Smith Normal Form and Combinatorics

Richard P. Stanley

Outline

Part I

- basics
- random matrices

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- random matrices

Part II: symmetric functions

- $\frac{\partial}{\partial p_1} p_1$ (operator)
- Jacobi-Trudi specializations

Smith normal form

A: $n \times n$ matrix over commutative ring ***R*** (with 1)

Suppose there exist $\boldsymbol{P}, \boldsymbol{Q} \in \mathrm{GL}(n, R)$ such that

$$PAQ := B = \mathrm{diag}(d_1, d_1 d_2, \dots d_1 d_2 \cdots d_n),$$

where $d_i \in R$. We then call B a **Smith normal form (SNF)** of A.

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NOTE. (1) Can extend to $m \times n$.

$$(2)\ \mathrm{unit} \cdot \det(A) = \det(B) = d_1^n d_2^{n-1} \cdots d_n.$$

Thus SNF is a refinement of $\det$.

Row and column operations

Can put a matrix into SNF by the following operations.

- Add a multiple of a row to another row.
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Over a field, SNF is **row reduced echelon form** (with all unit entries equal to 1).

Existence of SNF

PIR: principal ideal ring, e.g., $\mathbb{Z}$, $K[x]$, $\mathbb{Z}/m\mathbb{Z}$.

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If R is a PIR then A has a unique SNF up to units.

Otherwise A “typically” does not have a SNF but may have one in special cases.

Algebraic note

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Example. ring of entire functions and ring of all algebraic integers (not PIR's)

Open: every matrix over a Bézout domain has an SNF.

Algebraic interpretation of SNF

R: a PID

A: an $n \times n$ matrix over R with rows $v_1, \dots, v_n \in R^n$

$\mathrm{diag}(e_1, e_2, \dots, e_n)$: SNF of A

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Theorem.

$$R^n/(v_1, \dots, v_n) \cong (R/e_1R) \oplus \cdots \oplus (R/e_nR).$$

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$R^n/(v_1, \dots, v_n)$: **(Kasteleyn) cokernel** of A

An explicit formula for SNF

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$\mathrm{diag}(e_1, e_2, \ldots, e_n)$: SNF of A

Theorem. $e_1 e_2 \cdots e_i$ *is the gcd of all* $i \times i$ *minors of* A.

minor: determinant of a square submatrix.

Special case: e_1 is the gcd of all entries of A.

An example

Reduced Laplacian matrix of K_4:

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

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What about SNF?

An example (continued)

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & -1 \\ -4 & 4 & -1 \\ 8 & -4 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & -1 \\ -4 & 4 & 0 \\ 8 & -4 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 0 & 4 & 0 \\ 4 & -4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reduced Laplacian matrix of K_n

$$\begin{aligned} \boldsymbol{L_0(K_n)} &= nI_{n-1} - J_{n-1} \\ \det L_0(K_n) &= n^{n-2} \end{aligned}$$

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Trick: 2×2 submatrices (up to row and column permutations):

$$\begin{bmatrix} n-1 & -1 \\ -1 & n-1 \end{bmatrix}, \quad \begin{bmatrix} n-1 & -1 \\ -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

with determinants $n(n-2)$, $-n$, and 0. Hence $e_1 e_2 = n$. Since $\prod e_i = n^{n-2}$ and $e_i | e_{i+1}$, we get the SNF $\mathrm{diag}(1, n, n, \ldots, n)$. ∎

Laplacian matrices of general graphs

SNF of the Laplacian matrix of a graph: very interesting

connections with sandpile models, chip firing, abelian avalanches, etc.

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SNF of random matrices

Huge literature on random matrices, mostly connected with eigenvalues.

Very little work on SNF of random matrices over a PID.

Is the question interesting?

$\mathbf{Mat}_{\boldsymbol{k}}(\boldsymbol{n})$: all $n \times n$ $\mathbb{Z}$-matrices with entries in $[-k, k]$ (uniform distribution)

$\boldsymbol{p_k(n, d)}$: probability that if $M \in \mathrm{Mat}_k(n)$ and SNF$(M) = (e_1, \ldots, e_n)$, then $e_1 = d$.

