Stern's Triangle

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 & & & & 1 & & & & \\
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 & 1 & & & & & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

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$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & 1 & & 1 & & \\
 & 1 & & \color{red}{2} & & \color{red}{2} & & 1 & \\
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 & & & & \vdots & & & &
\end{array}
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1 & & \mathbf{2} & & \mathbf{3} & & \mathbf{3} & & \mathbf{3} & & \mathbf{3} & & \mathbf{2} & & 1 \\
 & & & & & & & \vdots & & & & & & &
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							1							
			1				1				1			
	1		1		2		1		2		1		1	
1	**1**	2	**1**	3	**2**	3	**1**	3	**2**	3	**1**	2	**1**	1
							⋮							

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1

1 1 1

1 1 2 1 2 1 1

1 1 2 1 3 2 3 1 3 2 3 1 2 1 1

⋮

Stern's triangle

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- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$ (so not really a triangle)

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Some properties

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- Sum of entries in row n: 3^n
- Largest entry in row n: F_{n+1} (Fibonacci number)
- Let $\left\langle {n \atop k} \right\rangle$ be the kth entry (beginning with $k = 0$) in row n. Write

$$P_n(x) = \sum_{k \geq 0} \left\langle {n \atop k} \right\rangle x^k.$$

Then $P_{n+1}(x) = (1 + x + x^2)P_n(x^2)$, since $x P_n(x^2)$ corresponds to bringing down the previous row, and $(1 + x^2)P_n(x^2)$ to summing two consecutive entries.

Stern's diatomic sequence

- **Corollary.** $P_n(x) = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right)$

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$$\begin{aligned} P(x) &= \prod_{i=0}^{\infty}\left(1 + x^{2^i} + x^{2\cdot 2^i}\right) \\ &:= \sum_{n\geq 0} \boldsymbol{b_{n+1}} x^n. \end{aligned}$$

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- The sequence $b_1, b_2, b_3, \ldots$ is **Stern's diatomic sequence**:

$$1,\ 1,\ 2,\ 1,\ 3,\ 2,\ 3,\ 1,\ 4,\ 3,\ 5,\ 2,\ 5,\ 3,\ 4,\ 1,\ \ldots$$

(often prefixed with 0)

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(often prefixed with 0)

- $b_1 = 1,\ b_{2n} = b_n,\ b_{2n+1} = b_n + b_{n+1}$

Partition interpretation

$$\sum_{n\geq 0} b_n x^n = \prod_{i\geq 0}\left(1 + x^{2^i} + x^{2\cdot 2^i}\right)$$

$\Rightarrow$ b_n is the number of partitions of n into powers of 2, where each power of 2 can appear at most twice.

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Note. If each power of 2 can appear at most once, then we obtain the (unique) binary expansion of n:

$$\frac{1}{1-x} = \prod_{i\geq 0}\left(1 + x^{2^i}\right).$$

Historical note

An essentially equivalent array is due to **Moritz Abraham Stern** around 1858 and is known as **Stern's diatomic array**:

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
								⋮								

Amazing property

Theorem (**Stern**, 1858). *Let $b_0, b_1, \ldots$ be Stern's diatomic sequence. Then every positive rational number occurs exactly once among the ratios b_i/b_{i+1}, and moreover this expression is in lowest terms.*

Sums of squares

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}$$

$$\boldsymbol{u_2(n)} := \sum_k \left\langle {n \atop k} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots$$

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$$

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$$u_2(n+1) = 5u_2(n) - 2u_2(n-1), \quad n \geq 1$$

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1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
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$$

$$
\sum_{n\geq 0} u_2(n)x^n = \frac{1-2x}{1-5x+2x^2}
$$

Sums of cubes

$$u_3(n) \coloneqq \sum_k \left\langle {n \atop k} \right\rangle^3 = 1,\ 3,\ 21,\ 147,\ 1029,\ 7203,\ \ldots$$

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Equivalently, if $\prod_{i=0}^{n-1}\left(1 + x^{2^i} + x^{2\cdot 2^i}\right) = \sum a_j x^j$, then

$$\sum a_j^3 = 3 \cdot 7^{n-1}.$$

Proof for $u_2(n)$

$$\begin{aligned} u_2(n+1) &= \cdots + \left\langle {n \atop k} \right\rangle^2 + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^2 + \left\langle {n \atop k+1} \right\rangle^2 + \cdots \\ &= 3u_2(n) + 2\sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle. \end{aligned}$$

