Euler Numbers

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$$\sec x + \tan x = \sum_{n\geq 0} E_n \frac{x^n}{n!}.$$

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Raabe (1851): introduced the term "Euler numbers"

Basic definitions

A sequence $a_1, a_2, \ldots, a_k$ of distinct integers is **alternating** if

$$a_1 > a_2 < a_3 > a_4 < \cdots ,$$

and **reverse alternating** if

$$a_1 < a_2 > a_3 < a_4 > \cdots .$$

Euler numbers

$\mathfrak{S}_n$: symmetric group of all permutations of $1, 2, \ldots, n$

$$\begin{aligned} A_n &= \#\{w \in \mathfrak{S}_n \ : \ w \text{ is alternating}\} \\ &= \#\{w \in \mathfrak{S}_n \ : \ w \text{ is reverse alternating}\} \end{aligned}$$

(via $a_1 \cdots a_n \mapsto n+1-a_1, \ldots, n+1-a_n$)

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E.g., $E_4 = 5$: $2143, 3142, 3241, 4132, 4231$

André's theorem

Theorem (**Désiré André**, 1879)

$$A_n = E_n$$

Naive proof of André's theorem

Show combinatorially that

$$\Rightarrow 2A_{n+1} = \sum_{k=0}^{n} \binom{n}{k} A_k A_{n-k}, \quad n \geq 1$$

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(There exist more conceptual proofs.)

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$$\begin{aligned}\tan x &= \sum_{n\geq 0} A_{2n+1}\frac{x^{2n+1}}{(2n+1)!}\\ \sec x &= \sum_{n\geq 0} A_{2n}\frac{x^{2n}}{(2n)!}.\end{aligned}$$

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$\Rightarrow$ **combinatorial trigonometry**

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Take coefficient of $x^{2n}/(2n)!$:

$$\sum_{k=0}^{n} \binom{2n}{2k} A_{2k} A_{2(n-k)} = \sum_{k=0}^{n-1} \binom{2n}{2k+1} A_{2k+1} A_{2n-2k-1},$$

etc.

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Another identity (exercise)

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - (\tan x)(\tan y)}$$

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Enumerative Combinatorics, vol. 2, Exercise 5.7

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From Greek *boustrophēdon* ($\beta o \upsilon \sigma \tau \rho o \varphi \eta \delta \acute{o} \nu$), turning like an ox while plowing: *bous*, ox + *strophē*, a turning (from *strephein*, to turn)

The boustrophedon array

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & \rightarrow & 1 & & & & & \\
 & & & & 1 & \leftarrow & 1 & \leftarrow & 0 & & & & \\
 & & & 0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 2 & & & \\
 & & 5 & \leftarrow & 5 & \leftarrow & 4 & \leftarrow & 2 & \leftarrow & 0 & & \\
 & 0 & \rightarrow & 5 & \rightarrow & 10 & \rightarrow & 14 & \rightarrow & 16 & \rightarrow & 16 & \\
61 & \leftarrow & 61 & \leftarrow & 56 & \leftarrow & 46 & \leftarrow & 32 & \leftarrow & 16 & \leftarrow & 0. \\
 & & & & & & \dots & & & & & &
\end{array}
$$

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61 & \leftarrow & 61 & \leftarrow & 56 & \leftarrow & \color{red}{46} & \leftarrow & \color{blue}{32} & \leftarrow & 16 & \leftarrow & 0. \\
 & & & & & & \cdots & & & & & &
\end{array}
$$

Boustrophedon entries

- last term in row n: E_{n-1}
- sum of terms in row n: E_n
- kth term in row n: number of alternating permutations in $\mathfrak{S}_n$ with first term k, the **Entringer number** $\boldsymbol{E_{n-1,k-1}}$.

