# Permutation Enumeration and Symmetric Functions

Richard P. Stanley
M.I.T. and U. Miami

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# Topics

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- class multiplication and characters
- commutators and characters
- alternating permutations and the Foulkes representation
- Lyndon symmetric functions
- generalized descent sets

# The class multiplication theorem

$G$: finite group with conjugacy classes $C_1, \ldots, C_t$

Let $i, j \in [t] = \{1, \ldots, t\}$.

$\chi^1, \ldots, \chi^t$: the irreducible (complex) characters of $G$

$d_r = \deg \chi^r$

$\chi_i^r$: $\chi^r(v)$ for any $v \in C_i$

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**Theorem.** *Let $w \in C_k$. Then*

$$\#\{(u, v) \in C_i \times C_j : uv = w\} = \frac{|C_i| \cdot |C_j|}{|G|} \sum_{r=1}^{t} \frac{1}{d_r} \chi_i^r \chi_j^r \bar{\chi}_k^r$$

## Reformulation for $G = \mathfrak{S}_n$

$(\boldsymbol{x})$: the variables $x_1, x_2, \ldots$, and similarly $(\boldsymbol{y}), (\boldsymbol{z})$

$\boldsymbol{H_\lambda}$: product of hook lengths of $\lambda$ for $\lambda \vdash n$

**Theorem.**

$$\sum_{\lambda \vdash n} H_\lambda\, s_\lambda(x) s_\lambda(y) s_\lambda(z) = \frac{1}{n!} \sum_{\substack{uvw = \mathrm{id} \\ \text{in } \mathfrak{S}_n}} p_{\rho(u)}(x) p_{\rho(v)}(y) p_{\rho(w)}(z),$$

where $\rho(u)$ is the cycle type of $u$.

## Sample application

**Theorem.**

$$\sum_{\lambda \vdash n} H_\lambda = \frac{1}{n!} \#\{(u, v, w) \in \mathfrak{S}_n^3 \, : \, u^2 v^2 w^2 = 1\}.$$

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**Idea of proof.** For $w \in \mathfrak{S}_n$ let $\mathbf{sq}(w) = \#\{u \in \mathfrak{S}_n \,:\, u^2 = w\}$. Let $\varphi\colon \Lambda_{\mathbb{Q}} \to \mathbb{Q}$ be the linear transformation defined by $\varphi(s_\lambda) = 1$.

**Well-known:** $p_\lambda = \sum_\mu \chi^\mu(\lambda) s_\mu$, so

$$\begin{aligned} \varphi(p_\lambda) &= \sum_\mu \chi^\mu(\lambda) \\ &= \mathrm{sq}(w), \end{aligned}$$

where $\rho(w) = \lambda$.

# Proof (concluded)

$$\varphi(s_\lambda) = 1, \quad \varphi(p_\lambda) = \mathrm{sq}(w) \text{ where } \rho(w) = \lambda$$

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Apply $\varphi$ separately to each set of variables in

$$\sum_{\lambda \vdash n} H_\lambda \, s_\lambda(x) s_\lambda(y) s_\lambda(z) = \frac{1}{n!} \sum_{\substack{uvw = \mathrm{id} \\ \text{in } \mathfrak{S}_n}} p_{\rho(u)}(x) p_{\rho(v)}(y) p_{\rho(w)}(z). \quad \square$$

## Straightforward generalization

**Theorem.** *Let $k \geq 1$. Then*

$$\sum_{\lambda \vdash n} H_\lambda^{k-2} = \frac{1}{n!} \#\{(w_1, \ldots, w_k) \in \mathfrak{S}_n^k \; : \; w_1^2 \cdots w_k^2 = 1\}.$$

# Commutators

$G$: finite group of order $g$

For $w \in G$, define

$$f(w) = \#\{(u, v) \in G \times G \,:\, w = uvu^{-1}v^{-1}\}.$$

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**Theorem.** $f = \displaystyle\sum_{\chi \in \mathrm{Irr}(G)} \frac{g}{\chi(1)}\chi.$

**Aside:** From representation theory, $\frac{g}{\chi(1)} \in \mathbb{P}$. Proof uses algebraic number theory. Is there a direct proof that $f$ is a character of $G$?

