$\boldsymbol{G}$: simple graph with d vertices

$\boldsymbol{V}$: vertex set of G

$\boldsymbol{E}$: edge set of G

Coloring of G:

$$\text{any } \boldsymbol{\kappa} : V \to \mathbb{P} = \{1, 2, \ldots\}$$

Proper coloring:

$$uv \in E \Rightarrow \kappa(u) \neq \kappa(v)$$

$\boldsymbol{\chi_G(n)}$: number of proper

$$\kappa : V \to [n] = \{1, 2, \ldots, n\}$$

(the **chromatic polynomial** of G)

$\chi_G(n)$ is a monic polynomial in n with integer coefficients, of degree d, ...

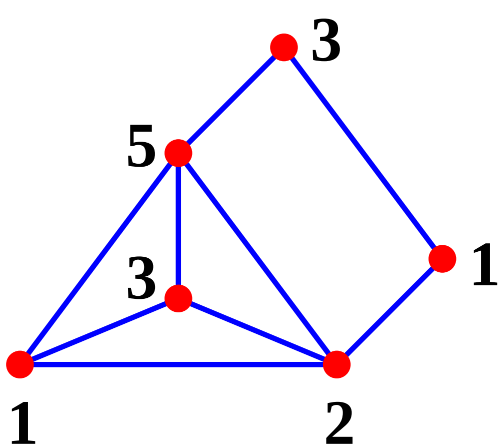
variables $\boldsymbol{x} = (x_1, x_2, \ldots)$

$$\boldsymbol{X_G} = \sum_{\text{proper } \kappa : V \to \mathbb{P}} x^\kappa,$$

the **chromatic symmetric function** of G, where

$$x^\kappa = \prod_{v \in V} x_{\kappa(v)}$$

$$= x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots .$$



$$x^\kappa = x_1^2 x_2 x_3^2 x_5$$

For any symmetric function $f(x)$, write

$$\boldsymbol{f(1^n)} = f(\underbrace{1,\ldots,1}_{n\ 1'\text{s}}).$$

Thus

$$X_G(1^n) = \chi_G(n).$$

Note. $X_G(x)$ is a homogeneous symmetric function (formal power series) of degree d with integer coefficients in the variables $x_1, x_2, \ldots$.

E.g., $X_{\text{point}} = x_1 + x_2 + x_3 + \cdots = e_1$. More generally, let

$$e_k = \sum_{1 \leq i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k},$$

the kth **elementary symmetric function**. Then

$$\begin{aligned} X_{K_n} &= n!\, e_n \\ X_{G+H} &= X_G \cdot X_H. \end{aligned}$$

Open: If T and T' are nonisomorphic trees, does $X_T \neq X_{T'}$? (presumably no, but true for $d \leq 9$)

Acyclic orientation: an orientation $\mathfrak{o}$ of the edges of G that contains no directed cycle.

Theorem (RS, 1973). *Let* $\boldsymbol{a(G)}$ *denote the number of acyclic orientations of* G. *Then*

$$a(G) = (-1)^d \chi_G(-1).$$

Easy to prove by induction, by deletion-contraction, bijectively, geometrically, etc.

Write $\boldsymbol{\lambda} \vdash \boldsymbol{d}$ if λ is a **partition** of d, i.e., $\lambda = (\lambda_1, \lambda_2, \ldots)$ where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \quad \sum \lambda_i = d.$$

Let

$$\boldsymbol{e_\lambda} = e_{\lambda_1} e_{\lambda_2} \cdots.$$

Fundamental theorem of symmetric functions. *Every symmetric function can be uniquely written as a polynomial in the e_i's, or equivalently as a linear combination of e_λ's.*

Note that if $\lambda \vdash d$, then

$$e_\lambda(1^n)|_{n=-1} = \prod \binom{-1}{\lambda_i} = (-1)^d.$$

Hence if $X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$, then

$$a(G) = \sum_{\lambda \vdash d} c_\lambda.$$

Sink of an acylic orientation (or digraph): vertex for which no edges point out (including an isolated vertex).