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Theorem. $\lim_{k\to\infty} p_k(n,d) = 1/d^{n^2}\zeta(n^2)$

Specifying some e_i

with **Yinghui Wang**

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Two general results.

- Let $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{P}$, $\alpha_i | \alpha_{i+1}$.

  $\boldsymbol{\mu_k(n)}$: probability that the SNF of a random $A \in \mathrm{Mat}_k(n)$ satisfies $e_i = \alpha_i$ for $1 \leq \alpha_i \leq n-1$.

$$\boldsymbol{\mu(n)} = \lim_{k \to \infty} \mu_k(n).$$

  Then $\mu(n)$ exists, and $0 < \mu(n) < 1$.

Second result

- Let $\alpha_n \in \mathbb{P}$.

  $\boldsymbol{\nu_k(n)}$: probability that the SNF of a random $A \in \mathrm{Mat}_k(n)$ satisfies $e_n = \alpha_n$.

  Then

$$\lim_{k \to \infty} \nu_k(n) = 0.$$

Sample result

$\boldsymbol{\mu_k(n)}$: probability that the SNF of a random $A \in \mathrm{Mat}_k(n)$ satisfies $e_1 = 2$, $e_2 = 6$.

$$\boldsymbol{\mu(n)} = \lim_{k \to \infty} \mu_k(n).$$

Conclusion

$$\mu(n) = 2^{-n^2}\left(1 - \sum_{i=(n-1)^2}^{n(n-1)} 2^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} 2^{-i}\right)$$

$$\cdot \frac{3}{2} \cdot 3^{-(n-1)^2}(1-3^{(n-1)^2})(1-3^{-n})^2$$

$$\cdot \prod_{p>3}\left(1 - \sum_{i=(n-1)^2}^{n(n-1)} p^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} p^{-i}\right).$$

Cyclic cokernel

$\boldsymbol{\kappa(n)}$: probability that an $n \times n$ $\mathbb{Z}$-matrix has SNF diag$(e_1, e_2, \ldots, e_n)$ with $e_1 = e_2 = \cdots = e_{n-1} = 1$.

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Theorem. $\kappa(n) = \dfrac{\prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n}\right)}{\zeta(2)\zeta(3)\cdots}$

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Corollary.
$$\lim_{n\to\infty} \kappa(n) = \frac{1}{\zeta(6)\prod_{j\geq 4}\zeta(j)}$$
$$\approx 0.846936\cdots .$$

Small number of generators

g: number of generators of cokernel (number of entries of SNF $\neq 1$) as $n \to \infty$

previous slide: Prob$(g = 1) = 0.846936\cdots$

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Example (**P. Q. Nguyen and I. E. Shparlinski**). Fix k, n. Choose a subgroup G of $\mathbb{Z}^n$ of index $\leq k$ uniformly.

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$$\boldsymbol{\rho_k(n)} : \text{ probability that } G \text{ is cyclic}$$

$$\lim_{n \to \infty} \lim_{k \to \infty} \rho_k(n) \approx 0.846936 \cdots ,$$

same probability of cyclic cokernel as $k, n \to \infty$ using previous distribution.

Part II: symmetric functions

- $\frac{\partial}{\partial p_1} p_1$ (operator)
- Jacobi-Trudi specializations

A down-up operator

In collaboration with Tommy Wuxing Cai.

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Par(n): set of all partitions of n

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V_n: real vector space with basis Par(n)

U

Define $\boldsymbol{U} = \boldsymbol{U_n} : V_n \to V_{n+1}$ by

$$U(\lambda) = \sum_{\mu} \mu,$$

where $\mu \in \mathrm{Par}(n+1)$ and $\mu_i \geq \lambda_i \ \ \forall i$.

Example.

$$U(42211) = 52211 + 43211 + 42221 + 422111$$

D

Dually, define $\boldsymbol{D} = \boldsymbol{D_n} : V_n \to V_{n-1}$ by

$$D(\lambda) = \sum_{\nu} \nu,$$

where $\nu \in \mathrm{Par}(n-1)$ and $\nu_i \le \lambda_i \;\; \forall i$.