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$$= 3u_2(n) + 2\sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle.$$

Thus define $\boldsymbol{u_{1,1}(n)} \coloneqq \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle$, so

$$u_2(n+1) = 3u_2(n) + 2u_{1,1}(n).$$

What about $u_{1,1}(n)$?

$$\begin{aligned} u_{1,1}(n+1) &= \cdots + \left(\left\langle {n \atop k-1} \right\rangle + \left\langle {n \atop k} \right\rangle\right)\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k} \right\rangle\left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right) \\ &\quad + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right)\left\langle {n \atop k+1} \right\rangle + \cdots \\ &= 2u_2(n) + 2u_{1,1}(n) \end{aligned}$$

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\end{aligned}
$$

Recall also $u_2(n+1) = 3u_2(n) + 2u_{1,1}(n)$.

Two recurrences in two unknowns

Let $\boldsymbol{A} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. Then

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

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$$\begin{aligned} (A^2 - 5A + 2I)A^{n-1} &= 0_{2\times 2} \\ \Rightarrow u_2(n+1) &= 5u_2(n) - 2u_2(n-1) \end{aligned}$$

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Also $u_{1,1}(n+1) = 5u_{1,1}(n) - 2u_{1,1}(n-1)$.

What about $u_3(n)$?

Now we need

$$\begin{aligned} u_{2,1}(n) &:= \sum_k \left\langle {n \atop k} \right\rangle^2 \left\langle {n \atop k+1} \right\rangle \\ u_{1,2}(n) &:= \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle^2 . \end{aligned}$$

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However, by symmetry about a vertical axis,

$$u_{2,1}(n) = u_{1,2}(n).$$

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However, by symmetry about a vertical axis,

$$u_{2,1}(n) = u_{1,2}(n).$$

We get

$$\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_3(n) \\ u_{2,1}(n) \end{bmatrix} = \begin{bmatrix} u_3(n+1) \\ u_{2,1}(n+1) \end{bmatrix}.$$

Unexpected eigenvalue

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Thus $u_3(n+1) = 7u_3(n)$ and $u_{2,1}(n+1) = 7u_{2,1}(n)$ $(n \geq 1)$.

In fact,

$$\begin{aligned} u_3(n) &= 3 \cdot 7^{n-1} \\ u_{2,1}(n) &= 2 \cdot 7^{n-1}. \end{aligned}$$

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Get a matrix of size $\lceil (r+1)/2 \rceil$, so expect a recurrence of this order.

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Proved by **David Speyer** (2018).

Basic idea of Speyer's proof

Theorem. *The matrix* A_r *is realized by the operator* $\phi\colon V_r \to V_r$ *defined by*

$$\phi(f)(x,y) = f(x+y,y) + f(x,x+y),$$

where $\boldsymbol{V_r}$ *is the space of homogeneous polynomials (over* $\mathbb{Z}$*) of degree* r *in the variables* x, y*, modulo the subspace generated by all* $f(x,y) - f(y,x)$*.*

General α

$$\alpha = (\alpha_0, \ldots, \alpha_{m-1})$$

$$u_\alpha(n) := \sum_k \left\langle {n \atop k} \right\rangle^{\alpha_0} \left\langle {n \atop k+1} \right\rangle^{\alpha_1} \cdots \left\langle {n \atop k+m-1} \right\rangle^{\alpha_{m-1}}$$

A closer look at $\alpha = (1, 1, 1, 1)$

$$u_{1,1,1,1}(n) = \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle \left\langle {n \atop k+2} \right\rangle \left\langle {n \atop k+3} \right\rangle$$

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$$u_{1,1,1,1}(n+1) =$$

$$\sum_k \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right) \left\langle {n \atop k+1} \right\rangle \left(\left\langle {n \atop k+1} \right\rangle + \left\langle {n \atop k+2} \right\rangle\right) \left\langle {n \atop k+2} \right\rangle$$
$$+ \sum_k \left\langle {n \atop k} \right\rangle \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right) \left\langle {n \atop k+1} \right\rangle \left(\left\langle {n \atop k+1} \right\rangle + \left\langle {n \atop k+2} \right\rangle\right)$$