$\boldsymbol{a_k(G)}$: number of acyclic orientations of G with k sinks

$\boldsymbol{\ell(\lambda)}$: length (number of parts) of λ

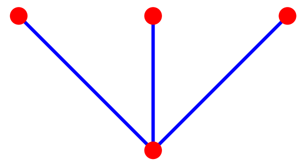
Theorem. *Let* $X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$. *Then*

$$\sum_{\substack{\lambda \vdash d \\ \ell(\lambda)=k}} c_\lambda = a_k(G).$$

Proof based on quasisymmetric functions.

Open: Is there a simpler proof?

Example. Let G be the **claw** K_{13}.



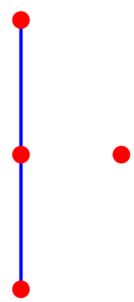
Then

$$X_G = 4e_4 + 5e_{31} - 2e_{22} + e_{211}.$$

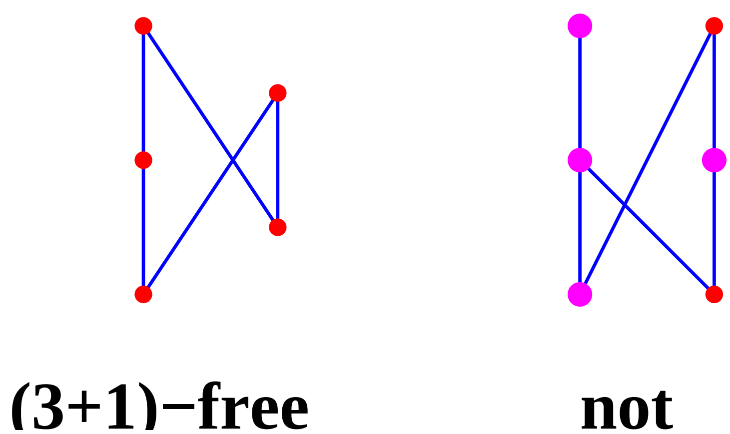
Thus $a_1(G) = 1$, $a_2(G) = 5 - 2 = 3$, $a_3(G) = 1$, $a(G) = 5$.

When is X_G ***e*-positive** (i.e., each $c_\lambda \geq 0$)?

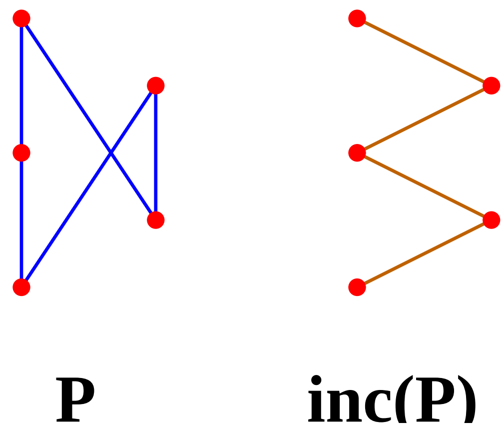
Let P be a finite poset. Let $\mathbf{3}+\mathbf{1}$ denote the disjoint union of a 3-element chain and 1-element chain:



P is **(3+1)-free** if it contains no **induced** $\mathbf{3}+\mathbf{1}$.



inc(P): incomparability graph of P (vertices are elements of P; uv is an edge if neither $u \le v$ nor $v \le u$)



Conjecture. If P is $(\mathbf{3} + \mathbf{1})$-free, then $X_{\mathrm{inc}(P)}$ is e-positive.

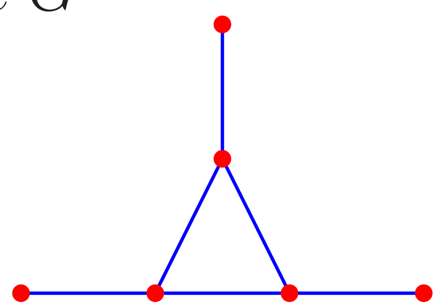
True if P is $\mathbf{3-free}$, i.e., X_G is e-positive if G is the complement of a bipartite graph. More generally, X_G is e-positive if G is the complement of a triangle-free (or $\boldsymbol{K_3 - \mathbf{free}}$) graph.

Suggests that for incomparability graphs of $(\mathbf{3}+\mathbf{1})$-free posets, c_λ counts acyclic orientations of G with $\ell(\lambda)$ sinks and some further property depending on λ.

