{2})\Delta t] \equiv u^{(0)}$. Thus, for example, $v_{m+1}^{n+1} = u^{(0)} + u_x^{(0)}\Delta x + u_t^{(0)}\frac{\Delta t}{2} + u_{xt}^{(0)}\frac{\Delta x\Delta t}{2} + u_{xx}^{(0)}\frac{\Delta x^2}{2} + u_{tt}^{(0)}\frac{\Delta t^2}{8} + O(\Delta^3)$. We therefore obtain on the left-hand side:

$$u_t^{(0)} + \frac{a}{2}\left(u_x^{(0)} + u_{xt}^{(0)}\frac{\Delta t}{2} + u_{xx}^{(0)}\frac{\Delta x}{2}\right) + \frac{a}{2}\left(u_x^{(0)} - u_{xt}^{(0)}\frac{\Delta t}{2} - u_{xx}^{(0)}\frac{\Delta x}{2}\right) + O(\Delta^2)$$

where the Δx and Δt terms cancel and we are left with $u_t^{(0)} + au_x^{(0)} + O(\Delta^2)$, and thus the scheme is consistent: it is second-order accurate compared to the exact PDE at $[m\Delta x, (n+\frac{1}{2})\Delta t]$.

(b) We want to show that this scheme is consistent with $u_t + au_{xxx} = 0$:

$$\frac{v_m^{n+1} - v_m^n}{\Delta t} + a\frac{v_{m+2}^n - 3v_{m+1}^n + 3v_m^n - v_{m-1}^n}{\Delta x^3} = 0.$$

Again, we Taylor expand for a smooth $u(x,t)$, this time around $u(m\Delta x, n\Delta t) \equiv u^{(0)}$. Note that $v_{m+p}^n = u([m+p]\Delta x, n\Delta t) = u^{(0)} + u_x^{(0)}p\Delta x + u_{xx}^{(0)}\frac{p^2\Delta x^2}{2} + u_{xxx}^{(0)}\frac{p^3\Delta x^3}{6} + O(\Delta x^4)$. Plugging this in, we find that (only) the $u^{(0)}$, $u_x^{(0)}$, and $u_{xx}^{(0)}$ terms cancel and we are left with:

$$u_t^{(0)} + O(\Delta t) + au_{xxx}^{(0)} + O(\Delta x),$$

i.e. a consistent approximation that is first-order in time and space.