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$$u_{1,1,1,1}(n+1) = \sum_k \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right) \left\langle {n \atop k+1} \right\rangle \left(\left\langle {n \atop k+1} \right\rangle + \left\langle {n \atop k+2} \right\rangle \right) \left\langle {n \atop k+2} \right\rangle + \sum_k \left\langle {n \atop k} \right\rangle \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right) \left\langle {n \atop k+1} \right\rangle \left(\left\langle {n \atop k+1} \right\rangle + \left\langle {n \atop k+2} \right\rangle \right)$$

$$A_{(1,1,1,1)} = \begin{bmatrix} 3 & 8 & 6 & 0 & 0 & 0 \\ 2 & 5 & 3 & 0 & 0 & 0 \\ 2 & 4 & 2 & 0 & 0 & 0 \\ 1 & 4 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 2 & 2 & 0 \end{bmatrix} \quad \begin{matrix} \mathbf{4} \\ \mathbf{3,1} \\ \mathbf{2,2} \\ \mathbf{1,2,1} \\ \mathbf{2,1,1} \\ \mathbf{1,1,1,1} \end{matrix}$$

A closer look at $\alpha = (1,1,1,1)$

$$u_{1,1,1,1}(n) = \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle \left\langle {n \atop k+2} \right\rangle \left\langle {n \atop k+3} \right\rangle$$

$$u_{1,1,1,1}(n+1) =$$

$$\sum_k \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right) \left\langle {n \atop k+1} \right\rangle \left(\left\langle {n \atop k+1} \right\rangle + \left\langle {n \atop k+2} \right\rangle \right) \left\langle {n \atop k+2} \right\rangle$$
$$+ \sum_k \left\langle {n \atop k} \right\rangle \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right) \left\langle {n \atop k+1} \right\rangle \left(\left\langle {n \atop k+1} \right\rangle + \left\langle {n \atop k+2} \right\rangle \right)$$

$$A_{(1,1,1,1)} = \left[\begin{array}{ccc|cc|c} 3 & 8 & 6 & 0 & 0 & 0 \\ 2 & 5 & 3 & 0 & 0 & 0 \\ 2 & 4 & 2 & 0 & 0 & 0 \\ \hline 1 & 4 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 2 & 1 & 0 \\ \hline 0 & 2 & 2 & 2 & 2 & 0 \end{array}\right] \quad \begin{array}{c} \mathbf{4} \\ \mathbf{3,1} \\ \mathbf{2,2} \\ \mathbf{1,2,1} \\ \mathbf{2,1,1} \\ \mathbf{1,1,1,1} \end{array}$$

Reduction to $\alpha = (r)$

$$\text{min. polynomial for } \alpha = (4): \quad (x+1)(2x^2 - 11x + 1)$$

$$\text{min. polynomial for } \alpha = (1,1,1,1): \quad (x-1)^2(x+1)(2x^2 - 11x + 1)$$

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Theorem. *Let $\alpha \in \mathbb{N}^m$ and $\sum \alpha_i = r$. Then $\mathrm{mp}(\alpha)$ has the form $x^{w_\alpha}(x-1)^{z_\alpha}\mathrm{mp}(r)$ for some $w_\alpha, z_\alpha \in \mathbb{N}$.*

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No conjecture for value of w_α, z_α.

A generalization

Let $\boldsymbol{p(x)}, \boldsymbol{q(x)} \in \mathbb{C}[x]$, $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{m-1}) \in \mathbb{N}^r$, and $\boldsymbol{b} \geq 2$. Set

$$q(x) \prod_{i=0}^{n-1} p(x^{b^i}) = \sum_k \left\langle {n \atop k} \right\rangle_{p,q,\alpha,b} x^k = \sum_k \left\langle {n \atop k} \right\rangle x^k$$

and

$$u_{p,q,\alpha,b}(n) = \sum_k \left\langle {n \atop k} \right\rangle^{\alpha_0} \left\langle {n \atop k+1} \right\rangle^{\alpha_1} \cdots \left\langle {n \atop k+m-1} \right\rangle^{\alpha_{m-1}}.$$

Main theorem

Theorem. *For fixed* p, q, α, b*, the function* $u_{p,q,\alpha,b}(n)$ *satisfies a linear recurrence with constant coefficients* ($n \gg 0$)*. Equivalently,* $\sum_n u_{p,q,\alpha,b}(n)x^n$ *is a rational function of* x*.*

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Note. $\exists$ multivariate generalization.