Example. $D(42211) = 32211 + 42111 + 4221$

Symmetric functions

NOTE. Identify V_n with the space $\boldsymbol{\Lambda}_{\mathbb{Q}}^{\boldsymbol{n}}$ of all homogeneous symmetric functions of degree n over $\mathbb{Q}$, and identify $\lambda \in V_n$ with the Schur function s_λ. Then

$$U(f) = p_1 f, \quad D(f) = \frac{\partial}{\partial p_1} f.$$

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Write

$$\begin{aligned} U &= \boldsymbol{U_n} : V_n \to V_{n+1} \\ D &= \boldsymbol{D_{n+1}} : v_{n+1} \to V_n. \end{aligned}$$

Commutation relation

Basic commutation relation: $DU - UD = I$

Allows computation of eigenvalues of
$DU : V_n \to V_n$.

Or note that the eigenvectors of $\frac{\partial}{\partial p_1} p_1$ are the p_λ's ($\lambda \vdash n$), with eigenvalue $1 + m_1(\lambda)$, where $\boldsymbol{m_1(\lambda)}$ is the number of parts of λ equal to 1.

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NOTE.

$$\#\{\lambda \vdash n \,:\, m_1(\lambda) = i\} = p(n+1-i) - p(n-i),$$

where $\boldsymbol{p(m)} = \#\mathrm{Par}(m) = \dim V_m$. ∎

Eigenvalues of DU

Theorem. *Let* $1 \leq i \leq n+1$, $i \neq n$. *Then* i *is an eigenvalue of* $D_{n+1}U_n$ *with multiplicity* $p(n+1-i) - p(n-i)$. *Hence*

$$\det D_{n+1}U_n = \prod_{i=1}^{n+1} i^{p(n+1-i)-p(n-i)}.$$

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$$\det D_{n+1}U_n = \prod_{i=1}^{n+1} i^{p(n+1-i)-p(n-i)}.$$

What about SNF of the matrix $[D_{n+1}U_n]$ (with respect to the basis $\mathrm{Par}(n)$)?

Conjecture of A. R. Miller, 2005

Conjecture (first form). The diagonal entries of the SNF of $[D_{n+1}U_n]$ are:

- $(n+1)(n-1)!$, with multiplicity 1
- $(n-k)!$ with multiplicity $p(k+1)-2p(k)+p(k-1)$, $3 \leq k \leq n-2$
- 1, with multiplicity $p(n)-p(n-1)+p(n-2)$.

Not a trivial result

NOTE. $\{p_\lambda\}_{\lambda \vdash n}$ is not an integral basis.

Another form

$\boldsymbol{m_1(\lambda)}$: number of 1's in λ

$\boldsymbol{\mathcal{M}_1(n)}$: multiset of all numbers $m_1(\lambda)+1$, $\lambda \in \mathrm{Par}(n)$

Let SNF of $[D_{n+1}U_n]$ be diag$(f_1, f_2, \dots, f_{p(n)})$.

Conjecture (second form). $f_{p(n)}$ is the product of the **distinct** entries of $\mathcal{M}_1(n)$; $f_{p(n)-1}$ is the product of the remaining **distinct** entries of $\mathcal{M}_1(n)$, etc.

An example: $n = 6$

$$\begin{aligned}
\mathrm{Par}(6) &= \{6, 51, 42, 33, 411, 321, 222, 3111, \\
&\qquad 2211, 21111, 111111\} \\
\mathcal{M}_1(6) &= \{1, 2, 1, 1, 3, 2, 1, 4, 3, 5, 7\} \\
(f_1, \ldots, f_{11}) &= (7 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,\ 3 \cdot 2 \cdot 1, \\
&\qquad 1, 1, 1, 1, 1, 1, 1, 1, 1) \\
&= (840, 6, 1, 1, 1, 1, 1, 1, 1, 1, 1)
\end{aligned}$$

Yet another form

Conjecture (third form). The matrix $[D_{n+1}U_n + xI]$ has an SNF over $\mathbb{Z}[x]$.

