Catalan Numbers

Richard P. Stanley

June 9, 2017

An OEIS entry

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A000108: $1, 1, 2, 5, 14, 42, 132, 429, \ldots$

$C_0 = 1,\ C_1 = 2,\ C_2 = 3,\ C_3 = 5,\ C_4 = 14, \ldots$

C_n is a **Catalan number**.

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Comments. ... This is probably the longest entry in OEIS, and rightly so.

Catalan monograph

R. Stanley, *Catalan Numbers*, Cambridge University Press, 2015.

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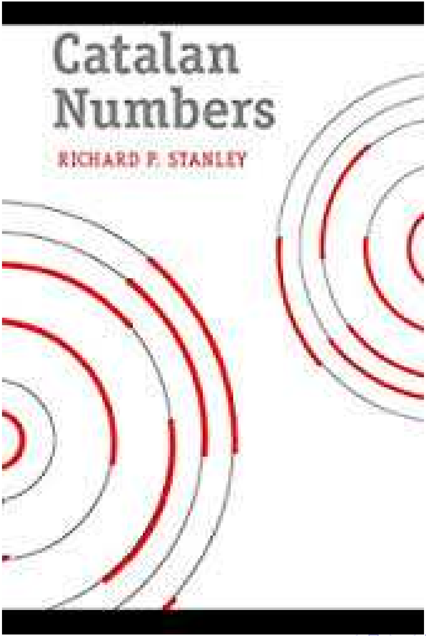
Includes 214 combinatorial interpretations of C_n and 68 additional problems.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History

Sharabiin Myangat, also known as **Minggatu**, **Ming'antu** (明安图), and **Jing An** (c. 1692–c. 1763): a Mongolian astronomer, mathematician, and topographic scientist who worked at the Qing court in China.

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No combinatorics, no further work in China.

Ming'antu

Manuscript of Ming'antu

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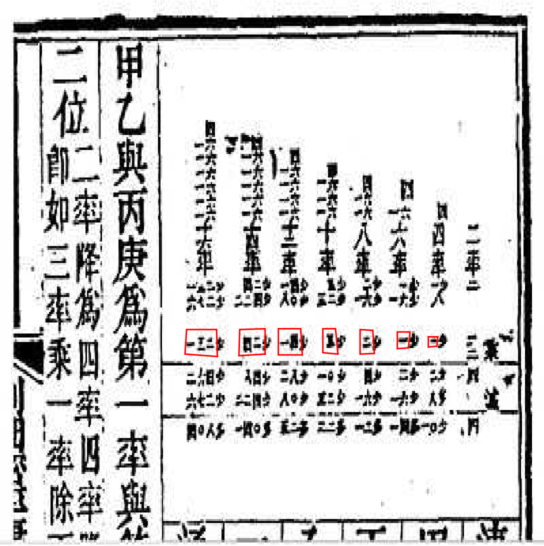

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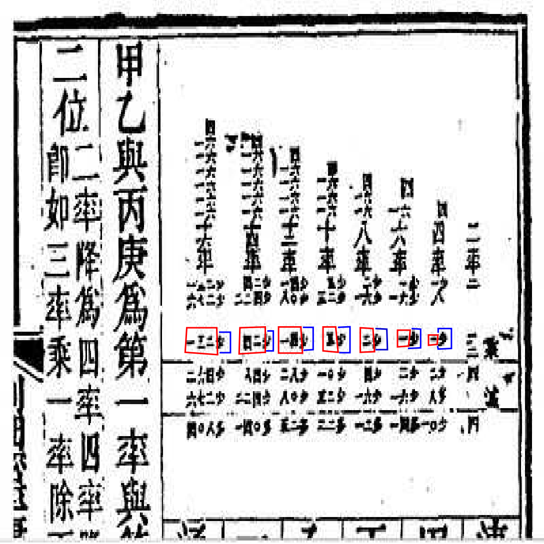

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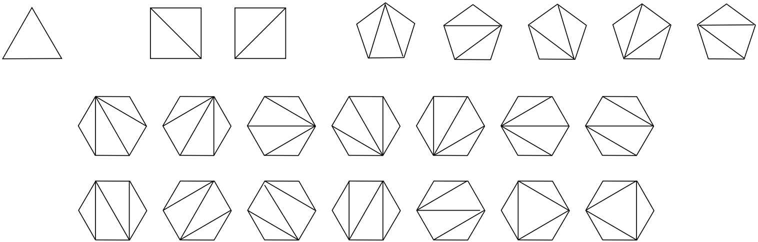
More history, via Igor Pak

- **Euler** (1751): conjectured formula for the number of triangulations of a convex $(n+2)$-gon. In other words, draw $n-1$ noncrossing diagonals of a convex polygon with $n+2$ sides.