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$$q(x) = 1, p(x) = 1 + x + x^2, b = 2, \alpha = (r)$$

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$1 + 3x + x^2$	$x^2 - 17x + 54$	$x^2 - 47x + 450$	$x^3 - \cdots - 30618$
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Aside. $30618 = 2 \cdot 3^7 \cdot 7, \quad 458752 = 2^{16} \cdot 7$

An example

Example. Let $p(x) = (1+x)^2$, $q(x) = 1$. Then

$$\begin{aligned} u_{p,(2),2}(n) &= \frac{1}{3}\left(2 \cdot 2^{3n} + 2^{n}\right) \\ u_{p,(3),2}(n) &= \frac{1}{2}\left(2^{4n} + 2^{2n}\right) \\ u_{p,(4),2}(n) &= \frac{1}{15}\left(6 \cdot 2^{5n} + 10 \cdot 2^{3n} - 2^{n}\right). \end{aligned}$$

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$$\begin{aligned} p(x)p(x^2)p(x^4)\cdots p(x^{2^{n-1}}) &= \left((1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{n-1}})\right)^2 \\ &= \left(1+x+x^2+x^3+\cdots+x^{2^n-1}\right)^2. \end{aligned}$$

The rest of the story

Example. Let

$$\left(1 + x + x^2 + x^3 + \cdots + x^{2^n - 1}\right)^3 = \sum_j a_j x^j.$$

What is $\sum_j a_j^r$?

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$$\begin{aligned}
(1 + x + \cdots + x^{m-1})^3 &= \left(\frac{1 - x^m}{1 - x}\right)^3 \\
&= \frac{1 - 3x^m + 3x^{2m} - x^{3m}}{(1 - x)^3}
\end{aligned}$$

$$= \sum_{k=0}^{m-1} \binom{k+2}{2} x^k + \sum_{k=m}^{2m-1} \left[\binom{k+2}{2} - 3\binom{k-m+2}{2}\right] x^k$$

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for some polynomial $P(m) \in \mathbb{Q}[m]$.

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So $P(2^n)$ is a $\mathbb{Q}$-linear combination of terms 2^{jn}, as desired.

Evenness and oddness

Fact. $P(m)$ is either even ($P(m) = P(-m)$) or odd ($P(m) = -P(-m)$) (depending on degree).

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Proof surprisingly tricky. Is there a simple proof?

Modular properties

Sample result for Pascal's triangle:

$$\#\{k : \binom{n}{k} \equiv 1 \pmod{2}\} = 2^{b(n)},$$

where $b(n)$ is the number of 1's in the binary expansion of n (**Lucas**).

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Behavior for Stern's triangle is entirely different!

Rationality

Let $0 \leq a < m$.

$$\mathbf{g_{m,a}(n)} = \#\left\{k \,:\, 0 \leq k \leq 2^{n+1} - 2,\ \left\langle {n \atop k} \right\rangle \equiv a \pmod m\right\}.$$

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Theorem (**Reznick**). *$G_{m,a}(x)$ is a rational function.*

The cases $m = 2$ and $m = 3$

$$
\begin{aligned}
G_{2,0}(x) &= \frac{2x^2}{(1-x)(1+x)(1-2x)} \\
G_{2,1}(x) &= \frac{1+2x}{(1+x)(1-2x)} \\
G_{3,0}(x) &= \frac{4x^3}{(1-x)(1-2x)(1+x+2x^2)} \\
G_{3,1}(x) &= \frac{1+x-4x^3-4x^4}{(1-x)(1-2x)(1+x+2x^2)} \\
G_{3,2}(x) &= \frac{2x^2+4x^4}{(1-x)(1-2x)(1+x+2x^2)}
\end{aligned}
$$

. . . and more ($m = 4$)

$$G_{4,0}(x) = \frac{4x^4}{(1-x)(1+x)(1-2x)(1-x+2x^2)}$$

$$G_{4,1}(x) = \frac{1+x-2x^2-4x^3}{(1-x)(1+x)(1-2x)}$$

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$$G_{4,3}(x) = \frac{4x^3}{(1-x)(1+x)(1-2x)}$$