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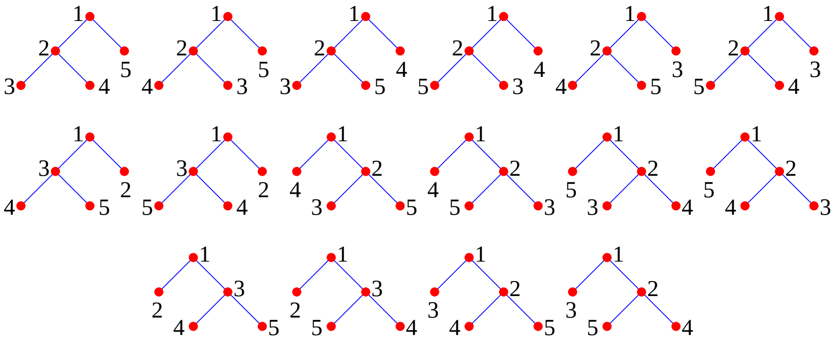
$$\sum_{m\geq 0}\sum_{n\geq 0} E_{m+n,[m,n]}\frac{x^m}{m!}\frac{y^n}{n!} = \frac{\cos x+\sin x}{\cos(x+y)},$$

$$[m,n] = \begin{cases} m, & m+n \text{ odd} \\ n, & m+n \text{ even.} \end{cases}$$

Some occurrences of E_n

(1) E_{2n+1} is the number of complete increasing binary trees on the vertex set $[2n+1] = \{1, 2, \ldots, 2n+1\}$.

Five vertices



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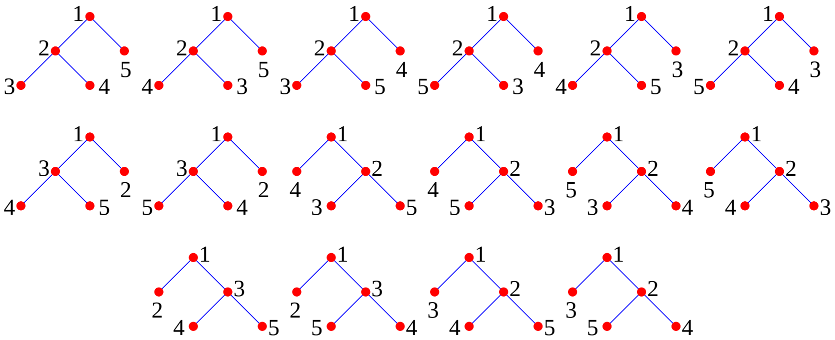

Slightly more complicated for E_{2n}

Proof for $2n + 1$

$$b_1 b_2 \cdots b_m : \text{ sequence of distinct integers}$$

$$b_i = \min\{b_1, \ldots, b_m\}$$

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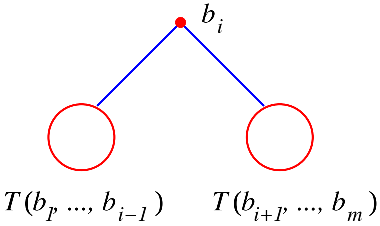
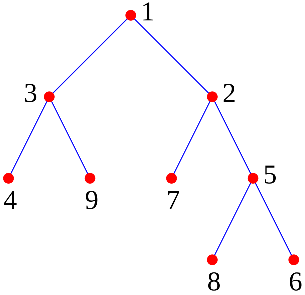
Define recursively a binary tree $T(b_1, \ldots, b_m)$ by



Completion of proof

Example. 439172856



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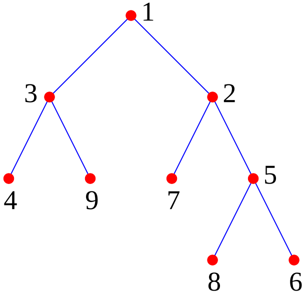

Let $w \in \mathfrak{S}_{2n+1}$. Then $T(w)$ is complete if and only if w is alternating, and the map $w \mapsto T(w)$ gives the desired bijection.

Orbits of mergings

(2) Start with n one-element sets $\{1\}, \ldots, \{n\}$.

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Proof omitted.