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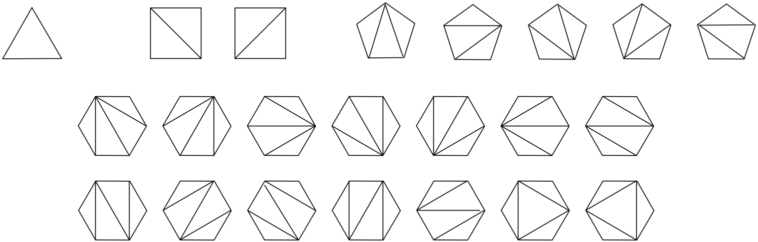

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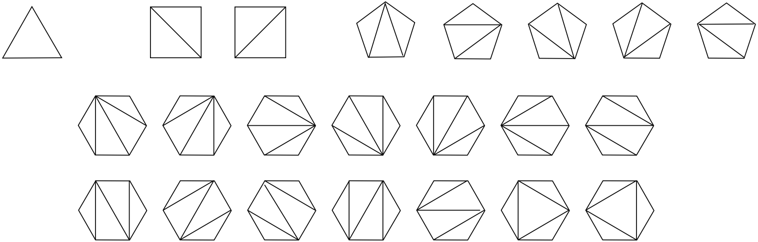

$$1, \quad 2, \quad 5, \quad 14, \quad \ldots$$

We **define** these numbers to be the Catalan numbers C_n.

Completion of proof

- **Goldbach and Segner** (1758–1759): helped Euler complete the proof, in pieces.
- **Lamé** (1838): first self-contained, complete proof.

Catalan

- **Eugène Charles Catalan** (1838): wrote C_n in the form $\frac{(2n)!}{n!\,(n+1)!}$ and showed it counted (nonassociative) **bracketings** (or **parenthesizations**) of a string of $n+1$ letters.

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Born in 1814 in Bruges (now in Belgium, then under Dutch rule). Studied in France and worked in France and Liège, Belgium. Died in Liège in 1894.

Why "Catalan numbers"?

- **John Riordan** (1948): introduced the term "Catalan number" in *Math Reviews*.

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- **Martin Gardner** (1976): used the term in his Mathematical Games column in *Scientific American*. Real popularity began.

The primary recurrence

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad C_0 = 1$$

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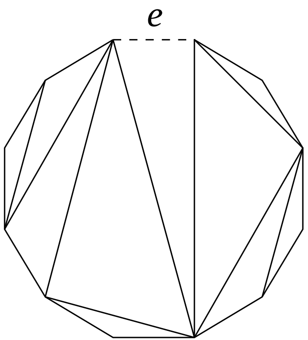

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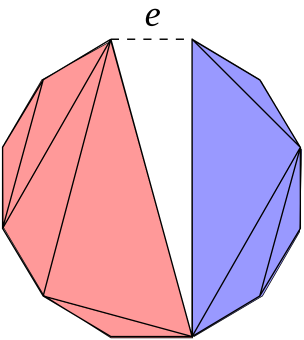

Solving the recurrence

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Let $\boldsymbol{y} = \sum_{n \geq 0} C_n x^n$ (**generating function**).

Multiply recurrence by x^n and sum on $n \geq 0$.

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Multiply recurrence by x^n and sum on $n \geq 0$.

After simplification, we get

$$xy^2 - y + 1 = 0.$$

Solving the quadratic equation

$$xy^2 - y + 1 = 0 \Rightarrow y = \frac{1 \pm \sqrt{1-4x}}{2x}$$

The minus sign turns out to be correct, so

$$y = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}.$$

A formula for C_n

We get

$$\begin{aligned} y &= \frac{1}{2x}(1-\sqrt{1-4x}) \\ &= \frac{1}{2x}\left(1-\sum_{n\geq 0}\binom{1/2}{n}(-4x)^n\right), \end{aligned}$$

where $\binom{1/2}{n} = \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-3}{2})}{n!}$.

