The Sperner Property

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Sperner's theorem

Theorem (**E. Sperner**, 1927). *Let* $S_1, S_2, \ldots, S_m$ *be subsets of an* n*-element set* X *such that* $S_i \not\subseteq S_j$ *for* $i \neq j$*, Then* $m \leq \binom{n}{\lfloor n/2 \rfloor}$*, achieved by taking all* $\lfloor n/2 \rfloor$*-element subsets of* X*.*

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Emanuel Sperner
9 December 1905 – 31 January 1980

Posets

A **poset** (partially ordered set) is a set $\boldsymbol{P}$ with a binary relation $\leq$ satisfying:

- **Reflexivity:** $t \leq t$
- **Antisymmetry:** $s \leq t,\ t \leq s \Rightarrow s = t$
- **Transitivity:** $s \leq t,\ t \leq u \Rightarrow s \leq u$

Graded posets

chain: $u_1 < u_2 < \cdots < u_k$

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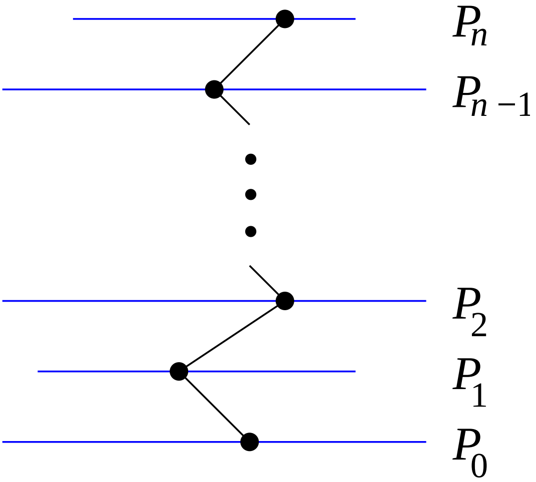
Assume P is **finite**. P is **graded of rank *n*** if

$$P = P_0 \cup P_1 \cup \cdots \cup P_n,$$

such that every maximal chain has the form

$$t_0 < t_1 < \cdots < t_n, \quad t_i \in P_i.$$

Diagram of a graded poset



Rank-symmetry and unimodality

Let $p_i = \#P_i$.

Rank-generating function: $F_P(q) = \sum_{i=0}^{n} p_i q^i$

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rank-unimodal and rank-symmetric $\Rightarrow j = \lfloor n/2 \rfloor$

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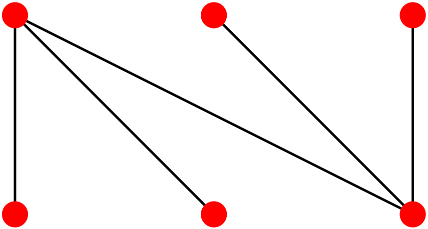
P is **Sperner** (or has the **Sperner property**) if

$$\max_A \#A = \max_i p_i$$

An example



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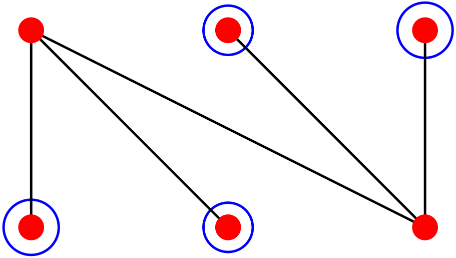

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The boolean algebra

$\boldsymbol{B_n}$: subsets of $\{1, 2, \ldots, n\}$, ordered by inclusion

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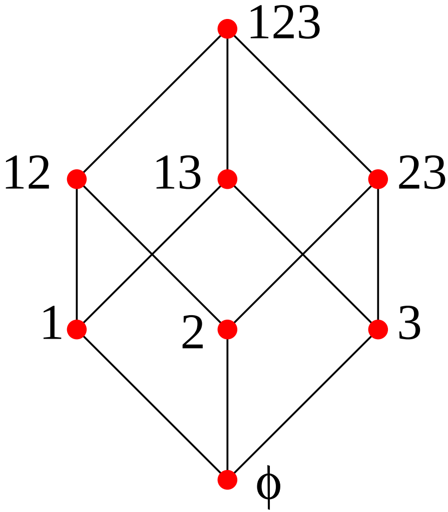
$$p_i = \binom{n}{i}, \quad F_{B_n}(q) = (1+q)^n$$

rank-symmetric, rank-unimodal

Diagram of B_3



Sperner's theorem, restated

Theorem. *The boolean algebra B_n is Sperner.*

Proof (**D. Lubell**, 1966).

