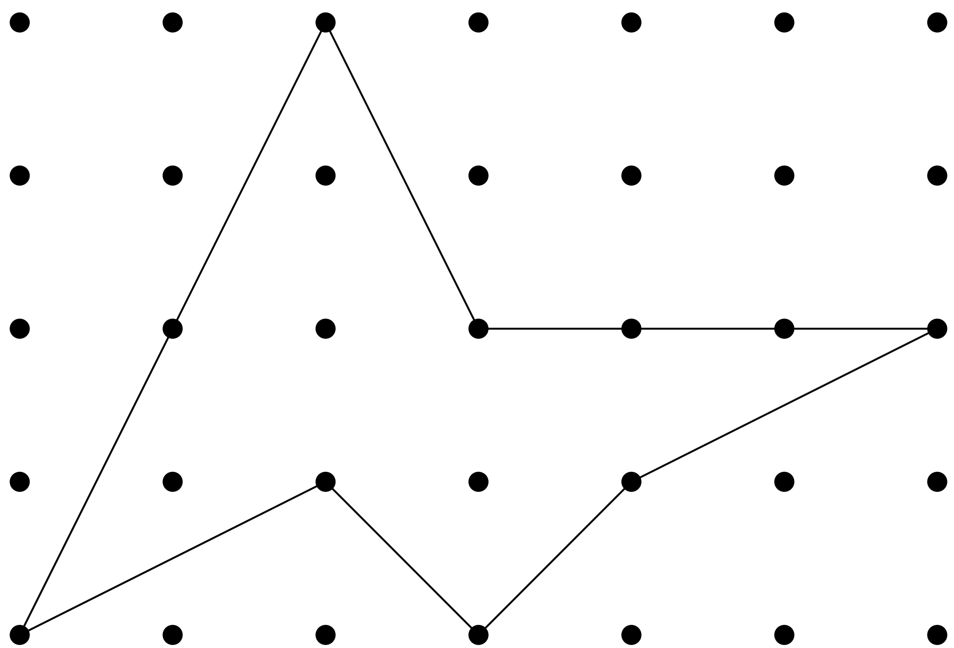
Lattice Points in Polytopes

Richard P. Stanley

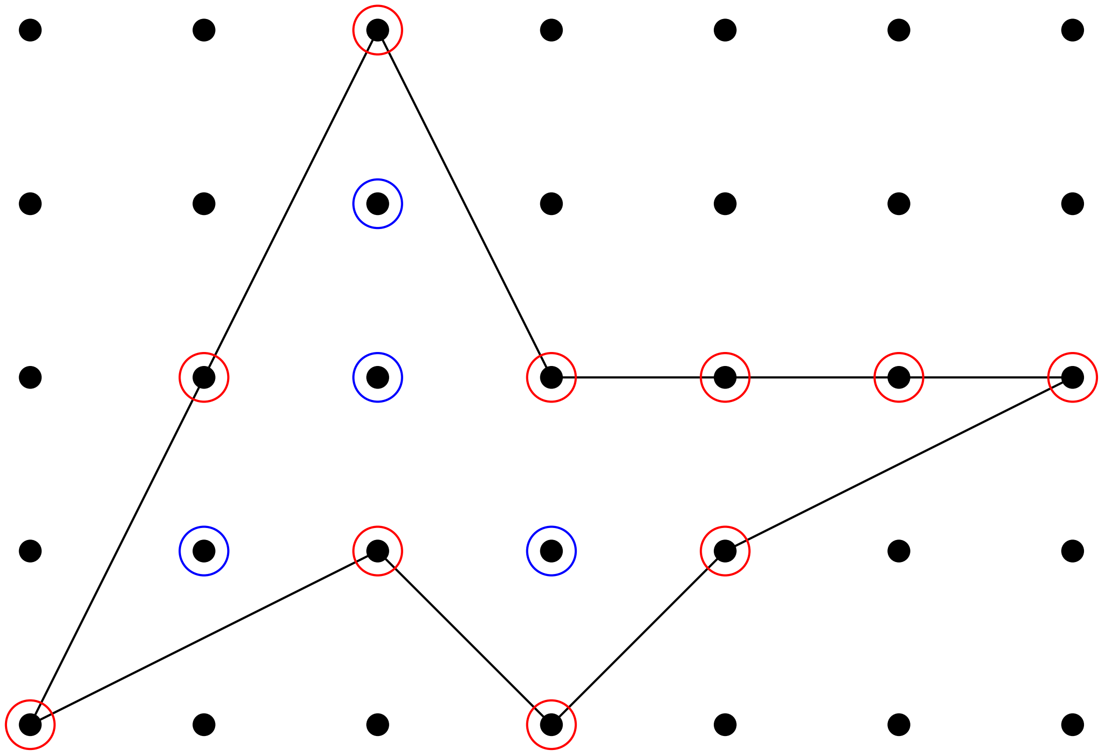
A lattice polygon

Georg Alexander Pick (1859–1942)

P: lattice polygon in $\mathbb{R}^2$
(vertices $\in \mathbb{Z}^2$, no self-intersections)



Boundary & interior lattice points



red: boundary lattice point
blue: interior lattice point

Pick's theorem

$$\begin{aligned} \boldsymbol{A} &= \text{area of } P \\ \boldsymbol{I} &= \#\text{ interior points of } P\ (=4) \\ \boldsymbol{B} &= \#\text{boundary points of } P\ (=10) \end{aligned}$$

Then

$$\boldsymbol{A} = \frac{2\boldsymbol{I} + \boldsymbol{B} - 2}{2}.$$

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Example on previous slide:

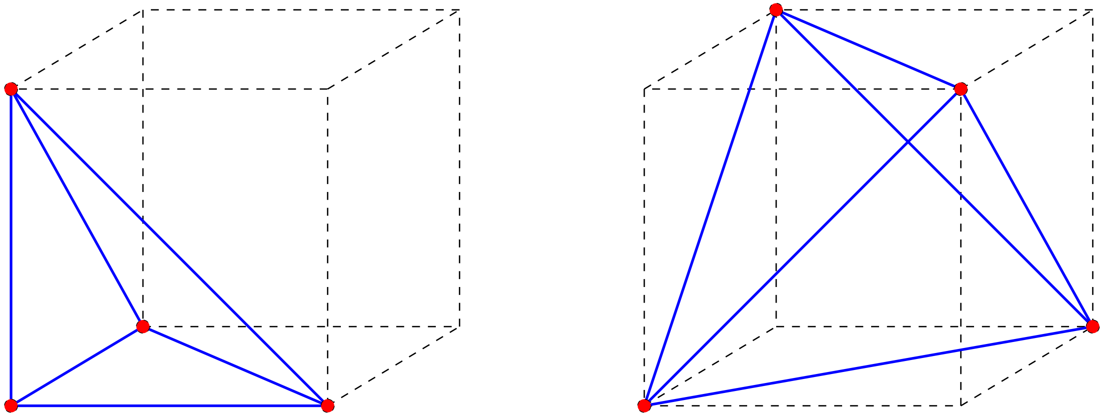
$$\boldsymbol{A} = \frac{2 \cdot \mathbf{4} + \mathbf{10} - 2}{2} = 9.$$

Two tetrahedra

Pick's theorem (seemingly) fails in higher dimensions. For example, let T_1 and T_2 be the tetrahedra with vertices

$$\begin{aligned}\mathrm{vert}(T_1) &= \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}\\ \mathrm{vert}(T_2) &= \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}.\end{aligned}$$

Failure of Pick's theorem in dim 3



Then

$$I(T_1) = I(T_2) = 0$$

$$B(T_1) = B(T_2) = 4$$

$$A(T_1) = 1/6, \quad A(T_2) = 1/3.$$

Convex hull

The **convex hull** **conv**(S) of $S \subseteq \mathbb{R}^n$:

$$\mathrm{conv}(S) = \bigcap_{\substack{T \supseteq S \\ T \text{ convex}}} T,$$

the smallest convex set containing S.