## Reformulation for $G = \mathfrak{S}_n$

**Theorem.** $$\frac{1}{n!}\sum_{u,v\in\mathfrak{S}_n} p_{\rho(uvu^{-1}v^{-1})} = \sum_{\lambda\vdash n} H_\lambda s_\lambda \qquad (*)$$

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**Sample application.** For $w \in \mathfrak{S}_n$, let $\kappa(w)$ be the number of cycles of $w$. Then

$$\frac{1}{n!} \sum_{u,v \in \mathfrak{S}_n} q^{\kappa(uvu^{-1}v^{-1})} = \sum_{\lambda \vdash n} \prod_{t \in \lambda} (q + c(t)),$$

where $c(t)$ denotes the content of the square $t$.

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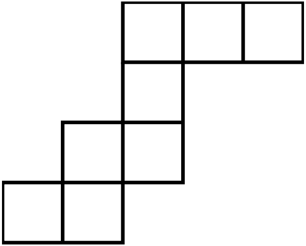
**Proof.** Let $q \in \mathbb{P}$. Set $x_1 = \cdots = x_q = 1$, other $x_i = 0$ in (*). Note that $p_{\rho(w)}(1^q) = q^{\kappa(w)}$ (since $p_i(1^q) = q$), etc. $\square$

# Border strips (or ribbons)

$$S = \{b_1 < b_2 < \cdots < b_k\} \subseteq [n-1] := \{1, 2, \ldots, n-1\}$$

$B_S$: the border strip with row lengths
$b_1, b_2 - b_1, b_3 - b_2, \ldots, n - b_k$.



$$B_{\{3,4,6\}}, n{=}8$$

# Theorems of Foulkes and Niven-de Bruijn

**Theorem** (**Foulkes**). *Let* $S, T \subseteq [n-1]$. *Then*

$$\langle s_{B_S}, s_{B_T} \rangle = \#\{w \in \mathfrak{S}_n \,:\, D(w) = S, D(w^{-1}) = T\},$$

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$\boldsymbol{\beta_n(S)} = \#\{w \in \mathfrak{S}_n : D(w) = S\}$

**Theorem (Niven, de Bruijn)** *Fix $n$. Then $\beta_n(S)$ is maximized by $S = \{1, 3, 5, \dots\}$ and $S = \{2, 4, 6, \dots\}$.*

## Gessel's conjecture

Recall

$$\langle s_{B_S}, s_{B_T} \rangle = \#\{w \in \mathfrak{S}_n : D(w) = S, D(w^{-1}) = T\}.$$

**Conjecture.** Fix $n$. Then $\langle s_{B_S}, s_{B_T} \rangle$ is maximized by $S = T = \{1, 3, 5, \dots\}$ and $S = T = \{2, 4, 6, \dots\}$.

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**Theorem.** *The maximum of value of $\langle s_{B_S}, s_{B_T} \rangle$ is achieved by some $S = T$.*

**Proof.** $\langle s_{B_S} - s_{B_T}, s_{B_S} - s_{B_T} \rangle \geq 0$

$$\Rightarrow \langle s_{B_S}, s_{B_S} \rangle + \langle s_{B_T}, s_{B_T} \rangle \geq 2\langle s_{B_S}, s_{B_T} \rangle,$$

so either $\langle s_{B_S}, s_{B_S} \rangle \geq \langle s_{B_S}, s_{B_T} \rangle$ or $\langle s_{B_T}, s_{B_T} \rangle \geq \langle s_{B_S}, s_{B_T} \rangle$. $\square$