. . . and even more $(m = 5)$

$$G_{5,0}(x) = \frac{4x^4}{(1-x)(1+x)(1-2x)(1-x+2x^2)}$$

$$G_{5,1}(x) = \frac{1-x^2-x^4-8x^5+5x^6-4x^7-16x^8+8x^9-32x^{10}-32x^{11}}{(1-x)(1+x)(1-2x)(1+x^2)(1-x+2x^2)(1-x^2+4x^4)}$$

$$G_{5,2}(x) = \frac{2x^2+8x^5+2x^6-4x^7+12x^8-16x^{10}}{(1+x)(1-2x)(1+x^2)(1-x+2x^2)(1-x^2+4x^4)}$$

$$G_{5,3}(x) = \frac{4x^3+4x^5+4x^6+12x^7-4x^8+16x^{10}}{(1+x)(1-2x)(1+x^2)(1-x+2x^2)(1-x^2+4x^4)}$$

$$G_{5,4}(x) = \frac{4x^4-4x^5+8x^6+8x^7+8x^8+16x^{10}+32x^{11}}{(1+x)(1-2x)(1+x^2)(1-x+2x^2)(1-x^2+4x^4)}$$

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  E.g., $(m, a) =$
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- Why are so many numerator coefficients a power of 2? (In all computed cases, coefficients are $\pm 2^k$ or $\pm 3 \cdot 2^k$.)

$g_{m,a,b}(n)$

For the proof of rationality, need to define

$$g_{m,a,b}(n) = \#\left\{k : \left\langle {n \atop k} \right\rangle \equiv a \pmod m \ \left\langle {n \atop k+1} \right\rangle \equiv b \pmod m\right\},$$

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$$\gcd\left(\left\langle {n \atop k} \right\rangle, \left\langle {n \atop k+1} \right\rangle\right) = 1$$

$$\Rightarrow g_{m,a,b} = 0 \text{ unless } \mathbb{Z}/m\mathbb{Z} = \langle a, b \rangle$$

A recurrence

Let $\langle a, b \rangle = \mathbb{Z}/m\mathbb{Z}$. How to get

$$\left\langle {n+1 \atop k} \right\rangle \equiv a \pmod m, \ \left\langle {n+1 \atop k+1} \right\rangle \equiv b \pmod m?$$

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Answer. Either

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Hence

$$g_{m,a,b}(n+1) = g_{m,a,b-a}(n) + g_{m,a-b,b}(n),$$

where we take $b-a$ and $a-b$ modulo m.

Example: $n = 3$

Write $g_{ab} = g_{3,a,b}$.

$$\begin{bmatrix} g_{01}(n+1) \\ g_{02}(n+1) \\ g_{11}(n+1) \\ g_{12}(n+1) \\ g_{22}(n+1) \\ g_{10}(n+1) \\ g_{20}(n+1) \\ g_{21}(n+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_{01}(n) \\ g_{02}(n) \\ g_{11}(n) \\ g_{12}(n) \\ g_{22}(n) \\ g_{10}(n) \\ g_{20}(n) \\ g_{21}(n) \end{bmatrix}$$

$$= A_3 v$$

Characteristic polynomial

$$\det(I - xA_3) = (1 - x)^3(1 - 2x)(1 + x + 2x^2)$$

Characteristic polynomial

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Explains why $G_{3,a}$ has denominator $(1-x)(1-2x)(1+x+2x^2)$.

Size of A_m

Size (number of rows and number of columns) of A_m is the number $\nu(m)$ of pairs $(a, b) \in \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ that generate $\mathbb{Z}/m\mathbb{Z}$.

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Theorem (nice exercise).

$$\begin{aligned} \nu(m) &= m^2 \prod_{p|m} \frac{(p-1)(p+1)}{p^2} \\ &= \phi(m)\psi(m), \end{aligned}$$

where ϕ is the Euler phi function and ψ is the **Dedekind psi function**.

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$$\begin{aligned} \phi(m) &= m \prod_{p|m} \frac{p-1}{p} \\ \psi(m) &= m \prod_{p|m} \frac{p+1}{p} \end{aligned}$$

The final slide

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