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where $\binom{1/2}{n} = \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-3}{2})}{n!}$.
Simplifies to $y = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}x^n$, so

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}$$

Other combinatorial interpretations

$$\begin{aligned} \mathcal{P}_n &:= \{\text{triangulations of convex } (n+2)\text{-gon}\} \\ \Rightarrow \#\mathcal{P}_n &= C_n \text{ (where } \#S = \text{number of elements of } S) \end{aligned}$$

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bijective proof: show that $C_n = \#\mathcal{S}_n$ by giving a bijection

$$\varphi\colon \mathcal{T}_n \to \mathcal{S}_n$$

(or $\mathcal{S}_n \to \mathcal{T}_n$), where we already know $\#\mathcal{T}_n = C_n$.

Bijection

Reminder: a **bijection** $\varphi\colon S \to T$ is a function that is one-to-one and onto, that is, for every $t \in T$ there is a unique $s \in S$ for which $\varphi(s) = t$.

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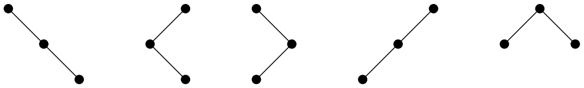
If S, T are finite and $\varphi\colon S \to T$ is a bijection, then $\#S = \#T$ (the "best" way to prove $\#S = \#T$).

Binary trees

4. Binary trees with n vertices (each vertex has a left subtree and a right subtree, which may be empty)



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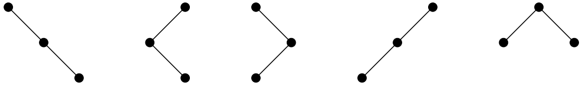

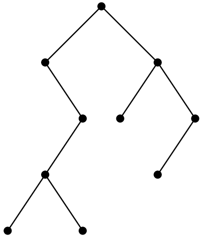

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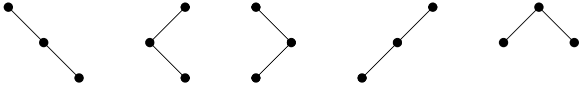
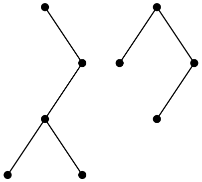
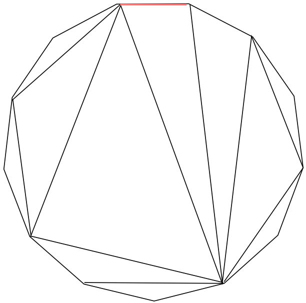




Bijection with triangulations



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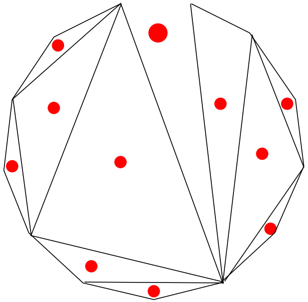

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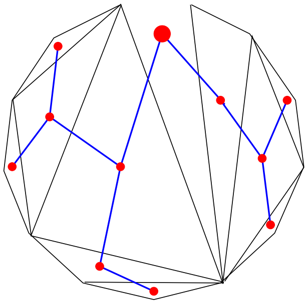

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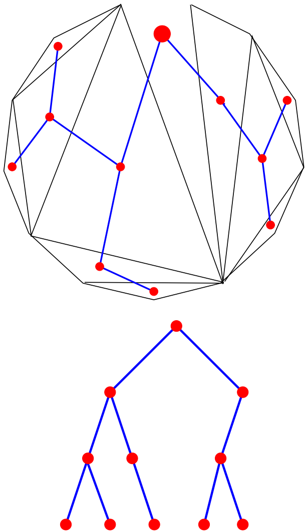

Binary parenthesizations

3. Binary **parenthesizations** or **bracketings** of a string of $n+1$ letters

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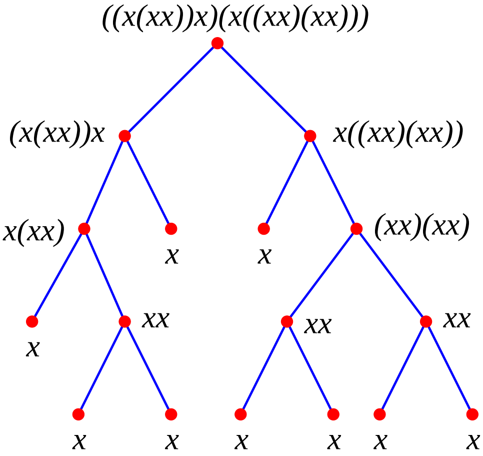
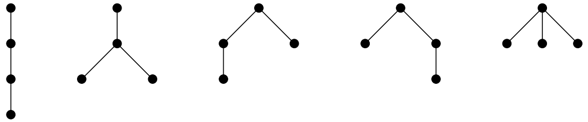