Orbit representatives for $n = 5$

12 – 3 – 4 – 5	123 – 4 – 5	1234 – 5
12 – 3 – 4 – 5	123 – 4 – 5	123 – 45
12 – 3 – 4 – 5	12 – 34 – 5	125 – 34
12 – 3 – 4 – 5	12 – 34 – 5	12 – 345
12 – 3 – 4 – 5	12 – 34 – 5	1234 – 5

Volume of a polytope

(3) Let $\mathcal{E}_n$ be the convex polytope in $\mathbb{R}^n$ defined by

$$\begin{aligned} x_i &\geq 0, \quad 1 \leq i \leq n \\ x_i + x_{i+1} &\leq 1, \quad 1 \leq i \leq n-1. \end{aligned}$$

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Theorem. *The volume of* $\mathcal{E}_n$ *is* $E_n/n!$.

Naive proof

$$\mathrm{vol}(\mathcal{E}_n) = \int_{x_1=0}^{1} \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1\, dx_2 \cdots dx_n$$

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$$f_n(t) := \int_{x_1=0}^{t} \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1\, dx_2 \cdots dx_n$$

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$$\begin{aligned} f_n'(t) &= \int_{x_2=0}^{1-t} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_2\, dx_3 \cdots dx_n \\ &= f_{n-1}(1-t). \end{aligned}$$

$F(y)$

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$$\begin{aligned} \mathbf{F}(\mathbf{y}) &= \sum_{n \geq 0} f_n(t) y^n \\ \Rightarrow \frac{\partial^2}{\partial t^2} F(y) &= -y^2 F(y), \end{aligned}$$

etc.

Conclusion of proof

$$\begin{aligned} F(y) &= (\sec y)(\cos(t-1)y + \sin ty) \\ \Rightarrow F(y)|_{t=1} &= \sec y + \tan y. \end{aligned}$$

Tridiagonal matrices

An $n \times n$ matrix $M = (m_{ij})$ is **tridiagonal** if $m_{ij} = 0$ whenever $|i - j| \geq 2$.

doubly-stochastic: $m_{ij} \geq 0$, row and column sums equal 1

$\mathcal{T}_n$: set of $n \times n$ tridiagonal doubly stochastic matrices

Polytope structure of $\mathcal{T}_n$

Easy fact: the map

$$\begin{array}{rcl} \mathcal{T}_n & \to & \mathbb{R}^{n-1} \\ M & \mapsto & (m_{12}, m_{23}, \dots, m_{n-1,n}) \end{array}$$

is a (linear) bijection from $\mathcal{T}$ to $\mathcal{E}_{n-1}$.

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Application (**Diaconis** et al.): random doubly stochastic tridiagonal matrices and random walks on $\mathcal{T}_n$

here??

Prelude: distribution of is(w)

$\mathrm{is}(w)$ = length of longest increasing subsequence of $w \in \mathfrak{S}_n$

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Vershik-Kerov, Logan-Shepp:

$$E(n) := \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{is}(w) \sim 2\sqrt{n}$$

Limiting distribution of is(w)

Baik-Deift-Johansson:

For fixed $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathrm{Prob}\left(\frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \le t\right) = F(t),$$

the **Tracy-Widom distribution**.

Longest alternating subsequences

$\mathrm{as}(w)$= length of longest alt. subseq. of w

$$w = 56218347 \Rightarrow \mathrm{as}(w) = 5$$

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$$D(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{as}(w) \sim ?$$

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w	$\mathrm{as}(w)$
123	1
132	2
213	3
231	2
312	3
321	2

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123	1
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$$a_1(3) = 1,\ a_2(3) = 3,\ a_3(3) = 2$$

The main lemma

Lemma. $\forall\, w \in \mathfrak{S}_n\ \exists$ *alternating subsequence of maximal length that contains* n*.*

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Lemma. $\forall\, w \in \mathfrak{S}_n\ \exists$ *alternating subsequence of maximal length that contains* n.