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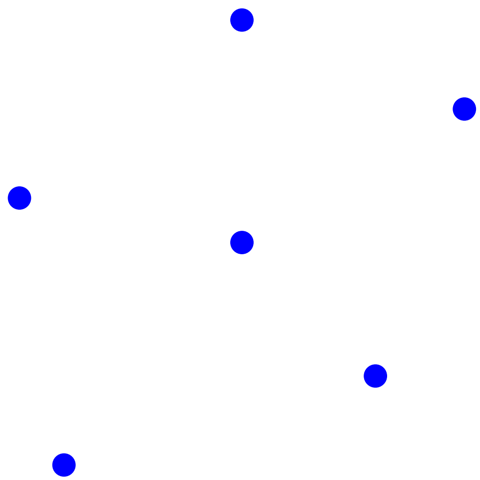

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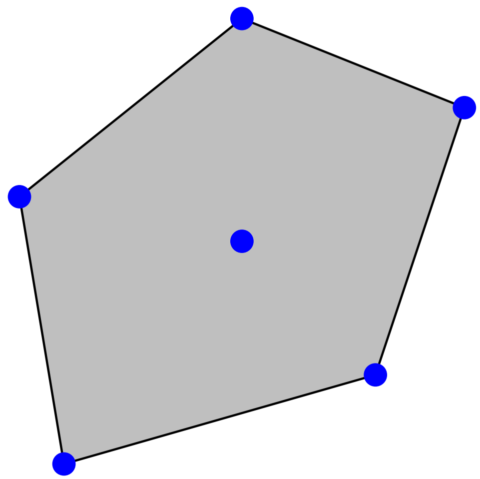

Polytope dilation

Let $\mathcal{P}$ be a convex polytope (convex hull of a finite set of points) in $\mathbb{R}^d$. For $n \geq 1$, let

$$\boldsymbol{n\mathcal{P}} = \{n\alpha \,:\, \alpha \in \mathcal{P}\}.$$

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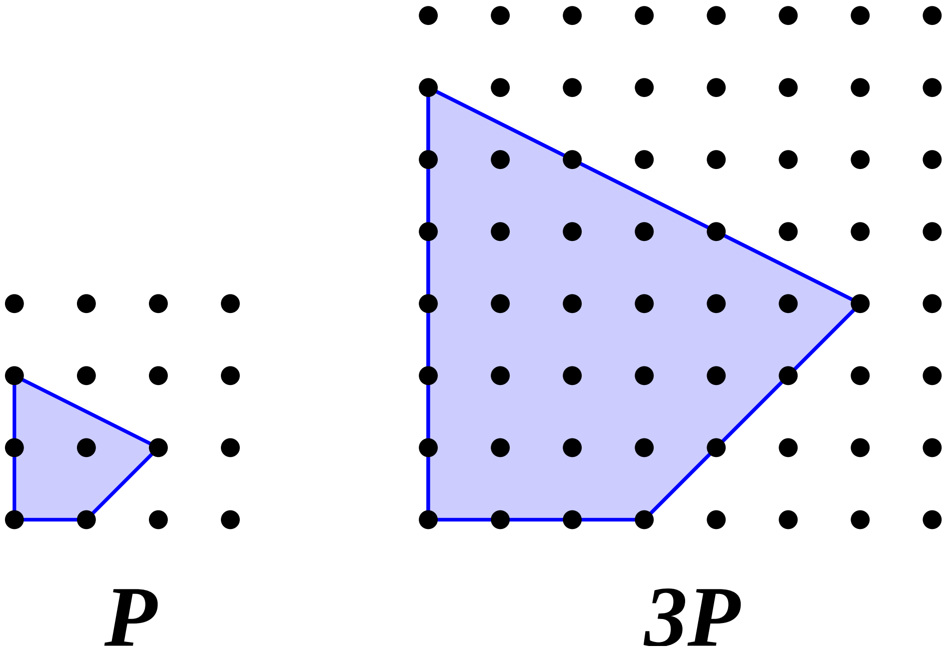


$i(\mathcal{P}, n)$

Let

$$
\begin{aligned}
i(\mathcal{P}, n) &= \#(n\mathcal{P} \cap \mathbb{Z}^d) \\
&= \#\{\alpha \in \mathcal{P} : n\alpha \in \mathbb{Z}^d\},
\end{aligned}
$$

the number of lattice points in $n\mathcal{P}$.