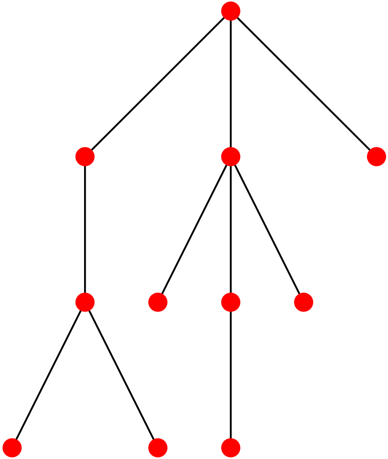
Plane trees

Plane tree: subtrees of a vertex are linearly ordered

6. Plane trees with $n+1$ vertices



Plane tree recurrence



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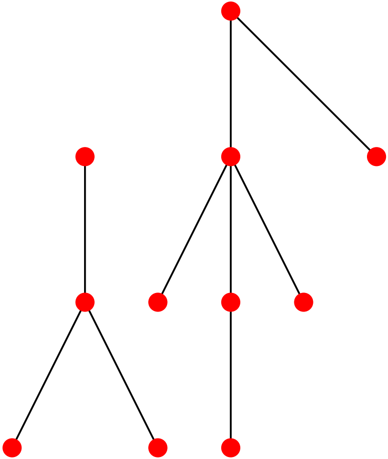

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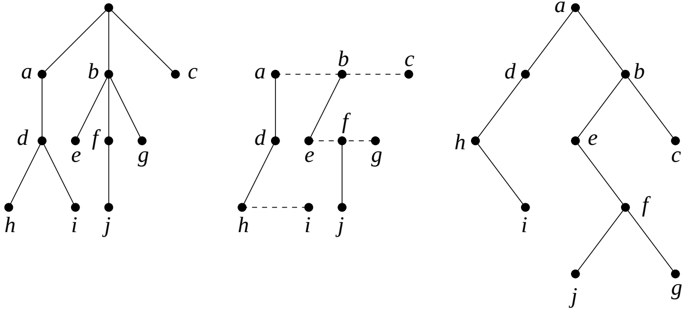

The ballot problem

Bertrand's ballot problem: first published by **W. A. Whitworth** in 1878 but named after **Joseph Louis François Bertrand** who rediscovered it in 1887 (one of the first results in probability theory).

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Special case: there are two candidates A and B in an election. Each receives n votes. What is the probability that A will never trail B during the count of votes?

Example. $AABABBBAAB$ is bad, since after seven votes, A receives 3 while B receives 4.

Definition of ballot sequence

Encode a vote for A by 1, and a vote for B by -1 (abbreviated $-$). Clearly a sequence $a_1 a_2 \cdots a_{2n}$ of n each of 1 and -1 is allowed if and only if $\sum_{i=1}^{k} a_i \geq 0$ for all $1 \leq k \leq 2n$. Such a sequence is called a **ballot sequence**.

Ballot sequences

77. Ballot sequences, i.e., sequences of n 1's and n -1's such that every partial sum is nonnegative (with -1 denoted simply as $-$ below)

$$111--- \quad 11-1-- \quad 11--1- \quad 1-11-- \quad 1-1-1-$$

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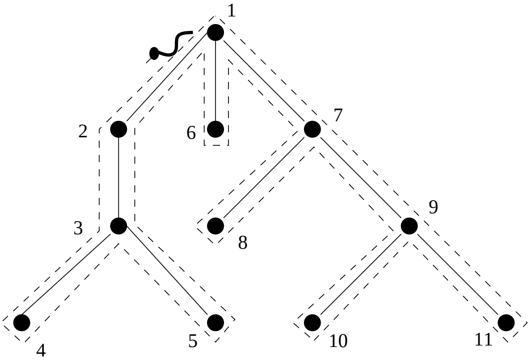
Note. Answer to original problem (probability that a sequence of n each of 1's and -1's is a ballot sequence) is therefore