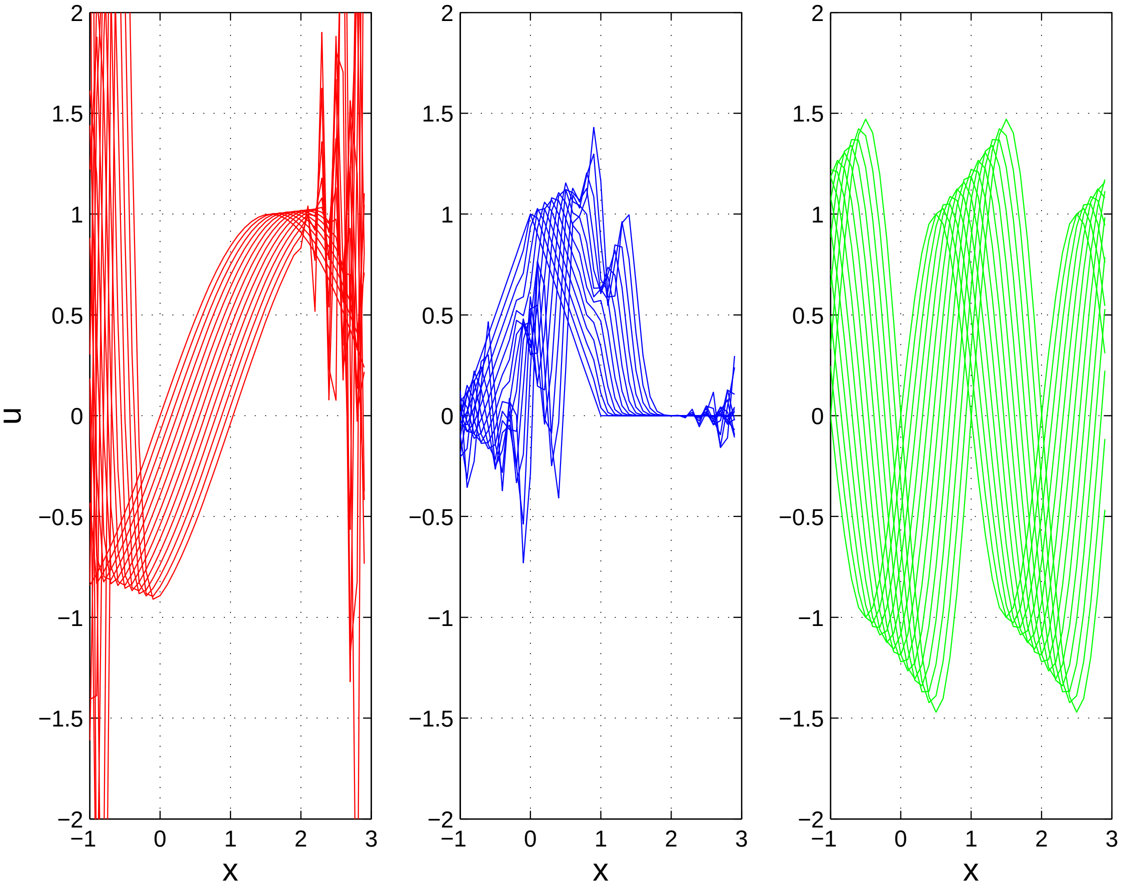

Figure 1: $u(x,t)$ vs. x for $t = 0, 0.08, 0.16, \cdots, 0.96$ (i.e. for $\Delta t = 0.08$), for three initial conditions: (left) $u_1 = \sin(x)$; (middle) $u_2 = (1 - |x|) \cdot (|x| \leq 1)$; (right) $u_3 = \sin(\pi x)$.

To analyze the stability, we plug in $v_m^n = g^n e^{im\theta}$ as usual, and find (via the usual binomial expansion formula):

$$g = 1 + a\nu e^{i\theta/2}(e^{i\theta/2} - e^{-i\theta/2})^3 = 1 + 8ia\nu e^{i\theta/2}\sin^3(\theta/2),$$

and thus:

$$|g|^2 = 1 + 64a^2\nu^2\sin^6(\theta/2) - 16a\nu\sin^4(\theta/2),$$

which simplifies to:

$$|g|^2 = 1 - 16a\nu\left(1 - 4a\nu\sin^2(\theta/2)\right)\sin^4(\theta/2).$$

If $\nu = \Delta t/\Delta x^3$ is constant, then for stability we must have $|g|^2 \leq 1$ for all θ (and in particular for the worst case of $\theta = \pi$), which from this expression is obviously only true when $a\nu \geq 0$ and $a\nu \leq \frac{1}{4}$. Q.E.D.

Problem 3: Instability

We want to implement the time-stepping scheme:

$$u_m^{n+1} = u_m^n - \frac{a\lambda}{2}\left(u_{m+1}^n - u_{m-1}^n\right).$$

Given a vector **u** in Matlab that stores $u(x,t)$ for some t and $x = -1, -1+\Delta x, \ldots, 3-\Delta x$, this time-step $u(x,t) \to u(x,t+\Delta t)$, with periodic boundaries, is given via the Matlab command:

```
u = u + C * ([u(2:end),u(1)] - [u(end),u(1:end-1)]);
```

for `C` given by $-a\lambda/2$. We then construct our initial conditions via:

```
x = [-1:0.1:2.9];
u1 = sin(x);
u2 = (abs(x) <= 1) .* (1 - abs(x));
u3 = sin(pi * x);
```

Note that we only go up to $x = 2.9$, since $x = 3$ is equivalent to $x = -1$ by the boundary conditions—we only want to store that point *once*.

(a) For each initial condition, we plotted $u(x,t)$ vs. x for $t = 0, 0.08, 0.16, \cdots, 0.96$ (i.e. for $\Delta t = 0.08$). This is shown in Fig. 1. We can clearly see two features. First, the curves are propagating to the right, as they should. Second, the curves are blowing up like crazy at the points of discontinuity, and the curves are blowing up more slowly elsewhere. Third, the most discontinuous function u_1 blows up the fastest (u_2 only has discontinuous slope, and $u_3$is smooth).

(b) The L_2 norm vs. time-step n is plotted in Fig. 2 on a semilog scale. Clearly, all three initial conditions result in norms that asymptote to straight lines—a straight line on a log scale corresponds to an exponential divergence g^n for some g's.

(c) As we showed in class, this scheme has $g(\theta) = 1 - ia\lambda \sin\theta$. The two discontinuous functions u_1 and u_2 (note that u_1 is discontinuous at the boundaries) contain *all* Fourier, and therefore they should diverge (eventually) according to the *worst* Fourier component, $\theta = \pi/2$. Thus, they diverge as $|1 - ia\lambda|^n \sim 1.2806^n$. This can be seen in Fig. 2, where the u_1 and u_2 curves clearly match the slope of the 1.2806^n line. On the other hand, the $u_3 = \sin(\pi x)$ has only *one* Fourier component at $\theta = \pi\Delta x$ (unlike u_1, u_3 has the correct periodicity). Thus, it diverges as $|g(\pi\Delta x)|^n \sim 1.0301^n$. Our calculated curve clearly matches this slope in Fig. 2.