$\bar{i}(\mathcal{P}, n)$

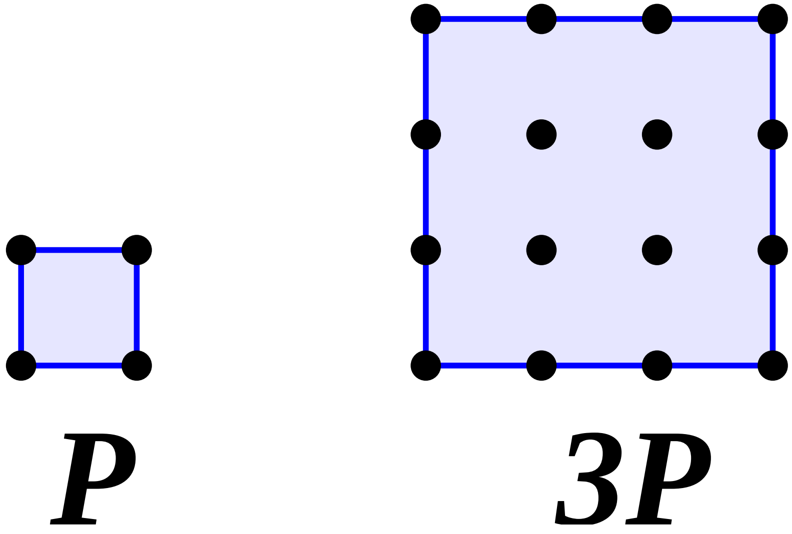
Similarly let

$$\mathcal{P}^\circ = \text{interior of } \mathcal{P} = \mathcal{P} - \partial\mathcal{P}$$

$$\begin{aligned} \bar{i}(\mathcal{P}, n) &= \#(n\mathcal{P}^\circ \cap \mathbb{Z}^d) \\ &= \#\{\alpha \in \mathcal{P}^\circ \,:\, n\alpha \in \mathbb{Z}^d\}, \end{aligned}$$

the number of lattice points in the **interior** of $n\mathcal{P}$.

An example



$$i(\mathcal{P}, n) \;=\; (n+1)^2$$

$$\bar{i}(\mathcal{P}, n) \;=\; (n-1)^2 = i(\mathcal{P}, -n).$$

Reeve's theorem

lattice polytope: polytope with integer vertices

Theorem (**Reeve**, 1957). *Let $\mathcal{P}$ be a three-dimensional lattice polytope. Then the volume $V(\mathcal{P})$ is a certain (explicit) function of $i(\mathcal{P},1)$, $\overline{i}(\mathcal{P},1)$, and $i(\mathcal{P},2)$.*

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Recall: $\bar{i}(P,1) =$ number of interior lattice points.

The main result

Theorem (**Ehrhart** 1962, **Macdonald** 1963). *Let*

$$\mathcal{P} = \text{lattice polytope in } \mathbb{R}^N, \ \dim \mathcal{P} = d.$$

*Then $i(\mathcal{P}, n)$ is a polynomial (the **Ehrhart polynomial** of $\mathcal{P}$) in n of degree d.*

Reciprocity and volume

Moreover,

$$\begin{aligned} i(\mathcal{P},0) &= 1 \\ \bar{i}(\mathcal{P},n) &= (-1)^d i(\mathcal{P},-n),\ n>0 \end{aligned}$$

(reciprocity).

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(reciprocity).