- B_n has $n!$ maximal chains.

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$$\sum_{S \in A} |S|!\,(n-|S|)! \leq n!.$$

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$$\sum_{S \in A} \frac{1}{\binom{n}{|S|}} \leq 1.$$

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- $\Rightarrow |A| \leq \binom{n}{\lfloor n/2 \rfloor} \quad \square$

A q-analogue

Lubell's proof carries over to some other posets.

Theorem. *Let $B_n(q)$ denote the poset of all subspaces of $\mathbb{F}_q^n$. Then $B_n(q)$ is Sperner.*

Linear algebra

$$\boldsymbol{P} = P_0 \cup \cdots \cup P_m : \quad \text{graded poset}$$

$$\mathbb{Q}\boldsymbol{P_i} : \quad \text{vector space with basis } \mathbb{Q}$$

$U \colon \mathbb{Q}P_i \to \mathbb{Q}P_{i+1}$ is **order-raising** if for all $s \in P_i$,

$$U(s) \in \mathrm{span}_{\mathbb{Q}}\{t \in P_{i+1} \,:\, s < t\}$$

Order-matchings

Order matching: $\mu\colon P_i \to P_{i+1}$: injective and $\mu(t) < t$

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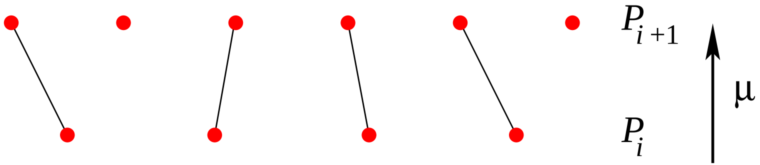

Order-raising and order-matchings

Key Lemma. *If* $U\colon \mathbb{Q}P_i \to \mathbb{Q}P_{i+1}$ *is injective and order-raising, then there exists an order-matching* $\mu\colon P_i \to P_{i+1}$.

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Proof. Consider the matrix of U with respect to the bases P_i and P_{i+1}.

Key lemma proof

$$\overbrace{\begin{matrix} t_1 & \cdots & t_m & \cdots & t_n \end{matrix}}^{P_{i+1}}$$

$$P_i \left\{ \begin{matrix} s_1 \\ \vdots \\ s_m \end{matrix} \right. \begin{bmatrix} \neq 0 & & & \Big| & * \\ & \ddots & & \Big| & * \\ & & \neq 0 & \Big| & * \end{bmatrix}$$

$$\mathbf{det \neq 0}$$

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\Rightarrow s_1 < t_1, \ldots, s_m < t_m \qquad \square
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Minor variant

Similarly if there exists **surjective** order-raising $U\colon \mathbb{Q}P_i \to \mathbb{Q}P_{i+1}$, then there exists an order-matching $\mu\colon P_{i+1} \to P_i$.

A criterion for Spernicity

$$\boldsymbol{P} = P_0 \cup \cdots \cup P_n : \text{ finite graded poset}$$

Proposition. *If for some j there exist order-raising operators*

$$\mathbb{Q}P_0 \stackrel{\text{inj.}}{\to} \mathbb{Q}P_1 \stackrel{\text{inj.}}{\to} \cdots \stackrel{\text{inj.}}{\to} \mathbb{Q}P_j \stackrel{\text{surj.}}{\to} \mathbb{Q}P_{j+1} \stackrel{\text{surj.}}{\to} \cdots \stackrel{\text{surj.}}{\to} \mathbb{Q}P_n,$$

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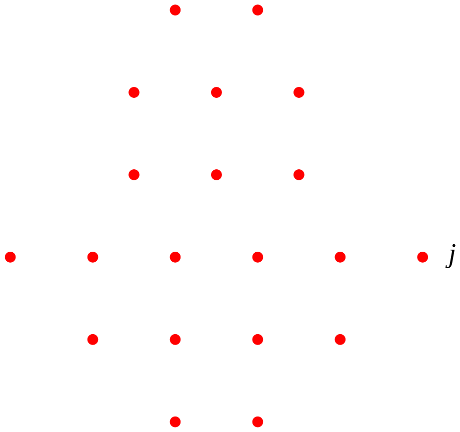
"Glue together" the order-matchings.