Open: What is this property?

Define a graph to be **clawfree** if it contains no induced K_{13}. Since $K_{13} = \mathrm{inc}(\mathbf{3}+\mathbf{1})$ and no other poset Q has $\mathrm{inc}(Q) = K_{13}$, an incomparability graph $\mathrm{inc}(P)$ is clawfree if and only if P is $(3+1)$-free .

Example: X_G need not be e-positive for clawfree G



$$X_G = 12e_6+18e_{51}+12e_{42}-6e_{33}+6e_{411}+6e_{321}$$

We can define the **Schur function** $\boldsymbol{s_\lambda}$ by

$$s_\lambda = \det \left(e_{\lambda'_i - i + j} \right)_{i,j=1}^{\ell(\lambda')},$$

where $\boldsymbol{\lambda'} = (\lambda'_1, \lambda'_2, \ldots)$ is the **conjugate partition** to λ (i.e., diagram is transposed).

E.g., $\lambda = (3, 2, 1, 1)$, $\lambda' = (4, 2, 1)$,

$$s_{3211} = \begin{vmatrix} e_4 & e_5 & e_6 \\ e_1 & e_2 & e_3 \\ 0 & 1 & e_1 \end{vmatrix}$$

$$= e_{61} + e_{421} - e_{43} - e_{511}.$$

Note. e_λ is s-positive, so e-positive $\Rightarrow$ s-positive (but not conversely).

Theorem (Gasharov). *If* P *is* $(\mathbf{3}+\mathbf{1})$*-free then* $X_{\mathrm{inc}(P)}$ *is s-positive.*

Conjecture (Gasharov, as a question) *If* G *is clawfree then* X_G *is s-positive.*

Refinement: Let $\#P = d$. A ***P*-tableau of shape $\boldsymbol{\lambda} \vdash \boldsymbol{d}$** is a map $\tau : P \to \mathbb{P}$ satisfying:

(a) For all i we have $\lambda_i = \#\tau^{-1}(i)$.

(b) τ is a proper coloring of $\mathrm{inc}(P)$, i.e., if $\tau(u) = \tau(v)$ then $u \leq v$ or $v \leq u$.

(c) By (b) the elements of the set $\tau^{-1}(i)$ form a chain, say $u_1 < u_2 < \cdots < u_{\lambda_i}$. Similarly let the elements of $\tau^{-1}(i+1)$ be $v_1 < v_2 < \cdots < v_{\lambda_{i+1}}$. Then for all i and all $1 \leq j \leq \lambda_{i+1}$, we require that $v_j \not< u_j$.

If P is a chain then $\exists$ simple bijection between P-tableaux of shape λ and standard Young tableaux (SYT) of shape λ. Thus a P-tableau of shape λ is a generalization of a SYT of shape λ.

$\boldsymbol{f^{\lambda}(P)}$: # P-tableaux of shape λ.

Theorem (Gasharov). *Let P be $(\mathbf{3}+\mathbf{1})$-free and $G = \mathrm{inc}(P)$.* Then

$$X_G = \sum_{\lambda \vdash d} f^{\lambda}(P) s_{\lambda}.$$

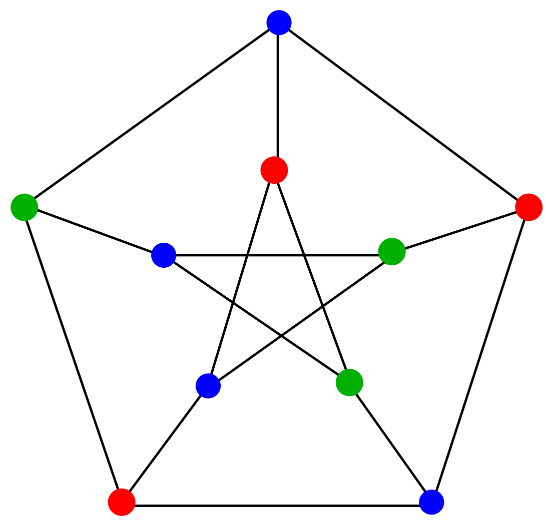
Proof: involution argument.

If P is a chain, RSK gives a bijective proof.