# Alternating permutations

$w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ is **alternating** if

$$a_1 > a_2 < a_3 > a_4 < \cdots a_n.$$

***$E_n$***: number of alternating $w \in \mathfrak{S}_n$ (**Euler number**)

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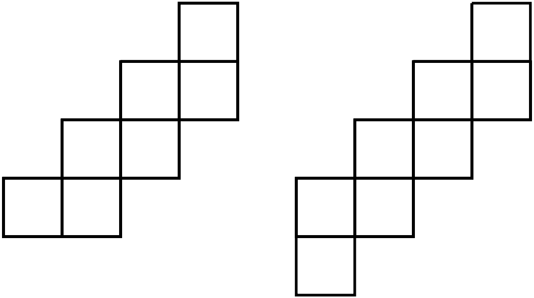
**Theorem** (**D. André**, 1879)

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

## Ribbon staircases

Let $R_n$ be the **ribbon staircase**: the border strip with row lengths $(1,2,2,\ldots,2,2,1)$ ($n$ even) or $(1,2,2,\ldots,2,2)$ ($n$ odd).



$R_7$ $\qquad$ $R_8$

# Another theorem of Foulkes

$\chi^{R_n}$: the (reducible) character of $\mathfrak{S}_n$ corresponding to $R_n$, i.e., $\mathrm{ch}(\chi^{R_n}) = s_{R_n}$. Equivalently,

$$s_{R_n} = \sum_{\mu \vdash n} z_\mu^{-1} \chi^{R_n}(\mu) p_\mu.$$

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**Theorem** (**Foulkes**). *Let* $\mu \vdash n = 2k+1$. *Then*

$$\chi^{R_n}(\mu) = \begin{cases} 0, & \text{if } \mu \text{ has an even part} \\ (-1)^{k+r} E_{2r+1}, & \text{if } \mu \text{ has } 2r+1 \text{ odd parts and} \\ & \quad \text{no even parts.} \end{cases}$$

Similar result for $n = 2k$.

# Sample application

$$L(t) = \frac{1}{2}\log\frac{1+t}{1-t} = t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots$$

$$f(n) = \#\{w \in \mathfrak{S}_n \ : \ w \text{ and } w^{-1} \text{ are alternating}\}$$

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**Theorem.** $\displaystyle\sum_{k\geq 0} f(2k+1)t^{2k+1} = \sum_{r\geq 0} E_{2r+1}^2 \frac{L(t)^{2r+1}}{(2r+1)!}.$

*Similar result for $f(2k)$.*

## Idea of proof.

Let **OP**$(n)$ be the set of partitions of $n$ with odd parts. Then for $n = 2k+1$,

$$\begin{aligned} f(n) &= \langle s_{R_n}, s_{R_n} \rangle \\ &= \left\langle \sum_{\mu \vdash n} z_\mu^{-1} \chi^{R_n}(\mu) p_\mu, \sum_{\mu \vdash n} z_\mu^{-1} \chi^{R_n}(\mu) p_\mu \right\rangle \\ &= \sum_{\mu \vdash n} z_\mu^{-1} \left( \chi^{R_n}(\mu) \right)^2 . \end{aligned}$$

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Use Foulkes' theorem on value of $\chi^{R_n}(\mu)$ to get

$$f(n) = \sum_{\mu \in \mathrm{OP}(n)} z_\mu^{-1} E_{2r+1}^2.$$

Now use elementary generating function manipulatorics. $\square$

# Lyndon symmetric functions

For $\lambda \vdash n$, let

$$K_\lambda = \{w \in \mathfrak{S}_n \,:\, \rho(w) = \lambda\},$$

a conjugacy class in $\mathfrak{S}_n$.

For $S \subset [n-1]$, define

$$F_S = \sum_{\substack{1 \le i_1 \le i_2 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_n},$$

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Define the **Lyndon symmetric function**

$$L_\lambda = \sum_{w \in K_\lambda} F_{D(w)},$$

a generating function for the number of permutations of cycle type $\lambda$ by descent set.

