Persification

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January 31, 2020

Dedication

Delving
Into
Algebraic
Combinatorics
Originates
New,
Intriguing
Stories

Thanks

Profoundly
Enhanced
Richard
Stanley's
Interests

Definition of Persification

per·si·fi·ca·tion *noun*

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1. Persianization, i.e., sociological process of cultural change in which something becomes "Persianate" (acclimated to Persian culture).

2. The process of turning a mathematical result into a "story" explaining how this result applies to a concrete or real world situation, usually related to probability theory, in the manner of Persi Diaconis.

per·si·fy *verb*

Persi:

PERSONIFICATION

of

Persi:

PERSON**IFICATION**

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PERS IFICATION

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PERSON**IFICATION**

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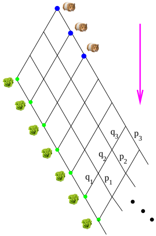
PERSIFICATION

I. Hungry hamsters



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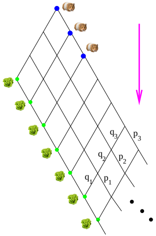

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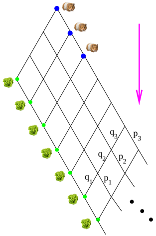

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- When distance i from food, they walk right with probability p_i and left with probability $q_i = 1 - p_i$.

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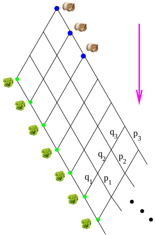
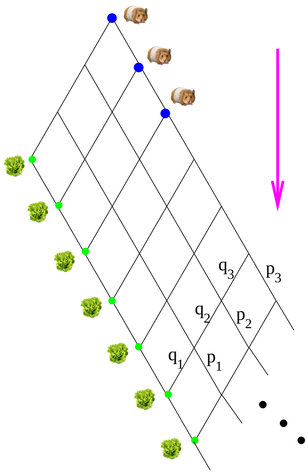


- Top hamster starts walking downhill at $t = 0$, next at $t = 1$, and next at $t = 2$ (for persification only).
- When distance i from food, they walk right with probability p_i and left with probability $q_i = 1 - p_i$.
- They are aggressive Syrian hamsters and hence very territorial. If they meet they will fight to the death.

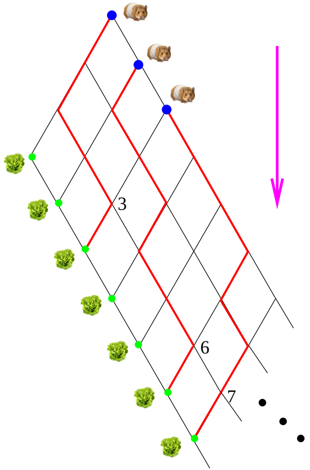
What is the probability P' all will reach food?

Larger figure



Last location

Specify last location of each hamster before they reach food.



Schur functions

Lindström-Wilf-Gessel-Viennot: probability of reaching a, b, c one step from food is $(q_2 q_3)^3 s_{c-3,b-2,a-1}(p_1, p_2, p_3)$ (Schur function).

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Probability of reaching food after one more step each:

$$q_1^3(q_2q_3)^3 s_{c-3,b-2,a-1}(p_1, p_2, p_3)$$

A Schur function sum

Probability $\boldsymbol{P}(\boldsymbol{p})$ of all hamsters reaching food:

$$P(p_1, p_2, p_3) = (q_1 q_2 q_3)^3 \sum_{\substack{\lambda \in \mathrm{Par} \\ \ell(\lambda) \leq 3}} s_\lambda(p_1, p_2, p_3)$$

$$= (q_1 q_2 q_3)^3 \sum_{\lambda \in \mathrm{Par}} s_\lambda(p_1, p_2, p_3)$$

$$= \frac{(q_1 q_2 q_3)^3}{\prod_{i=1}^{3}(1 - p_i) \cdot \prod_{1 \leq 1 < j \leq 3}(1 - p_i p_j)}$$

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Clearly generalizes to any (finite) number of hamsters.

$p_1 = p_2 = p_3$

Let $p_1 = p_2 = p_2 = \boldsymbol{p}$. Then

$$P(p,p,p) = \frac{(1-p)^9}{(1-p)^3(1-p^2)^3} = \left(\frac{1-p}{1+p}\right)^3.$$

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For n hamsters at distance n from food,

$$P(p,\ldots,p) = \left(\frac{1-p}{1+p}\right)^{\binom{n}{2}}.$$

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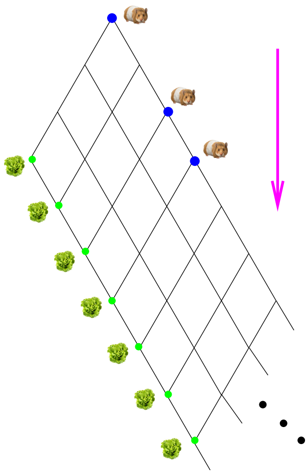
Simple reason?