Open: Find a bijective proof of Gasharov's theorem (special case by Sundquist, Wagner, & West).

Nice graphs and s-positivity

A **stable partition** of G of **type** $\boldsymbol{\lambda} \vdash \boldsymbol{d}$ is a partition of V into stable subsets (or independent subsets or co-cliques) of sizes $\lambda_1, \lambda_2, \ldots$.



type (4,3,3)

$\boldsymbol{\mu} \leq \boldsymbol{\lambda}$: $|\lambda| = |\mu|$ and

$$\forall i \ \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$$

(dominance order)

G is **nice** if whenever G has a stable partition of type λ and $\mu \leq \lambda$, then G has a stable partition of type μ,

Example. K_{13} is not nice ($\exists$ stable partition of type $(3,1)$ but not $(2,2)$).

Proposition. *If X_G is s-positive, then G is nice.*

Proposition. *G and all induced subgraphs are nice if and only if G is clawfree.*

Conjecture (Griggs). If B_n is a boolean algebra of rank n (subsets of an n-set ordered by inclusion) then $\mathrm{inc}(B_n)$ is nice.

Open: Is $X_{\mathrm{inc}(B_n)}$ s-positive? Is $X_{\mathrm{inc}(L)}$ s-positive for distributive lattices L (unlikely)?

Note. Modular lattices need not be nice.

Real zeros and s-positivity

$\boldsymbol{c_i = c_i(G)}$: number of i-element stable subsets of vertices

E.g.:

Turan's theorem. *If G has d vertices and no induced K_r then*

$$\#E \leq f(d,r)$$

for an explicit integer $f(d,r)$. Equivalently, if $c_1(\overline{G}) = d$ and $c_r(\overline{G}) = 0$ then

$$c_2(\overline{G}) \leq f(d,r).$$

Open: Characterize the the vectors $(c_1, c_2, \ldots, c_d)$ **(f-vectors of stable set complexes)**.

Stable set polynomial:

$$C_G(t) = \sum_{i \geq 0} c_i(G) t^i$$

Example. $C_{K_d}(t) = 1 + dt$

$$C_{dK_1}(t) = C_{\overline{K_d}}(t) = \sum \binom{d}{i} t^i = (1+t)^d$$

Conjecture (Hamidoune). If G is clawfree, then $C_G(t)$ has only real zeros.

Theorem (consequence of Aissen-Schoenberg-Whitney, Edrei, Thoma) *Let* $P(t) \in \mathbb{R}[t]$, $P(0) = 1$. *Define*

$$F_P(x) = \prod_i P(x_i).$$

TFAE:

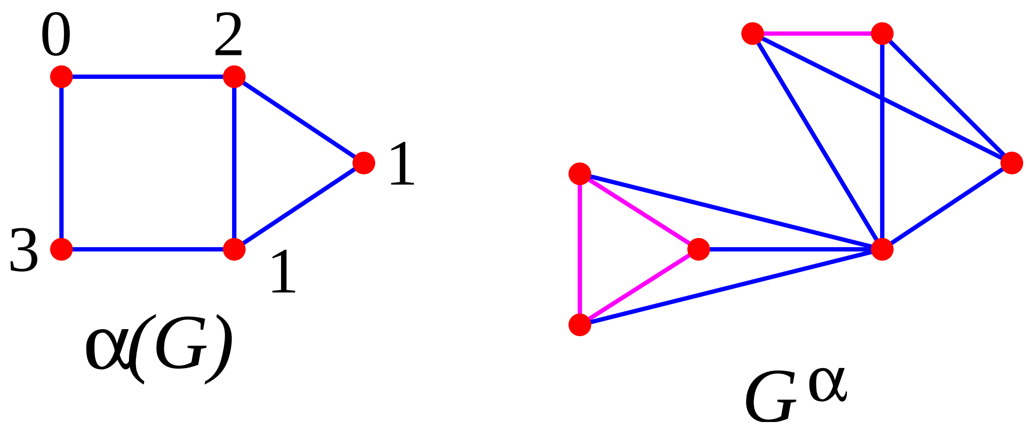
(a) $F_P(x)$ *is s-positive.*

(b) $F_P(x)$ *is e-positive.*

(c) *All the zeros of* $P(t)$ *are negative real numbers.*

Let $\alpha : V \to \mathbb{N}$.