If $d = N$ *then*

$$i(\mathcal{P},n) = V(\mathcal{P})n^d + \text{ lower order terms,}$$

where $\boldsymbol{V(\mathcal{P})}$ *is the volume of* $\mathcal{P}$*.*

Eugène Ehrhart

- April 29, 1906: born in Guebwiller, France
- 1932: begins teaching career in lycées
- 1959: Prize of French Sciences Academy
- 1963: begins work on Ph.D. thesis
- 1966: obtains Ph.D. thesis from Univ. of Strasbourg
- 1971: retires from teaching career
- January 17, 2000: dies

Photo of Ehrhart

Self-portrait

Generalized Pick's theorem

Corollary. *Let* $\mathcal{P} \subset \mathbb{R}^d$ *and* $\dim \mathcal{P} = d$. *Knowing any* d *of* $i(\mathcal{P}, n)$ *or* $\bar{i}(\mathcal{P}, n)$ *for* $n > 0$ *determines* $V(\mathcal{P})$.

Generalized Pick's theorem

Corollary. *Let* $\mathcal{P} \subset \mathbb{R}^d$ *and* $\dim \mathcal{P} = d$*. Knowing any* d *of* $i(\mathcal{P}, n)$ *or* $\bar{i}(\mathcal{P}, n)$ *for* $n > 0$ *determines* $V(\mathcal{P})$*.*

Proof. Together with $i(\mathcal{P}, 0) = 1$, this data determines $d + 1$ values of the polynomial $i(\mathcal{P}, n)$ of degree d. This uniquely determines $i(\mathcal{P}, n)$ and hence its leading coefficient $V(\mathcal{P})$. $\square$

An example: Reeve's theorem

Example. When $d = 3$, $V(\mathcal{P})$ is determined by

$$\begin{aligned} i(\mathcal{P},1) &= \#(\mathcal{P} \cap \mathbb{Z}^3) \\ i(\mathcal{P},2) &= \#(2\mathcal{P} \cap \mathbb{Z}^3) \\ \bar{i}(\mathcal{P},1) &= \#(\mathcal{P}^\circ \cap \mathbb{Z}^3), \end{aligned}$$

which gives Reeve's theorem.

Birkhoff polytope

Example. Let $\mathcal{B}_M \subset \mathbb{R}^{M\times M}$ be the **Birkhoff polytope** of all $M \times M$ **doubly-stochastic** matrices $A = (a_{ij})$, i.e.,

$$a_{ij} \geq 0$$

$$\sum_i a_{ij} = 1 \text{ (column sums 1)}$$

$$\sum_j a_{ij} = 1 \text{ (row sums 1).}$$

(Weak) magic squares

Note. $B = (b_{ij}) \in n\mathcal{B}_M \cap \mathbb{Z}^{M\times M}$ if and only if

$$b_{ij} \in \mathbb{N} = \{0, 1, 2, \ldots\}$$

$$\sum_i b_{ij} = n$$

$$\sum_j b_{ij} = n.$$

Example of a magic square

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 3 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 4 & 0 \end{bmatrix} \qquad (M = 4,\ n = 7)$$

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$$\in 7\mathcal{B}_4$$

$H_M(n)$

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$$= i(\mathcal{B}_M, n)$$

$$H_1(n) = 1$$
$$H_2(n) = n + 1$$

$$\begin{bmatrix} a & n-a \\ n-a & a \end{bmatrix}, \quad 0 \le a \le n.$$

The case $M = 3$

$$H_3(n) = \binom{n+2}{4} + \binom{n+3}{4} + \binom{n+4}{4}$$

(MacMahon)

Values for small n

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Anand-Dumir-Gupta, 1966:

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$$\sum_{M \geq 0} H_M(2) \frac{x^M}{M!^2} = \frac{e^{x/2}}{\sqrt{1-x}}$$

Anand-Dumir-Gupta conjecture

Theorem (**Birkhoff-von Neumann**). *The vertices of* $\mathcal{B}_M$ *consist of the* $M!$ $M \times M$ *permutation matrices. Hence* $\mathcal{B}_M$ *is a lattice polytope.*