# An example

**Example.** $n = 3,\ \lambda = (2,1)$

| $w$ | $D(w)$ |
|---|---|
| 213 | 1 |
| 132 | 2 |
| 321 | 1,2 |

$$L_{(2,1)} = F_1 + F_2 + F_{1,2} = s_{2,1} + s_{1,1,1}$$

## Gessel-Reutenauer theorem

**Theorem.** *$L_\lambda$ is a symmetric function given by*

$$\begin{aligned} L_n &= \frac{1}{n}\sum_{d|n}\mu(d)p_d^{n/d} \\ L_{\langle n^k\rangle} &= h_k[L_n] \quad \text{(plethysm)} \\ L_{\langle 1^{k_1}\,2^{k_2}\cdots\rangle} &= L_{\langle 1^{k_1}\rangle}L_{\langle 2^{k_2}\rangle}\cdots . \end{aligned}$$

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- $L_\lambda$ is Schur positive.
- $\sum_{\lambda\vdash n} L_\lambda = p_1^n$
- Let $d(n)$ be the codimension of the span of the $L_\lambda$'s, $\lambda \vdash n$, in $\Lambda^n_{\mathbb{Q}}$. **Open:** what is $d(n)$?

| $n$ | 1–3 | 4–6 | 7 | 8 | 9–11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|
| $d(n)$ | 0 | 1 | 2 | 3 | 4 | 7 | 10 | 12 | 15 |

# A consequence of Gessel-Reutenauer

**Theorem** (**Gessel-Reutenauer**). *Let* $\lambda \vdash n$ *and* $S \subset [n-1]$. *Then*

$$\langle L_\lambda, s_{B_S} \rangle = \#\{w \in \mathfrak{S}_n \,:\, \rho(w) = \lambda,\ D(w) = S\}.$$

## Sample application

**Theorem** (**Gessel-Reutenauer**) *The number of involutions in* $\mathfrak{S}_n$ *with descent set* $S$ *equals the number of involutions in* $\mathfrak{S}_n$ *with descent set* $\bar{S} = [n-1] - S$.

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**Proof.** The set of involutions in $\mathfrak{S}_n$ is a union of conjugacy classes. Now

$$\sum_{\substack{w \in \mathfrak{S}_n \\ w^2 = 1}} F_{\rho(w)} = \prod_i \frac{1}{1 - x_i} \prod_{i<j} \frac{1}{1 - x_i x_j} = \sum_\lambda s_\lambda,$$

which is invariant under $\omega$. Moreover, $\omega s_{B_S} = s_{B_{\bar{S}}}$. The proof follows from

$$\left\langle \sum_\lambda s_\lambda, s_{B_S} \right\rangle = \left\langle \omega \sum_\lambda s_\lambda, \omega s_{B_S} \right\rangle = \left\langle \sum_\lambda s_\lambda, s_{B_{\bar{S}}} \right\rangle.$$

# A sample result on alternating permutations

$$f(n) = \#\{w \in \mathfrak{S}_{2n} \,:\, \rho(w) = (2,2,\ldots,2),\ D(w) = \{1,3,5,\cdots\}\}$$

Thus $f(n) = \left\langle L_{\langle 2^n \rangle}, s_{R_{2n}} \right\rangle$. Using

$$L_{\langle 2^n \rangle} = h_n\left[\frac{1}{2}(p_1^2 - p_2)\right] = \frac{1}{2}\left(p_1^{2n} - p_2^n\right)$$

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**Theorem.** *Let $E$ be an indeterminate. Let $\Omega$ be the linear operator sending $E^k$ to the Euler number $E_k$. Then*

$$\sum_{n \geq 0} f(n) t^n = \Omega \left( \frac{1+t}{1-t} \right)^{(E^2+1)/4}.$$

# Computation of $\Omega\left(\frac{1+t}{1-t}\right)^{(E^2+1)/4}$

$$
\begin{aligned}
\Omega\left(\frac{1+t}{1-t}\right)^{\frac{E^2+1}{4}} &= \Omega\left(1+\frac{1}{2}(E^2+1)t+\frac{1}{8}(E^4+2E^2+1)t^2+\cdots\right) \\
&= 1+\frac{1}{2}(E_2+1)t+\frac{1}{8}(E_4+2E_2+1)t^2+\cdots \\
&= 1+\frac{1}{2}(1+1)t+\frac{1}{8}(5+2\cdot 1+1)t^2+\cdots \\
&= 1+t+t^2+\cdots
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\end{aligned}$$

E.g., the unique $w \in \mathfrak{S}_4$ that is alternating and has cycle type $(2,2)$ is 2143.