Corollary.

$$\Rightarrow a_k(n) = \sum_{j=1}^{n} \binom{n-1}{j-1}$$

$$\sum_{2r+s=k-1} \left(a_{2r}(j-1) + a_{2r+1}(j-1)\right) a_s(n-j)$$

The main generating function

$$\mathbf{A}(\mathbf{x}, \mathbf{t}) = \sum_{k,n \geq 0} a_k(n) t^k \frac{x^n}{n!}$$

Theorem.

$$A(x,t) = (1-t)\left(\frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho}\right),$$

where $\rho = \sqrt{1-t^2}$.

Formulas for $b_k(n)$

Corollary.

$$\Rightarrow \quad \begin{aligned} a_1(n) &= 1 \\ a_2(n) &= n-1 \\ a_3(n) &= \tfrac{1}{4}(3^n - 6n + 3) \\ a_4(n) &= \tfrac{1}{8}(4^n - 2\cdot 3^n - (2n-4)2^n + 8n - 6) \\ &\vdots \end{aligned}$$

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$$

No such formulas for longest **increasing** subsequences.

Mean (expectation) of $\mathrm{as}(w)$

$$D(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{as}(w) = \frac{1}{n!} \sum_{k=1}^{n} k \cdot a_k(n),$$

the **expectation** of $\mathrm{as}(w)$ for $w \in \mathfrak{S}_n$

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the **expectation** of $\mathrm{as}(w)$ for $w \in \mathfrak{S}_n$

Recall

$$\begin{aligned} A(x,t) &= \sum_{k,n \geq 0} a_k(n) t^k \frac{x^n}{n!} \\ &= (1-t)\left(\frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho} \right). \end{aligned}$$

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$$\Rightarrow D(n) = \frac{4\boldsymbol{n}+1}{6},\ \ n\geq 2$$

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$$\Rightarrow D(n)=\frac{4n+1}{6},\ \ n\geq 2$$

Compare $E(n)\sim 2\sqrt{n}$.

Simple proof

Is there a simple proof that $D(n) = \frac{4n+1}{6}$, $n > 1$?

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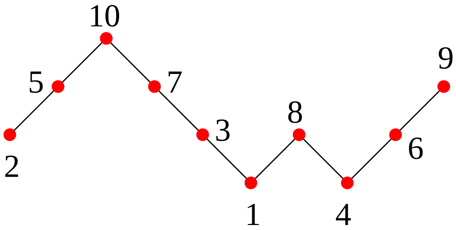

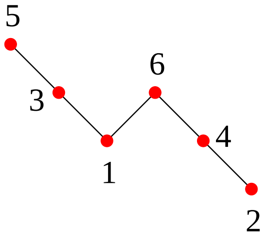

Simple proof

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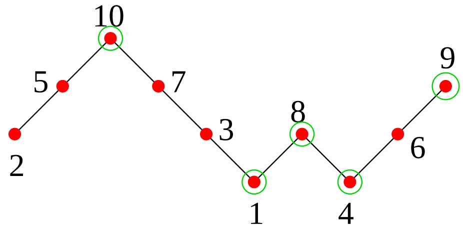

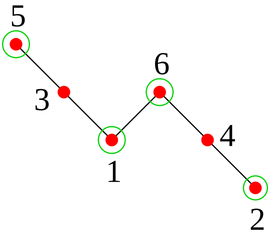

Simple proof (cont.)

$$\begin{aligned} w &= a_1 a_2 \cdots a_n \\ \mathrm{Prob}(a_1 > a_2) &= 1/2 \\ \mathrm{Prob}(a_i \text{ peak or valley}) &= 2/3,\ 2 \leq i \leq n-1 \\ \mathrm{Prob}(a_n > a_{n-1} \text{ or } a_n < a_{n-1}) &= 1 \end{aligned}$$

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$$\begin{aligned} \Rightarrow D(n) &= \frac{1}{2} + (n-2)\frac{2}{3} + 1 \\ &= \frac{4n+1}{6} \end{aligned}$$

Variance of $\mathrm{as}(w)$

$$V(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \left(\mathrm{as}(w) - \frac{4n+1}{6} \right)^2, \quad n \geq 2$$

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similar results for higher moments

A new distribution?

$$P(t) = \lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}(w) - 2n/3}{\sqrt{n}} \leq t\right)$$

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Stanley distribution?