Note that $\mathbb{Z}[x]$ is not a PID.

Resolution of conjecture

Theorem. *The conjecture of Miller is true.*

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Proof (first step). Rather than use the basis $\{s_\lambda\}_{\lambda\in\mathrm{Par}(n)}$ (Schur functions) for $\Lambda^n_{\mathbb{Q}}$, use the basis $\{h_\lambda\}_{\lambda\in\mathrm{Par}(n)}$ (complete symmetric functions). Since the two bases differ by a matrix in $SL(p(n),\mathbb{Z})$, the SNF's stay the same.

Conclusion of proof

(second step) Row and column operations.

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A generalization

$m_j(\lambda)$: number of j's in λ

$\mathcal{M}_j(n)$: multiset of all numbers $j(m_j(\lambda)+1)$, $\lambda \in \mathrm{Par}(n)$

p_j: power sum symmetric function $\sum x_i^j$

Let SNF of the operator $f \mapsto j\frac{\partial}{\partial p_j} p_j f$ with respect to the basis $\{s_\lambda\}$ be diag$(g_1, g_2, \ldots, g_{p(n)})$.

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Theorem (**Zipei Nie**). $g_{p(n)}$ is the product of the **distinct** entries of $\mathcal{M}_j(n)$; $g_{p(n)-1}$ is the product of the remaining **distinct** entries of $\mathcal{M}_j(n)$, etc.

Two remarks

- The operators D, U and identity $DU - UD = I$ extend to any differential poset P . Miller and Reiner have conjectures for the SNF of DU. Nie has a conjecture on the structure of P which would prove the Miller-Reiner conjecture.

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- More general operators:

$$\frac{\partial^2}{\partial p_1^2} p1^2, \quad 2\frac{\partial}{\partial p_1}\frac{\partial}{\partial p_2} p_2 p_1, \ \text{etc.}$$

No conjecture known for SNF.

Jacobi-Trudi specialization

Jacobi-Trudi identity:

$$s_\lambda = \det[h_{\lambda_i - i + j}],$$

where s_λ is a **Schur function** and h_i is a **complete symmetric function**.

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We consider the specialization $x_1 = x_2 = \cdots = x_n = 1$, other $x_i = 0$. Then

$$h_i \to \binom{n+i-1}{i}.$$

Specialized Schur function

$$s_\lambda \to \prod_{u\in\lambda} \frac{n+c(u)}{h(u)}.$$

$c(u)$: **content** of the square u

0	1	2	3	4
−1	0	1	2	
−2	−1	0	1	
−3	−2			

Diagonal hooks $D_1, \ldots, D_m$

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$$\lambda = (5,4,4,2)$$

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D_1

Diagonal hooks $D_1, \ldots, D_m$

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D_2

Diagonal hooks $D_1, \ldots, D_m$

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D_3

SNF result

$$\boldsymbol{R} = \mathbb{Q}[n] \quad \text{(a PID)}$$

Let

$$\mathrm{SNF}\left[\binom{n+\lambda_i-i+j-1}{\lambda_i-i+j}\right] = \mathrm{diag}(e_1,\ldots,e_m).$$

Theorem.

$$e_i = \prod_{u\in D_{m-i+1}} \frac{n+c(u)}{h(u)}$$

Idea of proof

$$f_i = \prod_{u \in D_{m-i+1}} \frac{n + c(u)}{h(u)}$$

Want to prove $e_i = f_i$. Note that $f_1 f_2 \cdots f_i$ is the value of the lower-left $i \times i$ minor. (Special argument for 0 minors.)

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Every $i \times i$ minor is a specialized skew Schur function $s_{\mu/\nu}$. Let s_α correspond to the lower left $i \times i$ minor.