Anand-Dumir-Gupta conjecture

Theorem (**Birkhoff-von Neumann**). *The vertices of $\mathcal{B}_M$ consist of the $M!$ $M \times M$ permutation matrices. Hence $\mathcal{B}_M$ is a lattice polytope.*

Corollary (**Anand-Dumir-Gupta conjecture**). *$H_M(n)$ is a polynomial in n (of degree $(M-1)^2$).*

$H_4(n)$

Example. $H_4(n) = \frac{1}{11340}\left(11n^9 + 198n^8 + 1596n^7\right.$

$+7560n^6 + 23289n^5 + 48762n^5 + 70234n^4 + 68220n^2$

$\left.+40950n + 11340\right).$

Reciprocity for magic squares

Reciprocity $\Rightarrow \pm H_M(-n) =$

$\#\{M \times M$ matrices B of **positive** integers, line sum $n\}$

But every such B can be obtained from an $M \times M$ matrix A of **nonnegative** integers by adding 1 to each entry.

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Corollary.

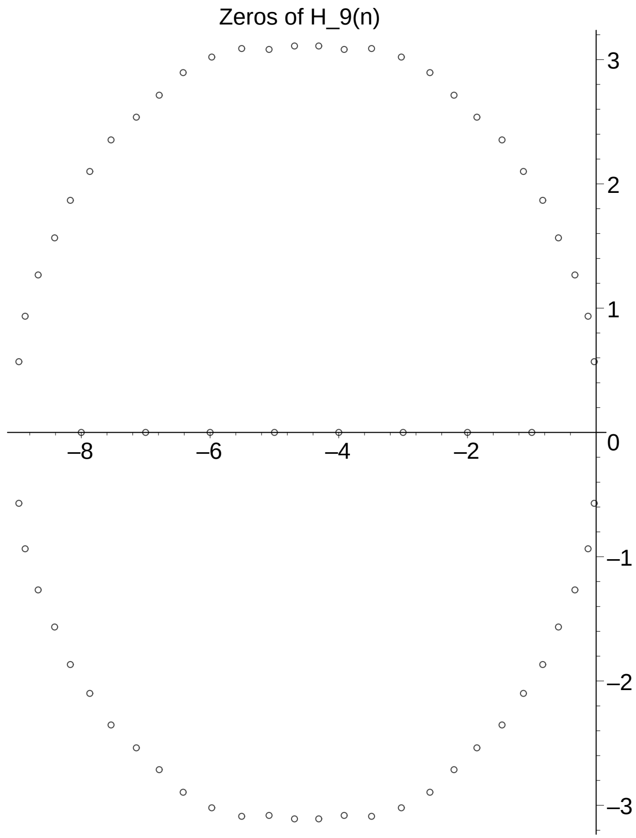
$$H_M(-1) = H_M(-2) = \cdots = H_M(-M+1) = 0$$

$$H_M(-M-n) = (-1)^{M-1} H_M(n)$$

Two remarks

- Reciprocity greatly reduces computation.
- Applications of magic squares, e.g., to statistics (contingency tables).