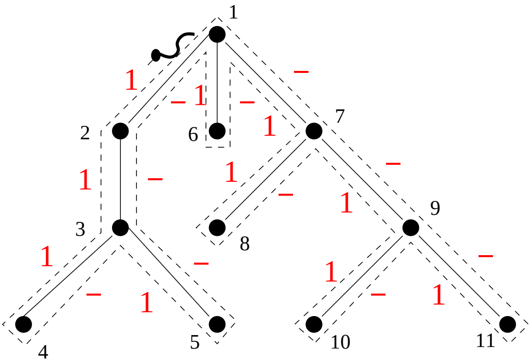
$$\frac{C_n}{\binom{2n}{n}}=\frac{\frac{1}{n+1}\binom{2n}{n}}{\binom{2n}{n}}=\frac{1}{n+1}.$$

Bijection with plane trees



depth first order or **preorder**

Bijection with plane trees

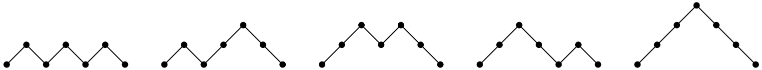


down an edge: $+1$, up an edge: -1

$$1\,1\,1 - 1 - - - 1 - 1\,1 - 1\,1 - 1 - - -$$

Dyck paths

25. Dyck paths of length $2n$, i.e., lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis



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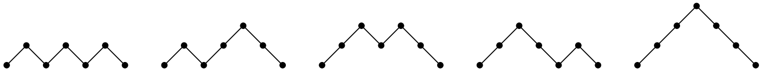

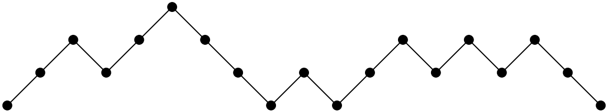

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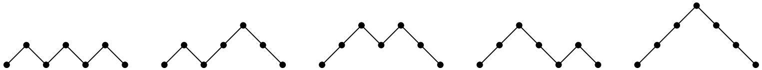

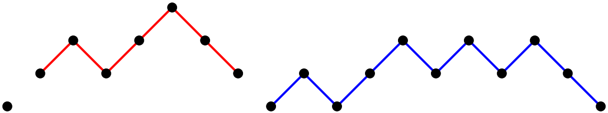

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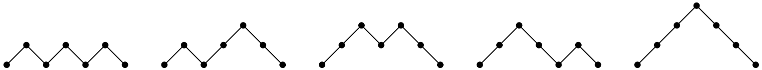
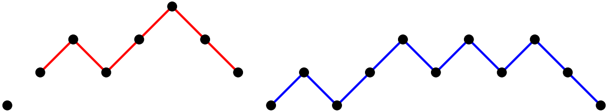
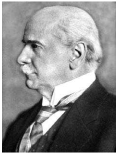
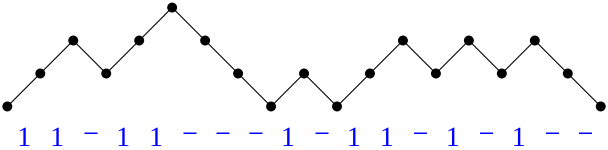




Walther von Dyck (1856–1934)



Bijection with ballot sequences

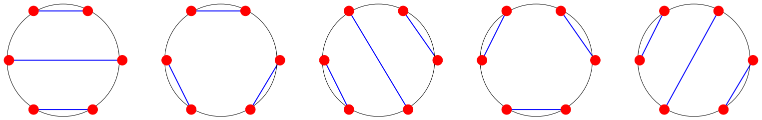


For each upstep, record 1.

For each downstep, record -1.