Conclusion of proof

Let $s_{\mu/\nu} = \sum_{\rho} c^{\mu}_{\nu\rho} s_{\rho}$. By Littlewood-Richardson rule,

$$\begin{aligned} c^{\mu}_{\nu\rho} \neq 0 &\Rightarrow \alpha \subseteq \rho \\ &\Rightarrow \{\text{contents of } \alpha\} \subseteq \{\text{contents of } \rho\} \\ &\qquad \text{(as multisets).} \end{aligned}$$

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Let $s_{\mu/\nu} = \sum_{\rho} c^{\mu}_{\nu\rho} s_{\rho}$. By Littlewood-Richardson rule,

$$\begin{aligned} c^{\mu}_{\nu\rho} \neq 0 \ &\Rightarrow\ \alpha \subseteq \rho \\ &\Rightarrow\ \{\text{contents of } \alpha\} \subseteq \{\text{contents of } \rho\} \\ &\qquad\qquad \text{(as multisets).} \end{aligned}$$

Hence $f_1 \cdots f_i = \gcd(i \times i \text{ minors}) = e_1 \cdots e_i. \quad \square$

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An example (cont.)

A "random" 3×3 minor of JT_λ:

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Jacobi-Trudi matrix for $s_{653/21}$

An example (concluded)

		1	1	1	1
		2	2	2	
		3			

Every LR-filling contains 1,1,1,1,2,2,2,3. Thus if $\langle s_{653/21}, s_\rho \rangle > 0$, then $431 \subseteq \rho$. Therefore

$$\prod_{u \in 431} (n + c(u)) \mid \prod_{u \in \rho} (n + c(u))$$

$$\Rightarrow \prod_{u \in 431} (n + c(u)) \mid s_{653/21}(1^n).$$

A q-analogue

"Natural" q-analogue of $f(1^n)$ is $f(1, q, \ldots, q^{n-1})$.

$$h_i(1, q, \ldots, q^{n-1}) = \binom{n+i-1}{i}_q$$

$$s_\lambda(1, q, \ldots, q^{n-1}) = q^* \prod_{u \in \lambda} \frac{1 - q^{n+c(u)}}{1 - q^{h(u)}}.$$

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Before we had $R=\mathbb{Q}[n]$. Now $R=\mathbb{Q}[q]$. Putting $q=1$ doesn't reduce second situation to the first.

What to do?

Set $\boldsymbol{y} = q^n$. Thus for instance

$$\begin{aligned} h_3(1, q, \ldots, q^{n-1}) &= \frac{(1-q^{n+2})(1-q^{n+1})(1-q^n)}{(1-q^3)(1-q^2)(1-q)} \\ &= \frac{(1-q^2y)(1-qy)(1-y)}{(1-q^3)(1-q^2)(1-q)}. \end{aligned}$$

Work over the field $\mathbb{Q}(q)[y]$ (a PID).

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Previous proof carries over (using a couple of tricks).

Notation

Write

$$(\boldsymbol{i}) = \frac{1-q^i}{1-q}.$$

E.g., $(\mathbf{-3}) = -q^{-1} - q^{-2} - q^{-3}$ *and* $(\mathbf{0}) = 0$. *For* $k \geq 1$ *let*

$$\boldsymbol{f}(\boldsymbol{k}) = \frac{y(y+(\mathbf{1}))(y+(\mathbf{2}))\cdots(y+(\boldsymbol{k}-\mathbf{1}))}{(\mathbf{1})(\mathbf{2})\cdots(\boldsymbol{k})}.$$

Set $f(0) = 1$ *and* $f(k) = 0$ *for* $k < 0$.

The final result

Theorem. *Define*

$$\mathbf{JT}(\boldsymbol{q})_{\boldsymbol{\lambda}} = [f(\lambda_i - i + j)]_{i,j=1}^{t},$$

where $\ell(\lambda) \leq t$. *Let the SNF of* $\mathrm{JT}(q)_\lambda$ *over the ring* $\mathbb{Q}(q)[y]$ *have main diagonal* $(\gamma_1, \gamma_2, \ldots, \gamma_t)$. *Then we can take*

$$\gamma_i = \prod_{u \in D_{t-i+1}} \frac{y + \boldsymbol{c}(\boldsymbol{u})}{\boldsymbol{h}(\boldsymbol{u})}.$$

The last slide

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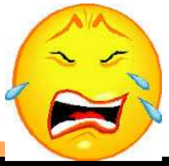

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