18.336 Problem Set 2

Due Thursday, 9 March 2006.

Note that this problem set closely follows the notes from previous terms. (See 18.336 on Open-CourseWare for the lecture notes, which is linked to from http://math.mit.edu/~stevenj/18.336)

Problem 1: Crank-Nicolson

(a) Prove that the Crank-Nicolson scheme for $u_t = -au_x$ is unconditionally stable. The C-N scheme for the discretized solution $u(m\Delta x, n\Delta t) \approx v_m^n$ is

$$v_m^{n+1} = v_m^n - a\Delta t \left[\alpha \frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2\Delta x} + (1-\alpha)\frac{v_{m+1}^n - v_{m-1}^n}{2\Delta x} \right]$$

where $\alpha = 0.5$.

(b) For what other values of α is this unconditionally stable?

Problem 2: Consistency and Stability

(a) Show that the following scheme is consistent with $u_t + au_x = 0$.

$$\frac{v_m^{n+1} - v_m^n}{\Delta t} + \frac{a}{2}\left(\frac{v_{m+1}^{n+1} - v_m^{n+1}}{\Delta x} + \frac{v_m^n - v_{m-1}^n}{\Delta x} \right) = 0.$$

(b) Show that this scheme is consistent with $u_t + au_{xxx} = 0$:

$$\frac{v_m^{n+1} - v_m^n}{\Delta t} + a\frac{v_{m+2}^n - 3v_{m+1}^n + 3v_m^n - v_{m-1}^n}{\Delta x^3} = 0$$

If $\nu = \Delta t/\Delta x^3$ is constant, show it is stable when $0 \le a\nu \le \frac{1}{4}$.

Problem 3: Instability

Use the unstable forward-time centered-space scheme (see notes, example 2, section 1.3) to solve $u_t = -au_x$ on the interval $[-1, 3]$, with periodic boundary conditions $u(x,t) = u(x+4,t)$, for the following three sets of initial data:

- $u_1(x, 0) = \sin x$
- $u_2(x, 0) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & \text{otherwise.} \end{cases}$
- $u_3(x, 0) = \sin(\pi x)$

Use $a = 1$, $\Delta x = 0.1$, and $\Delta t/\Delta x = \lambda = 0.8$.

(a) For each initial condition, plot $u(x,t)$ versus x for $0 \le t \le 1$ (one plot per initial condition, with one curve per time step).

(b) Compute & plot the L_2 norm for $0 \le t \le 10$ on a semilog scale, show that each one diverges as g^n for some constant g (n = time step).

(c) Predict g from Von-Neumann analysis as the class/notes, and compare.

(d) Which initial condition diverges most quickly, and why?

Hint: Given a vector **u** in Matlab that stores $u(x,t)$ for some t and $x = -1, -1+\Delta x, \ldots, 3-\Delta x$, you should be able to implement the time-step $u(x,t) \to u(x,t+\Delta t)$ via the Matlab command:

```
u = u + C * ([u(2:end),u(1)] - [u(end),u(1:end-1)]);
```

for some constant `C`.