Noncrossing chords

59. n nonintersecting chords joining $2n$ points on the circumference of a circle



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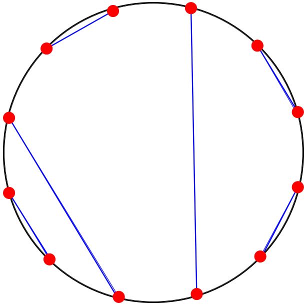

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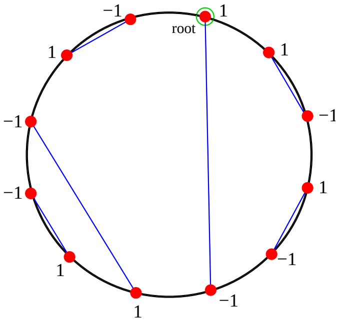

$$1\,1\,-\,1\,-\,-\,1\,1\,-\,-\,1\,-$$

312-avoiding permutations

116. Permutations $a_1 a_2 \cdots a_n$ of $1, 2, \ldots, n$ for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (called **312-avoiding** permutations)

$$123 \qquad 132 \qquad 213 \qquad 231 \qquad 321$$

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$$34251768$$

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part of the subject of **pattern avoidance**

321-avoiding permutations

Another example of pattern avoidance:

115. Permutations $a_1 a_2 \cdots a_n$ of $1, 2, \ldots, n$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$), called **321-avoiding** permutations

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more subtle: no obvious decomposition into two pieces

Bijection with Dyck paths

$$w = 412573968$$

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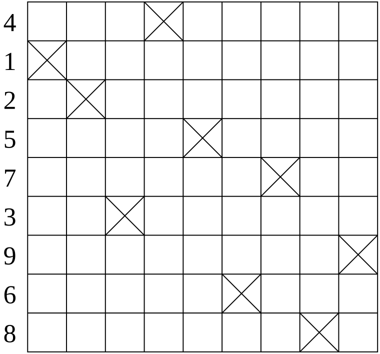

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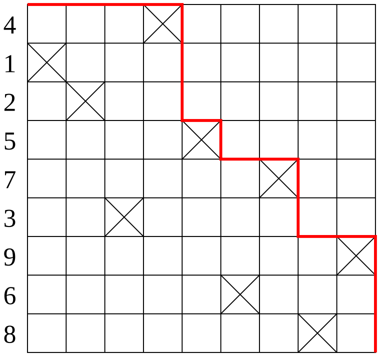

An unexpected interpretation

92. n-tuples $(a_1, a_2, \ldots, a_n)$ of integers $a_i \geq 2$ such that in the sequence $1a_1a_2\cdots a_n1$, each a_i divides the sum of its two neighbors

14321 13521 13231 12531 12341

Bijection with ballot sequences

remove largest; insert bar before the element to its left; continue until only 1's remain; then replace bar with 1 and an original number with -1, except last two

$$1 \quad 2 \quad 5 \quad 3 \quad 4 \quad 1$$

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$$\begin{array}{ccccccccccc} | & 1 & | & | & 2 & 5 & | & 3 & 4 & 1 \\ 1 & - & 1 & 1 & - & - & 1 & - & & \end{array}$$

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$$\begin{array}{cccccccccc}
| & 1 & | & | & 2 & 5 & | & 3 & 4 & 1 \\
1 & - & 1 & 1 & - & - & 1 & - & &
\end{array}$$

tricky to prove

Analysis

A65.(b)

$$\sum_{n\geq 0} \frac{1}{C_n} = ??$$

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$$1 + 1 + \frac{1}{2} + \frac{1}{5} = 2.7$$

Analysis

A65.(b)

$$\sum_{n\geq 0} \frac{1}{C_n} = 2 + \frac{4\sqrt{3}\pi}{27}$$

$$1 + 1 + \frac{1}{2} + \frac{1}{5} = 2.7$$

Why?

A65.(a)

$$\sum_{n\geq 0} \frac{x^n}{C_n} = \frac{2(x+8)}{(4-x)^2} + \frac{24\sqrt{x}\sin^{-1}\left(\frac{1}{2}\sqrt{x}\right)}{(4-x)^{5/2}}.$$

Why?

A65.(a)

$$\sum_{n\geq 0}\frac{x^n}{C_n}=\frac{2(x+8)}{(4-x)^2}+\frac{24\sqrt{x}\sin^{-1}\left(\frac{1}{2}\sqrt{x}\right)}{(4-x)^{5/2}}.$$

Based on a (difficult) calculus exercise: let

$$y=2\left(\sin^{-1}\frac{1}{2}\sqrt{x}\right)^2.$$

Then $y=\displaystyle\sum_{n\geq 1}\frac{x^n}{n^2\binom{2n}{n}}$. Use $\sin^{-1}x=\sum_{n\geq 0}4^{-n}C_nx^{2n+1}$.

The last slide

The last slide 

The last slide