(d) Clearly, either u_1 or u_2 has to diverge the fastest, since they both contain the $\theta = \pi/2$ Fourier component that results in the largest $g(\theta)$. These both diverge at the same asymptotic *rate,* but the constant factors differ. Which one of those two diverges first (i.e. with the biggest constant factor) depends on the *magnitude* of the $\pi/2$ Fourier component, which is determined by the magnitude of the discontinuity. Since u_1 is discontinuous and u_2 only has a discontinuous slope, u_1 has the bigger Fourier component at $\theta = \pi/2$ and thus diverges first, as we can clearly see in Fig. 2 as well as in Fig. 1.

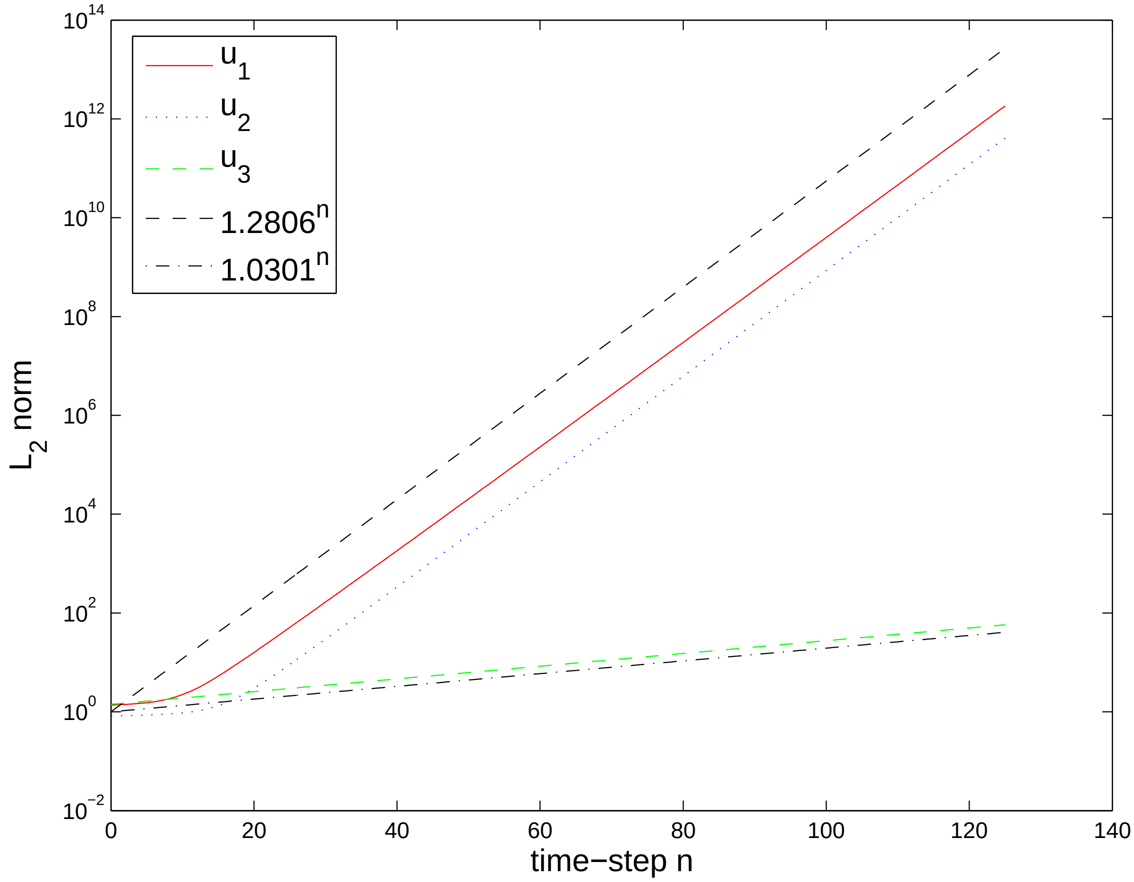

Figure 2: L_2 norm vs. time step n for three initial conditions u_1, u_2, and u_3, on a semilog scale (straight line = exponential divergence). Also shown, for reference, are the predicted $|g(\pi/2)|^n \sim 1.2806^n$ and $|g(\pi\Delta x)|^n \sim 1.0301^n$ divergences.