$\boldsymbol{G^\alpha}$: replace $v \in V$ with $K_{\alpha(v)}$, and connect a vertex of $K_{\alpha(u)}$ with one of $K_{\alpha(v)}$ if $uv \in E$.



Easy facts.

- If P is $(\mathbf{3}+\mathbf{1})$-free and $G = \text{inc}(P)$, then $G^\alpha = \text{inc}(P^\alpha)$ for some $(\mathbf{3}+\mathbf{1})$-free P^α.
- If G is clawfree, then G^α is clawfree.
- $\displaystyle\prod_i C_G(x_i) = \sum_{\alpha:V\to\mathbb{N}} X_{G^\alpha}$

Corollary. (a) *If P is $(\mathbf{3}+\mathbf{1})$-free and $G = \mathrm{inc}(P)$, then $C_G(t)$ has only real zeros.*

(b) *Let G be clawfree. If Gasharov's question whether X_G is s-positive has an affirmative answer, then Hamidoune's conjecture that $C_G(t)$ has only real zeros is true.*

General question. What polynomials $P(t)$ can be proved "combinatorially" to have only real zeros by showing that $\prod P(x_i)$ is s-positive or e-positive?

Jacobi-Trudi immanants

$\boldsymbol{A = (a_{ij})}$: $n \times n$ matrix

$\boldsymbol{\chi^{\lambda}}$: irreducible character of $\mathfrak{S}_n$ indexed by $\lambda \vdash n$

Define the **immanant**

$$\mathbf{Imm}_{\boldsymbol{\lambda}}\boldsymbol{(A)} = \sum_{w \in \mathfrak{S}_n} \chi^{\lambda}(w) \prod_{i=1}^{n} a_{i,w(i)}.$$

$$\lambda = (1^n) \Rightarrow \mathrm{Imm}_{\lambda}(A) = \det(A)$$

$$\lambda = (n) \Rightarrow \mathrm{Imm}_{\lambda}(A) = \mathrm{per}(A).$$

Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ be partitions with $\nu \subseteq \mu$.

Define

$$\begin{aligned} \boldsymbol{H_{\mu/\nu}} &= \boldsymbol{H_{\mu/\nu}(x)} \\ &= \left[h_{\mu_i - \nu_j + j - i}(x) \right]_{1 \leq i,j \leq n}, \end{aligned}$$

the **Jacobi-Trudi matrix** corresponding to μ/ν.

Fact: $\det H_{\mu/\nu} = s_{\mu/\nu}$

$\boldsymbol{m_\lambda}$: monomial symmetric function

Theorem (conjectured by Goulden-Jackson, proved by Greene). $\mathrm{Imm}_\lambda(H_{\mu/\nu})$ *is m-positive.*

Theorem (Haiman). $\mathrm{Imm}_\lambda(H_{\mu/\nu})$ *is s-positive.*

Haiman's proof is based on Kazhdan-Lusztig theory.

Define

$$\begin{aligned} \boldsymbol{F_{\mu/\nu}(x,y)} &= \sum_{\lambda\vdash n} \mathrm{Imm}_\lambda(H_{\mu/\nu})(x)s_\lambda(y) \\ &= \sum_{w\in\mathfrak{S}_n} h_{\mu+\delta-w(\nu+\delta)}(x)p_{\rho(w)}(y). \end{aligned}$$

Define $E^\theta_{\mu/\nu}$ for $\theta \vdash N := |\mu/\nu|$ by

$$F_{\mu/\nu}(x,y) = \sum_{\theta\vdash N} s_\theta(x)\boldsymbol{E^\theta_{\mu/\nu}}(y).$$

Conjecture (Stembridge, equivalently). $E^{\theta}_{\mu/\nu}$ is h-positive.

Theorem (RS & Stembridge). *TFAE:*

- $E^{(N)}_{\mu/\nu}$ *is h-positive.*
- *Let P be $(\mathbf{3}+\mathbf{1})$-free and $\mathbf{2}+\mathbf{2}$-free, i.e., a **semiorder** or **unit interval order**, and let $G = \mathrm{inc}(P)$. Then X_G is e-positive.*