Gluing example



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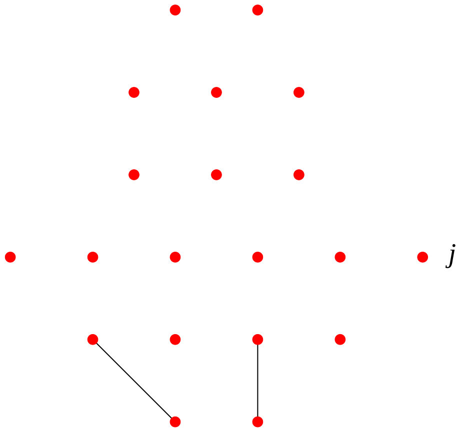

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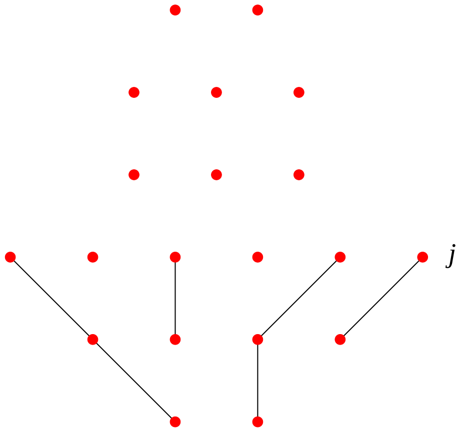

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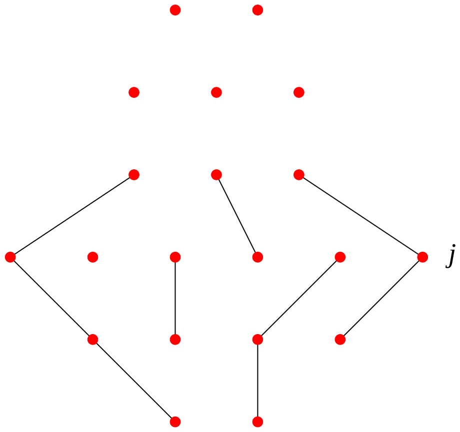

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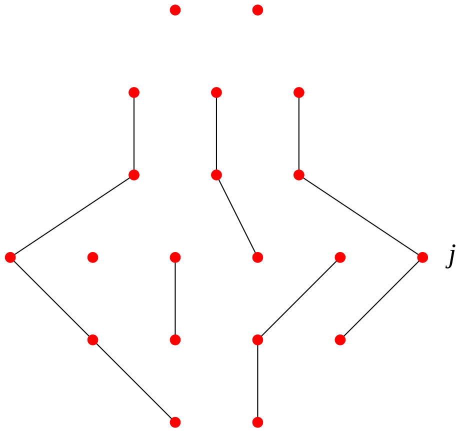

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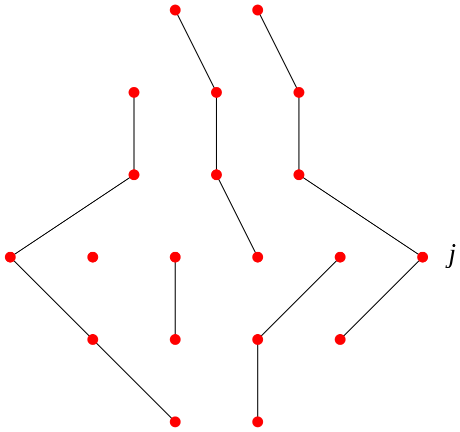

A chain decomposition

$$P = C_1 \cup \cdots \cup C_{p_j} \quad \text{(chains)}$$

$$A = \text{ antichain}, C = \text{ chain} \Rightarrow \#(A \cap C) \leq 1$$

$$\Rightarrow \#A \leq p_j. \qquad \square$$

Back to B_n

Explicit order matching $(B_n)_i \to (B_n)_{i+1}$ for $i < n/2$:

Example. $S = \{1, 4, 6, 7, 11\} \in (B_{13})_5$

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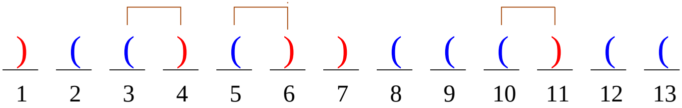

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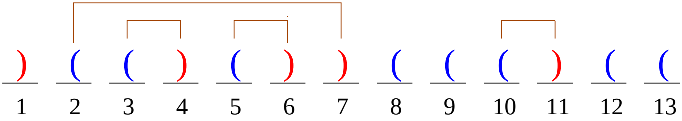

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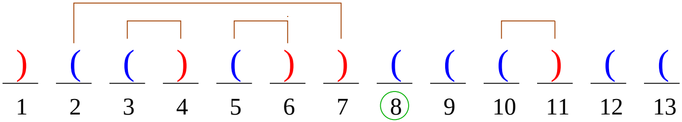

Order-raising for B_n

Define

$$\boldsymbol{U} : \mathbb{Q}(B_n)_i \to \mathbb{Q}(B_n)_{i+1}$$

by

$$U(S) = \sum_{\substack{\#T=i+1 \\ S\subset T}} T, \quad S \in (B_n)_i.$$

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Note. UD is positive semidefinite, and hence has nonnegative real eigenvalues, since the matrices of U and D with respect to the bases $(B_n)_i$ and $(B_n)_{i+1}$ are *transposes*.

A commutation relation

Lemma. *On* $\mathbb{Q}(B_n)_i$ *we have*

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Corollary. *If* $i < n/2$ *then* U *is injective.*

Proof. UD has eigenvalues $\theta \geq 0$, and eigenvalues of DU are $\theta + n - 2i > 0$. Hence DU is invertible, so U is injective. $\square$

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Corollary. B_n *is Sperner.*

What's the point?

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The symmetric group $\mathfrak{S}_n$ acts on B_n by

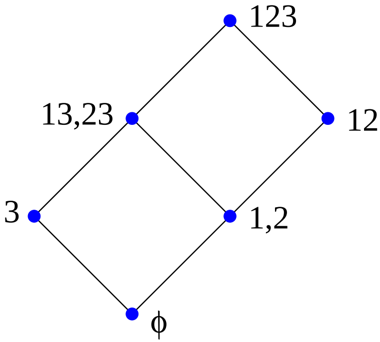
$$w \cdot \{a_1, \dots, a_k\} = \{w \cdot a_1, \dots, w \cdot a_k\}.$$

If G is a subgroup of $\mathfrak{S}_n$, define the **quotient poset** B_n/G to be the poset on the orbits of G (acting on B_n), with

$$\mathfrak{o} \le \mathfrak{o}' \quad \Leftrightarrow \quad \exists S \in \mathfrak{o}, T \in \mathfrak{o}', \quad S \subseteq T.$$

An example

$$n = 3, \quad G = \{(1)(2)(3), (1,2)(3)\}$$



Spernicity of B_n/G

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Crux of proof. The action of $w \in G$ on B_n commutes with U, so we can "transfer" U to B_n/G, preserving injectivity on the bottom half.

An interesting example

R: set of squares of an $m \times n$ rectangle of squares.

$\boldsymbol{G_{mn}} \subset \mathfrak{S}_R$: can permute elements in each row, and permute rows among themselves, so $\#G_{mn} = n!^m m!$.

$$G_{mn} \cong \mathfrak{S}_n \wr \mathfrak{S}_m \quad (\textbf{wreath product})$$

Young diagrams

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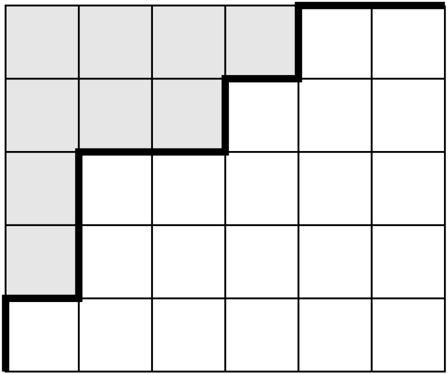