More general starting points



A difficult sum

$$P = (q_1 q_2 q_3)^3 \sum_{\substack{(1,1,0)\subseteq\lambda \\ \ell(\lambda)\leq 3}} s_{\lambda/(1,1,0)}(p_1, p_2, p_3)$$

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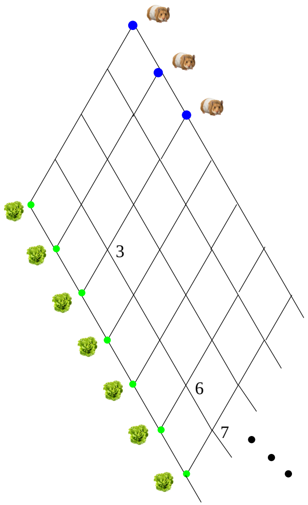
There is a nice formula for $\sum_{(1,1,0)\subseteq\lambda} s_{\lambda/(1,1,0)}$, but it is no longer true that

$$s_{\lambda/(1,1,0)}(x_1, x_2, x_3) \neq 0 \Rightarrow \ell(\lambda) \leq 3.$$

Farther from food



Another difficult sum

$$(q_1 q_2 q_3 q_4)^3 \sum_{\substack{\lambda \in \mathrm{Par} \\ \ell(\lambda) \leq 3}} s_\lambda(p_1, p_2, p_3, p_4)$$

Another difficult sum

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Again no longer true that

$$s_\lambda(x_1, x_2, x_3, x_4) \neq 0 \Rightarrow \ell(\lambda) \leq 3.$$

A formula of Gessel

Gessel showed

$$\sum_{\ell(\lambda)\leq 2m+1} s_\lambda = h \cdot \det(c_{i-j} - c_{i+j})_{i,j=1}^{m},$$

where $h = h_0 + h_1 + h_2 + \cdots$ and

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Difficult to extract probability.

II. Graded posets

P: finite graded poset of rank n with $\hat{0}$ and $\hat{1}$, so every maximal chain has the form

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ρ: rank function of P, so $\rho(t_i) = i$ above. In particular,

$$\rho(\hat{0}) = 0, \quad \rho(\hat{1}) = n.$$

Flag f-vectors

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$$\begin{aligned} \alpha_P(\emptyset) &= 1 \\ \alpha_P(\{i\}) &= \#\{t \in P \,:\, \rho(t) = i\} \\ \alpha_P([n-1]) &= \#(\text{maximal chains}) \end{aligned}$$

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Many nice properties and applications.

The boolean algebra B_n

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Theorem. *Let $S \subseteq [n-1]$. Then*

$$\beta_n(S) = \#\{w = w_1 \cdots w_n \in \mathfrak{S}_n : D(w) = S\},$$

*where $\boldsymbol{D(w)} = \{i : w_i > w_{i+1}\}$ (**descent set**).*

Quasisymmetric functions

Let $\mathbb{P} = \{1, 2, 3, \ldots\}$. A power series $F(x_1, x_2, \ldots)$ (over $\mathbb{Q}$, say) of bounded degree is **quasisymmetric** if for all $(a_1, \ldots, a_k) \in \mathbb{P}^k$ and all $1 \leq j_1 < \cdots < j_k$,

$$[x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}]F = [x_1^{a_1} \cdots x_k^{a_k}]F,$$

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where $[\cdots]$ denotes "coefficient of."

Example. $x_1^2 x_2 + x_1^2 x_3 + 2x_1 x_2^2 + 2x_1 x_3^2 + \cdots$ is quasisymmetric (so far), but

$$x_1^2 x_2 + 2x_3^2 x_5 + \cdots$$

is not.

Gessel's fundamental quasisymmetric function

Gessel: Fix n, and let $S \subseteq [n-1]$. Define the **fundamental quasisymmetric function** L_S in the variables $x_1, x_2, \dots$ by

$$L_S = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} \cdots x_{i_n}.$$

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Let $n = 3$. Then

$$\begin{aligned} L_\emptyset &= \textstyle\sum_{1 \le a \le b \le c} x_a x_b x_c, & L_1 &= \textstyle\sum_{1 \le a < b \le c} x_a x_b x_c \\ L_2 &= \textstyle\sum_{1 \le a \le b < c} x_a x_b x_c, & L_{1,2} &= \textstyle\sum_{1 \le a < b < c} x_a x_b x_c. \end{aligned}$$

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Note. $\{L_S : S \subseteq [n-1]\}$ is a $\mathbb{Q}$-basis for all homogeneous quasisymmetric functions of degree n.