## Descent set enumeration in the alternating group

$\mathfrak{A}_n$: alternating group of degree $n$

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Recall

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Recall the notation: let $\rho(w) = \lambda$. Then $\varepsilon_\lambda = \mathrm{sgn}(w)$. Hence:

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**Theorem.** $\gamma_n(S) = \left\langle \sum_{\substack{\lambda \vdash n \\ \varepsilon_\lambda = 1}} L_\lambda, s_{B_S} \right\rangle$

## A formula for $\sum_{\substack{\lambda \vdash n \\ \varepsilon_\lambda = 1}} L_\lambda$

**Theorem.**

$$\sum_{\substack{\lambda \vdash n \\ \varepsilon_\lambda = 1}} L_\lambda = \begin{cases} \frac{1}{2}\left(p_1^n + p_2^{n/2}\right), & \text{if } n \text{ is even} \\ \frac{1}{2}\left(p_1^n + p_1 p_2^{(n-1)/2}\right), & \text{if } n \text{ is odd.} \end{cases}$$

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**Theorem.**

$$\sum_{\substack{\lambda \vdash n \\ \varepsilon_\lambda = 1}} L_\lambda = \begin{cases} \frac{1}{2}\left(p_1^n + p_2^{n/2}\right), & \text{if } n \text{ is even} \\ \frac{1}{2}\left(p_1^n + p_1 p_2^{(n-1)/2}\right), & \text{if } n \text{ is odd.} \end{cases}$$

Proof is a computation based on the Gessel-Reutenauer formula

$$\begin{aligned} L_n &= \frac{1}{n}\sum_{d|n} \mu(d) p_d^{n/d} \\ L_{\langle n^k \rangle} &= h_k[L_n] \quad \text{(plethysm)} \\ L_{\langle 1^{k_1}\, 2^{k_2} \cdots \rangle} &= L_{\langle 1^{k_1} \rangle} L_{\langle 2^{k_2} \rangle} \cdots . \end{aligned}$$

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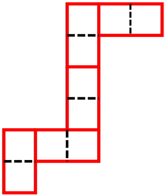
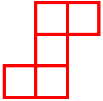
Is there a more conceptual proof?

# Half a border strip

Let $B_S$ be a border strip of even size $2m$. Tile it uniquely with $m$ dominos. Shrink each domino to a square to get $\boldsymbol{B_{S/2}}$.



$S = \{3,4,5,6,9\}$



$S/2 = \{2,3\}$

# A formula for $\gamma_n(S)$, $n$ even

$B_S$: a border strip of size $n = 2m$

$v(B_S)$: number of vertical dominos in the unique tiling of $B_S$ by $m$ dominos

Recall: $\beta_n(S) = \#\{w \in \mathfrak{S}_n \ : \ D(w) = S\}$

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More complicated formula when $n$ is odd.

## Sketch of proof.

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**Proof** (sketch).

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\gamma_n(S) &= \left\langle s_{B_S}, \sum_{\substack{\lambda \vdash n \\ \varepsilon_\lambda = 1}} L_\lambda \right\rangle \\
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Evaluate $\langle s_{B_S}, p_2^m \rangle$ by the Murnaghan-Nakayama rule.

## Completion of proof.

$\langle s_{B_S}, p_2^m \rangle$ is the number of border-strip tableau of type $2^m$. There is a unique tiling by dominos. A border strip tableaux is an ordering of these dominos so that removing them in that order from the lower right boundary always leaves a skew shape. This corresponds to a (reverse) standard Young tableau of shape $B_{S/2}$, of which there are $\beta_m(S/2)$. The sign is $(-1)^{v(B_S)}$. $\square$

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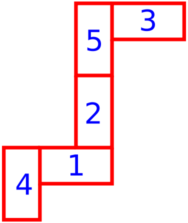


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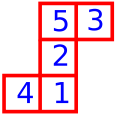