B_R/G_{mn}

Lemma. *Each orbit* $\mathfrak{o} \in B_R/G_{mn}$ *contains exactly one Young diagram.*

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Proof. Let $S \in B_R$. Let $\boldsymbol{\alpha_i}$ be the number of elements of S in row i. Let $\boldsymbol{\lambda_1} \geq \cdots \geq \boldsymbol{\lambda_m}$ be the decreasing rearrangement of $\alpha_1, \ldots, \alpha_m$. Then the unique Young diagram in the orbit containing S has λ_i elements in row i. $\square$

Poset structure of B_r/G_{mn}

$\boldsymbol{Y_{\mathfrak{o}}}$: Young diagram in the orbit $\mathfrak{o}$

Easy **:** $\mathfrak{o} \leq \mathfrak{o}'$ in B_R/G_{mn} if and only if $Y_{\mathfrak{o}} \subseteq Y_{\mathfrak{o}'}$ (containment of Young diagram).

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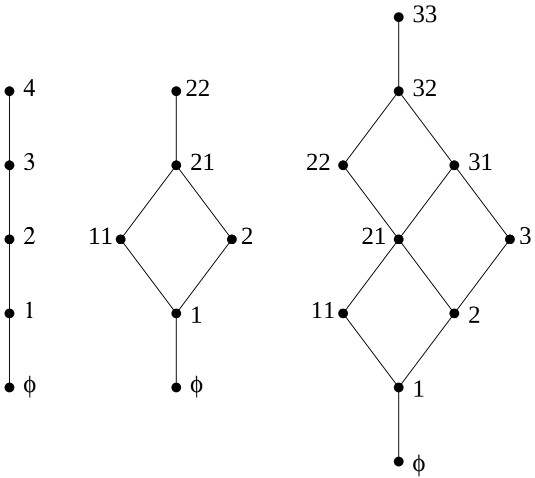
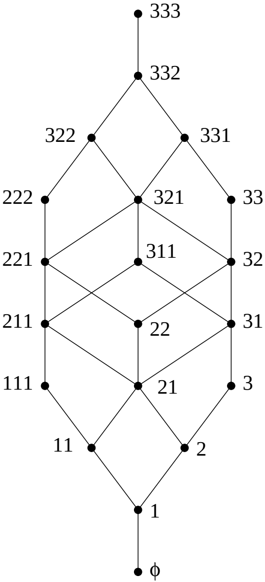
$L(m, n)$: poset of Young diagrams in an $m \times n$ rectangle

Corollary. $B_R/G_{mn} \cong L(m, n)$

Examples of $L(m,n)$



$L(3,3)$



q-binomial coefficients

For $0 \le k \le n$, define the **q-binomial coefficient**

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}.$$

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Example. $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 1 + q + 2q^2 + q^3 + q^4$

Properties

- $\begin{bmatrix} n \\ k \end{bmatrix} \in \mathbb{N}[q]$

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- Not an order-matching. Still open to find an explicit order-matching $L(m,n)_i \to L(m,n)_{i+1}$.

Algebraic geometry

$\boldsymbol{X}$: smooth complex projective variety of dimension n

$\boldsymbol{H^*(X;\mathbb{C})} = H^0(X;\mathbb{C}) \oplus H^1(X;\mathbb{C}) \oplus \cdots \oplus H^{2n}(X;\mathbb{C})$: cohomology ring, so $H^i \cong H^{2n-i}$.

Hard Lefschetz Theorem. *There exists* $\boldsymbol{\omega} \in H^2$ *(the class of a generic hyperplane section) such that for* $0 \leq i \leq n$*, the map*

$$\omega^{n-2i} \colon H^i \to H^{2n-i}$$

is a bijection. Thus $\omega \colon H^i \to H^{i+1}$ *is injective for* $i \leq n$ *and surjective for* $i \geq n$*.*

Cellular decompositions

X has a **cellular decomposition** if $X = \sqcup C_i$, each $C_i \cong \mathbb{C}^{d_i}$ (as affine varieties), and each $\bar{C}_i$ is a union of C_j's.