Ehrenborg's generating function for $\beta_P(S)$

Ehrenborg: Let P be a finite poset, graded of rank n, with $\hat{0}$ and $\hat{1}$. Define

$$\mathbf{E_P} = \sum_{\hat{0}=t_0\leq t_1\leq\cdots\leq t_{k-1}<t_k=\hat{1}} x_1^{\rho(t_1)-\rho(t_0)} x_2^{\rho(t_2)-\rho(t_1)} \cdots x_k^{\rho(t_k)-\rho(t_{k-1})}$$

$$= \sum_{S\subseteq[n-1]} \beta_P(S) L_S \text{ (homogeneous of degree } n).$$

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In general, difficult to extract information form E_P. Nicest situation: E_P is a **symmetric function**.

The QS-distribution

Let $p_1, p_2, \cdots \geq 0$, $\sum p_i = 1$ (probability distribution on $\mathbb{P}$)

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Let $p_1, p_2, \cdots \geq 0$, $\sum p_i = 1$ (probability distribution on $\mathbb{P}$)

Given n, choose a random sequence $\boldsymbol{x} = (x_1, \ldots, x_n)$, each x_j independently from this distribution.

Standardize $\boldsymbol{x}$, e.g.,

4	1	1	3	7	4	7	1	7
4	**1**	**2**	3	7	4	7	**3**	7
4	**1**	**2**	**4**	7	4	7	**3**	7
5	**1**	**2**	**4**	7	**6**	7	**3**	7
5	**1**	**2**	**4**	**7**	**6**	**8**	**3**	**9**

Defines a probability distribution on $\mathfrak{S}_n$, the **QS-distribution** (with respect to p).

Probabilistic interpretation of E_P

Easy theorem 1. *The probability of obtaining* w *under the QS-distribution is* $L_{D(w^{-1})}(p_1, p_2, \dots)$ (degree n).

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Persification (easy). *Let* $\mathbb{E}_w$ *denote expectation with respect to the QS-distribution on* $w \in \mathfrak{S}_n$. *Then*

$$E_P(p_1, p_2, \dots) = \mathbb{E}_w \left(\frac{\beta_P(D(w))}{\beta_n(D(w))} \right).$$

A trivial example

Example. $P = B_n$. Then $E_{B_n} = h_1^n = (x_1 + x_2 + \cdots)^n$, so $E_{B_n}(p_1, p_2, \ldots) = 1$.

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Clear since we are computing

$$\mathbb{E}_w\left(\frac{\beta_n(D(w))}{\beta_n(D(w))}\right).$$

Products of chains

C_j: chain of **length** j (or with $j+1$ elements)

For $\lambda = (\lambda_1, \ldots, \lambda_k)$, define

$$C_\lambda = C_{\lambda_1} \times \cdots \times C_{\lambda_k}.$$

E.g., $C_{1,\ldots,1} \cong B_n$ (n 1's).

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Theorem. *Let* $\sum \lambda_i = n$, $M = \{1^{\lambda_1}, \ldots, k^{\lambda_k}\}$ *(multiset), and* $S \subseteq [n-1]$. *Then*

$$\beta_{C_\lambda}(S) = \#\{w \in \mathfrak{S}_M \,:\, D(w) = S\}.$$

E_{C_λ}

Easy fact:

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the **complete symmetric function** indexed by λ.

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Corollary. *Let* $\sum \lambda_i = n$. *Then*

$$\mathbb{E}_w \left(\frac{\beta_{C_\lambda}(D(w))}{\beta_n(D(w))} \right) \left(\frac{1}{k}, \ldots, \frac{1}{k} \right) = \frac{1}{n^k} \prod_{i=1}^{k} \binom{\lambda_i + k - 1}{\lambda_i}.$$

Majorization

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Define $(P_1, \dots, P_n) \preceq (R_1, \dots, R_n)$ if

$$P_1^\uparrow + \cdots + P_i^\uparrow \leq R_1^\uparrow + \cdots + R_i^\uparrow, \ 1 \leq i \leq n,$$

the **majorization order**.

Schur convexity and concavity

$f(P_1, \ldots, P_n)$ is **Schur convex** if

$$\boldsymbol{P} \preceq \boldsymbol{R} \Rightarrow f(\boldsymbol{P}) \leq f(\boldsymbol{R})$$

and **Schur-concave** if

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Example. The elementary symmetric function $e_k(P_1, \ldots, P_n)$ is Schur **concave** on $P_i \geq 0$, and thus so is any e-positive symmetric function.

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Similarly, the complete symmetric function $h_k(P_1, \ldots, P_n)$ is Schur **convex** on $P_i \geq 0$.