Zeros of $H_9(n)$ in complex plane



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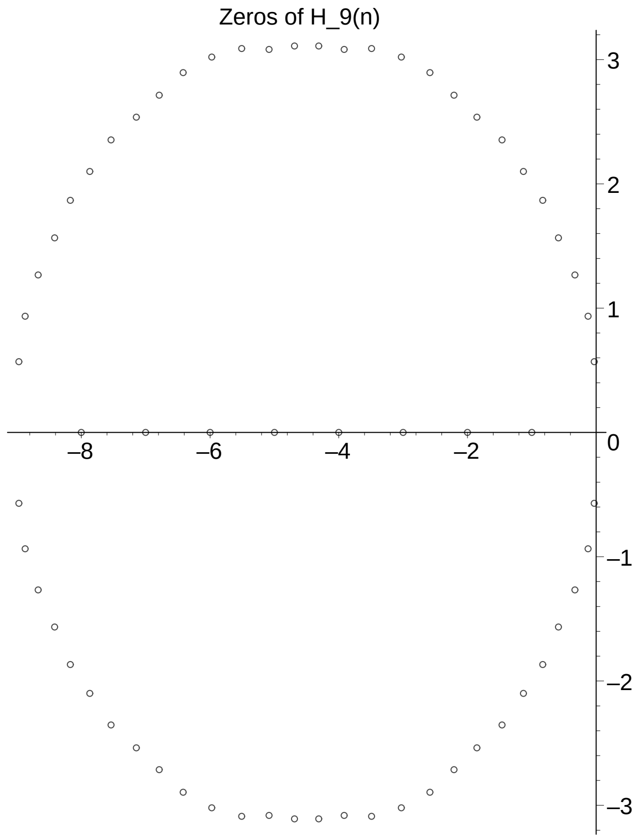

No explanation known.

Coefficients of $i(\mathcal{P}, n)$

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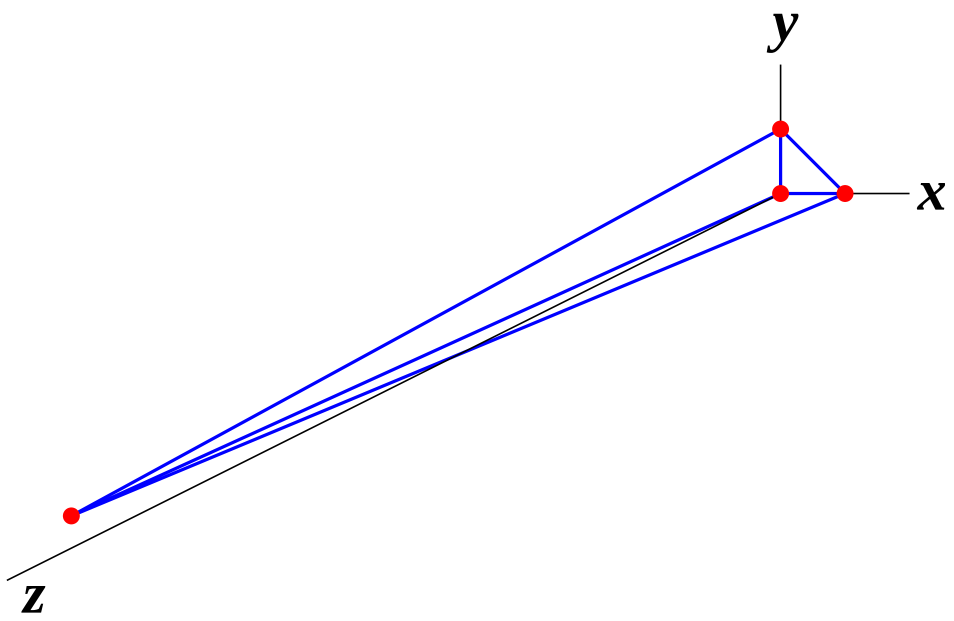
Let $\mathcal{P}$ denote the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,13)$. Then