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Fact. If X has a cellular decomposition and $[C_i] \in H^{2(n-i)}$ denotes the corresponding cohomology classes, then the $[C_i]$'s form a $\mathbb{C}$-basis for H^*.

The cellular decomposition poset

Let $X = \sqcup C_i$ be a cellular decomposition. Define a poset $\boldsymbol{P_X} = \{C_i\}$, by

$$C_i \leq C_j \text{ if } C_i \subseteq \bar{C}_j$$

(closure in Zariski or classical topology).

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- P_X is rank-unimodal (hard Lefschetz)

Spernicity of P_X

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$\Rightarrow$ **Theorem.** P_X has the Sperner property.

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rational canonical form $\Rightarrow P_{\mathrm{Gr}(m+n,m)} \cong L(m,n)$

“Best” special case

$G = \mathrm{SO}(2n+1, \mathbb{C})$, $Q =$ “spin” maximal parabolic subgroup

$\boldsymbol{M(n)} := P_{G/Q} \cong \mathfrak{B}_n/\mathfrak{S}_n$, where $\boldsymbol{\mathfrak{B}_n}$ is the hyperoctahedral group (symmetries of n-cube) of order $2^n n!$, so $\# M(n) = 2^n$

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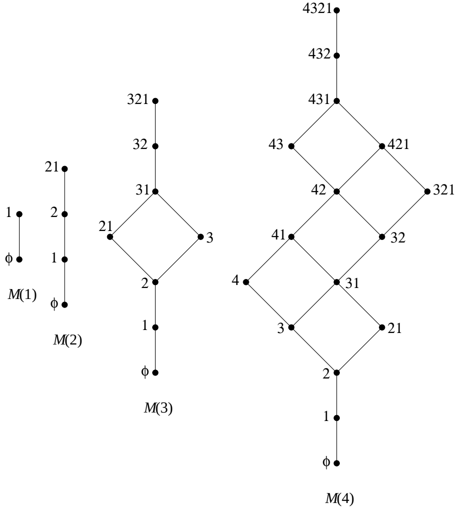
$M(n)$ is isomorphic to the set of all subsets of $\{1, 2, \dots, n\}$ with the ordering

$$\{a_1 > a_2 > \cdots > a_r\} \leq \{b_1 > b_2 > \cdots > b_s\},$$

if $r \leq s$ and $a_i \leq b_i$ for $1 \leq i \leq r$.

Examples of $M(n)$



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$$\Rightarrow F(M(n), q) := \sum_{i=0}^{\binom{n}{2}} |M(n)_i| \cdot q^i = (1+q)(1+q^2)\cdots(1+q^n)$$

Corollary. *The polynomial* $(1+q)(1+q^2)\cdots(1+q^n)$ *has unimodal coefficients.*

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No combinatorial proof known, though can be done with just elementary linear algebra (**Proctor**).

The function $f(S, \alpha)$

Let $S \subset \mathbb{R}$, $\#S < \infty$, $\alpha \in \mathbb{R}$.

$$\boldsymbol{f(S, \alpha)} = \#\{T \subseteq S : \sum_{i \in T} i = \alpha\}$$

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Example. $f(\{1, 2, 4, 5, 7, 10\}, 7) = 3$:

$$7 = 2 + 5 = 1 + 2 + 4$$

The Erdős-Moser conjecture for $\mathbb{R}^+$

Let $\mathbb{R}^+ = \{i \in \mathbb{R} \,:\, i > 0\}$.