Limiting distribution

Theorem (Pemantle, Widom, (Wilf)).

$$\lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}(w)-2n/3}{\sqrt{n}} \leq t\right)$$

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Umbral enumeration

Umbral formula: involves E^k, where E is an indeterminate (the **umbra**). Replace E^k with the Euler number E_k. (Technique from 19th century, modernized by **Rota** et al.)

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Example.

$$\begin{aligned}(1+E^2)^3 &= 1+3E^2+3E^4+E^6\\ &= 1+3E_2+3E_4+E_6\\ &= 1+3\cdot 1+3\cdot 5+61\\ &= 80\end{aligned}$$

Another example

$$\begin{aligned}
(1+t)^E &= 1 + Et + \binom{E}{2} t^2 + \binom{E}{3} t^3 + \cdots \\
&= 1 + Et + \frac{1}{2} E(E-1) t^2 + \cdots \\
&= 1 + E_1 t + \frac{1}{2} (E_2 - E_1)) t^2 + \cdots \\
&= 1 + t + \frac{1}{2} (1-1) t^2 + \cdots \\
&= 1 + t + O(t^3).
\end{aligned}$$

Alt. fixed-point free involutions

fixed point free involution $w \in \mathfrak{S}_{2n}$: all cycles of length two
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Let $\boldsymbol{f(n)}$ be the number of **alternating** fixed-point free involutions in $\mathfrak{S}_{2n}$.

$$\begin{aligned} n = 3: \quad & 214365 = (1,2)(3,4)(5,6) \\ & 645231 = (1,6)(2,4)(3,5) \end{aligned}$$

$$f(3) = 2$$

An umbral theorem

Theorem.

$$F(x) = \sum_{n \geq 0} f(n) x^n$$

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Theorem.

$$\mathbf{F}(\mathbf{x}) = \sum_{n\geq 0} f(n)x^n$$

$$= \left(\frac{1+x}{1-x}\right)^{(E^2+1)/4}$$

Proof idea

Proof. Uses representation theory of the symmetric group $\mathfrak{S}_n$.

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Now use known results on combinatorial properties of characters of $\mathfrak{S}_n$.

Entry 16 of Ramanujan's second notebook

As x *tends to* $0+$,

$$2\sum_{n=0}^{\infty}(-1)^n\left(\frac{1-x}{1+x}\right)^{n(n+1)} \sim 1+x+x^2+2x^3+5x^4+17x^5+\cdots .$$

LHS is a mock theta function. This is an **analytic** (nonformal identity).

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Computed first 50 coefficients and noticed they were all positive integers. **Brent** showed positivity (easy) and **Galway** (1997) integrality by a difficult argument.

Connection with alternating permutations

Recall: ***f*(*n*)**: number of alternating fixed fixed-point free involutions in $\mathfrak{S}_{2n}$

$$\begin{aligned}F(x) &:= \sum_{n\geq 0} f(n)x^n \\ &= \left(\frac{1+x}{1-x}\right)^{(E^2+1)/4} \\ &= \left(\frac{1+x}{1-x}\right)^{1/4} \exp\left(\frac{E^2}{4}\log\frac{1+x}{1-x}\right) \\ &= \left(\frac{1+x}{1-x}\right)^{1/4} \sum_{n=0}^{\infty} \frac{E_{2n}}{2^{2n} n!} \log^n\left(\frac{1+x}{1-x}\right),\end{aligned}$$

the series of Berndt.

A formal identity

Corollary (via **Ramanujan**, **Andrews**).

$$F(x) = 2\sum_{n\geq 0} q^n \frac{\prod_{j=1}^{n}(1-q^{2j-1})}{\prod_{j=1}^{2n+1}(1+q^j)},$$

where $\boldsymbol{q} = \left(\frac{1-x}{1+x}\right)^{2/3}$, *a* formal *identity*.

Generalizations?

What can replace $\frac{1+x}{1-x}$ in

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What about $\dfrac{1+ax}{1-bx}$?

The final slide

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