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# Generalized descent sets

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**$X$-descent set $\mathrm{XDes}(w)$**: set of $X$-descents

**Example.** (a) $X = \{(i,j) : n-1 \geq i > j \geq 1\}$: $\mathrm{XDes} = D$

(b) $X = \{(i,j) \in [n] \times [n] : i \neq j\}$: $\mathrm{XDes}(w) = [n-1]$

# A generating function for the XDescent set

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**Example.** $X = \{(1,3),(2,1),(3,1),(3,2)\}$

| $w$ | $\mathrm{XDes}(w)$ |
|---|---|
| 123 | $\emptyset$ |
| 132 | $\{1,2\}$ |
| 213 | $\{1,2\}$ |
| 231 | $\{2\}$ |
| 312 | $\{1\}$ |
| 321 | $\{1,2\}$ |

$$U_X = F_\emptyset + F_1 + F_2 + 3F_{1,2} = p_1^3 - p_2p_1 + p_3 = s_3 + s_{21} + 2s_{111}$$

## Two theorems

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**record set** $\mathrm{rec}(w)$ for $w = a_1 \cdots a_n \in \mathfrak{S}_n$:
$\mathrm{rec}(w) = \{0 \leq i \leq n-1 \,:\, a_i > a_j \text{ for all } j < i\}$. Thus always $0 \in \mathrm{rec}(w)$.

**record partition** $\mathrm{rp}(w)$: if $\mathrm{rec}(w) = \{r_0, \ldots, r_j\}_<$, then $\mathrm{rp}(w)$ is the numbers $r_1 - r_0, r_2 - r_1, \ldots, n - r_j$ arranged in decreasing order.

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**Theorem** (conjectured by **RS**, proved by **I. Gessel**) *Let $X$ have the property that if $(i,j) \in X$ then $i > j$. Then*

$$U_X = \sum_{\substack{w \in \mathfrak{S}_n \\ \mathrm{XDes}(w) = \emptyset}} p_{\mathrm{rp}(w)}.$$

*In particular, $U_X$ is p-positive.*

## Connection with chromatic symmetric functions

$\boldsymbol{P}$: partial ordering of $[n]$

$\boldsymbol{Y_P} = \{(i,j) : i >_P j\}$

$\mathbf{inc}(\boldsymbol{P})$: incomparability graph of $P$, i.e., vertex set $[n]$, edges $ij$ if $i \parallel j$ in $P$

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**Theorem.** $U_{Y_P} = X_{\mathrm{inc}(P)}$

# Reverse succession-free permutations

Let $X = \{(2,1),(3,2),\ldots,(n,n-1)\}$.

$f_n = \#\{w \in \mathfrak{S}_n : \mathrm{XDes}(w) = \emptyset\}$ (**rs-free** permutations)

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**Example.** $n=4$: $U_X = 11s_4 + 3s_{31} + s_{211} + s_{1111}$

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Right-hand side: Use

$$s_{i,1^{n-i}} = \sum_{S \in \binom{[n-1]}{n-i}} F_S.$$

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**Example.** $w = 3247651$, so $S = \{1, 4, 5\}$, $n = 7$, $i = 4$. Factor $w$:

$$w = 32 \cdot 4 \cdot 765 \cdot 1.$$

Let $1 \to 1$, $32 \to 2$, $4 \to 3$, $765 \to 4$. get

$$w \to 2341 = u. \quad \square$$

# A $q$-analogue for $X = \{(2,1),(3,2),\ldots,(n,n-1)\}$

Let $U_X(q) = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{des}(w^{-1})} F_{\mathrm{XDes}(w)}$, where des denotes the number of (ordinary) descents.

$U_X(q)$ is the generating function for $w \in \mathfrak{S}_n$ by positions of reverse successions and by $\mathrm{des}(w^{-1})$.

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# The final slide

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Thanks to **Ron Adin, Yuval Roichman, and Uzi Vishne** and the rest of the organizing committee, and to **Francesco Brenti, Roy Meshulam, Rosa Orellana, Dan Romik** and the rest of the program committee, for putting together such a successful meeting under difficult covidian circumstances!

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