Erdős-Moser Conjecture for $\mathbb{R}^+$

$$S \subset \mathbb{R}^+,\ \#S = n$$
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Note. $\frac{1}{2}\binom{n+1}{2} = \frac{1}{2}(1+2+\cdots+n)$

The proof

Proof. Suppose $S = \{a_1, \ldots, a_k\}$, $a_1 > \cdots > a_k$. Let

$$a_{i_1} + \cdots + a_{i_r} = a_{j_1} + \cdots + a_{j_s},$$

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Now $\{i_1, \ldots, i_r\} \geq \{j_1, \ldots, j_s\}$ in $M(n)$

$$\begin{aligned} &\Rightarrow \quad r \geq s,\ i_1 \geq j_1, \ldots, i_s \geq j_s \\ &\Rightarrow \quad a_{i_1} \geq b_{j_1},\ \ldots, a_{i_s} \geq b_{j_s} \\ &\Rightarrow \quad r = s,\ a_{i_k} = b_{i_k}\ \forall k. \end{aligned}$$

Conclusion of proof

Thus $a_{i_1} + \cdots + a_{i_r} = b_{j_1} + \cdots + b_{j_s}$

$$\Rightarrow \{i_1, \ldots, i_r\} \text{ and } \{j_1, \ldots, j_s\} \text{ are incomparable or equal in } M(n)$$

$$\Rightarrow \#S \leq \max_A \#A = f\left(\{1, \ldots, n\}, \left\lfloor \frac{1}{2}\binom{n+1}{2} \right\rfloor\right) \quad \square$$

The weak order on $\mathfrak{S}_n$

$s_i :== (i, i+1) \in \mathfrak{S}_n$, $1 \leq i \leq n-1$ (**adjacent transposition**)

For $w \in \mathfrak{S}_n$,

$$\begin{aligned} \ell(w) &:= \#\{(i,j) : i < j, w(i) > w(j)\} \\ &= \min\{p : w = s_{i_1} \cdots s_{i_p}\}. \end{aligned}$$

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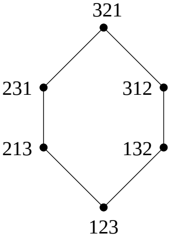
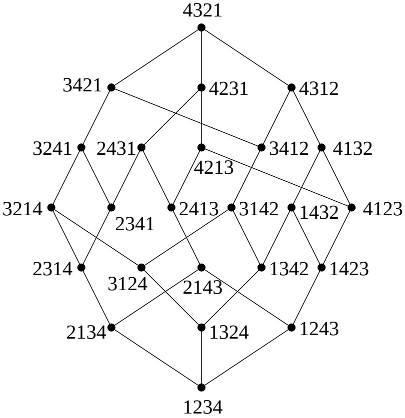
A. Björner (1984): does W_n have the Sperner property?

Examples of weak order



W_3



W_4

An order-raising operator

theory of Schubert polynomials suggests:

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Fact (**Macdonald**, **Fomin-S**). Let $u < v$ in W_n, $\ell(v) - \ell(u) = \mathbf{p}$. The coefficient of v in $U^p(u)$ is

$$p!\, \mathfrak{S}_{vu^{-1}}(1, 1, \ldots, 1),$$

where $\mathfrak{S}_w(x_1, \ldots, x_{n-1})$ is a **Schubert polynomial**.

A down operator

C. Gaetz and **Y. Gao** (2018): constructed $D\colon \mathbb{Q}(W_n)_i \to \mathbb{Q}(W_n)_{i-1}$ such that

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Suffices for Spernicity.

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Suffices for Spernicity.

Note. D is order-lowering on the **strong** Bruhat order. Leads to duality between weak and strong order.

Another method

Z. Hamaker, **O. Pechenik**, **D. Speyer**, and **A. Weigandt** (2018): for $k < \frac{1}{2}\binom{n}{2}$, let

$$\boldsymbol{D(n,k)} = \text{matrix of } U^{\binom{n}{2}-2k} : \mathbb{Q}(W_n)_k \to \mathbb{Q}(W_n)_{\binom{n}{2}-k}$$

with respect to the bases $(W_n)_k$ and $(W_n)_{\binom{n}{2}-k}$ (in some order).
Then (conjectured by RS):

$$\det D(n,k) = \pm \left(\binom{n}{2} - 2k \right)!^{\#(W_n)_k} \prod_{i=0}^{k-1} \left(\frac{\binom{n}{2} - (k+i)}{k-i} \right)^{\#(W_n)_i}.$$

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Also suffices to prove Sperner property (just need $\det D(n,k) \neq 0$).

An open problem

The weak order $W(G)$ can be defined for any (finite) Coxeter group G. Is $W(G)$ Sperner?

The final slide

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