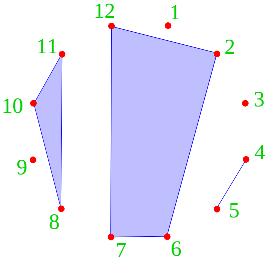
Schur convexity of E_{C_λ}

Corollary. *E_{C_λ} is h-positive, hence Schur convex on probability distributions $p_1, p_2, \ldots$. Therefore $\mathbb{E}_w\left(\frac{\beta_{C_\lambda}(D(w))}{\beta_n(D(w))}\right)(p_1, \ldots, p_k)$ is* **minimized** *for $p_i = 1/k$.*

Noncrossing partitions

A **noncrossing partition** of $\{1, 2, \ldots, n\}$ is a partition $\{B_1, \ldots, B_k\}$ of $\{1, \ldots, n\}$ such that

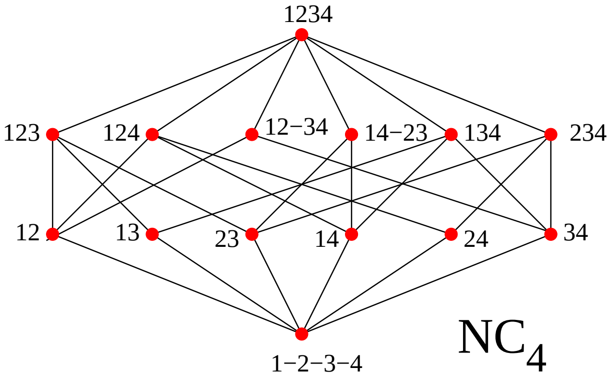
$$a < b < c < d,\ a, c \in B_i,\ b, d \in B_j \Rightarrow i = j.$$



Theorem (**H. W. Becker**, 1948–49) *The number of noncrossing partitions of* $\{1, \ldots, n\}$ *is the* **Catalan number** $C_n = \frac{1}{n+1}\binom{2n}{n}$.

The noncrossing partition lattice

$\mathbf{NC}_n$: noncrossing partitions of $\{1,\ldots,n\}$, ordered by refinement.
NC_n is graded of rank $n-1$.



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Corollary. $\mathbb{E}_w \left(\frac{\beta_{\mathrm{NC}_{n+1}}(D(w))}{\beta_n(D(w))} \right) \left(\frac{1}{k}, \ldots, \frac{1}{k} \right) = \frac{1}{(n+1)k^n} \binom{k(n+1)}{n}$

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Note. $\beta_{\mathrm{NC}_{n+1}}(S)$ is equal to the number of parking functions of length n and descent set $[n-1] - S$.

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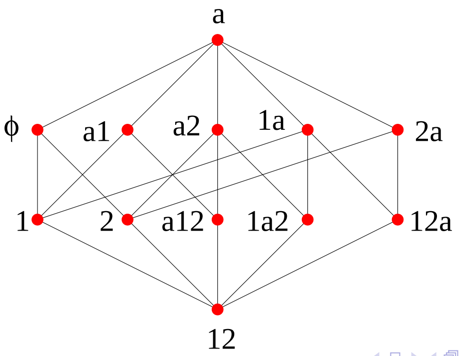
A **parking function** of length n is a sequence $(a_1, \ldots, a_n) \in \mathbb{P}^n$ whose increasing rearrangement $b_1 \leq \cdots \leq b_n$ satisfies $b_i \leq i$.

Schur concavity of $E_{\mathrm{NC}_{n+1}}$

Corollary. *$E_{\mathrm{NC}_{n+1}}$ is e-positive, hence Schur concave on probability distributions $p_1, p_2, \ldots$. Therefore $\mathbb{E}_w \left(\frac{\beta_{\mathrm{NC}_{n+1}}(D(w))}{\beta_n(D(w))} \right) (p_1, \ldots, p_k)$ is* **maximized** *for $p_i = 1/k$.*

Shuffle posets

C. Greene (1988): let $\alpha = a_1 \cdots a_j$ and $\beta = b_1 \cdots b_k$ be disjoint words. Define the **shuffle poset** W_{mn} to consist of all shuffles of subwords of α and β, with $u < v$ if we can get from u to v by deleting elements of α and adding elements of β. W_{mn} is graded of rank $m + n$ with $\hat{0} = \alpha$ and $\hat{1} = \beta$.



$E_{W_{mn}}$

Simion-S., 1999:

$$E_{W_{mn}} = \sum_{j\geq 0} \binom{m}{j} \binom{n}{j} e_1^{m+n-2j} e_2^j$$

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Corollary. *$E_{W_{mn}}$ is e-positive, hence Schur concave on probability distributions $p_1, p_2, \dots$. Therefore $\mathbb{E}_w \left(\frac{W_{mn}(D(w))}{\beta_n(D(w))} \right) (p_1, \dots, p_k)$ is* **maximized** *for $p_i = 1/k$.*

The final slide

The final slide