$$i(\mathcal{P}, n) = \frac{13}{6}n^3 + n^2 - \frac{1}{6}n + 1.$$

The “bad” tetrahedron



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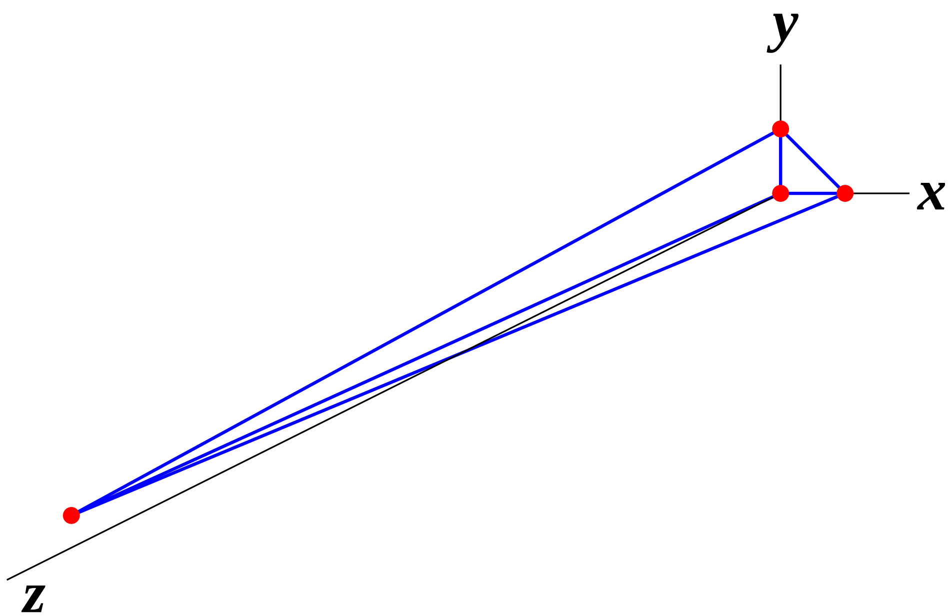

Thus in general the coefficients of Ehrhart polynomials are not “nice.” There is a better basis (not given here).

Zonotopes

Let $\boldsymbol{v_1, \dots, v_k} \in \mathbb{R}^d$. The **zonotope** $Z(v_1, \dots, v_k)$ **generated** by $v_1, \dots, v_k$:

$$\boldsymbol{Z(v_1, \dots, v_k)} = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \,:\, 0 \le \lambda_i \le 1\}$$

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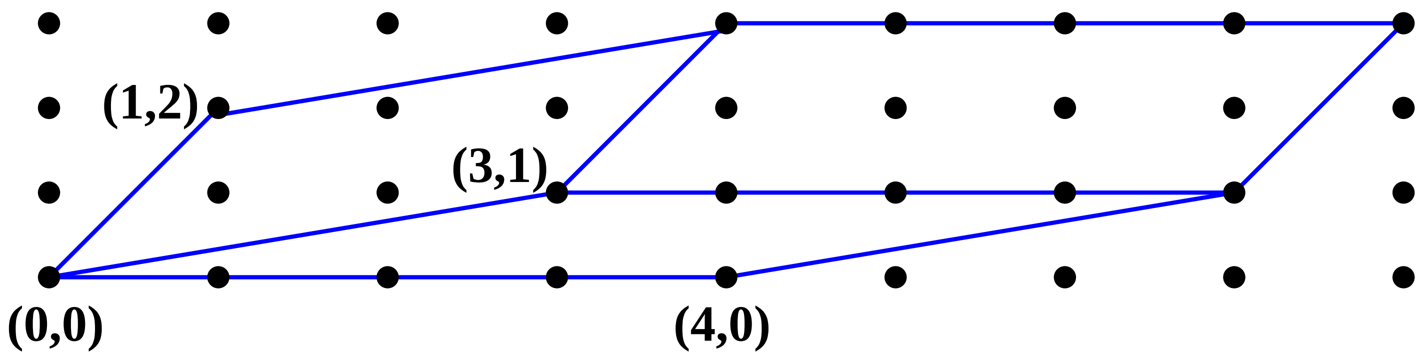
Example. $v_1 = (4,0)$, $v_2 = (3,1)$, $v_3 = (1,2)$



Lattice points in a zonotope

Theorem. *Let*

$$\boldsymbol{Z} = Z(v_1, \ldots, v_k) \subset \mathbb{R}^d,$$

where $v_i \in \mathbb{Z}^d$. *Then the coefficient of* n^j *in* $i(Z, n)$ *is given by* $\sum_X h(X)$, *where* X *ranges over all linearly independent* j*-element subsets of* $\{v_1, \ldots, v_k\}$, *and* $h(X)$ *is the gcd of all* $j \times j$ *minors of the matrix whose rows are the elements of* X.

An example

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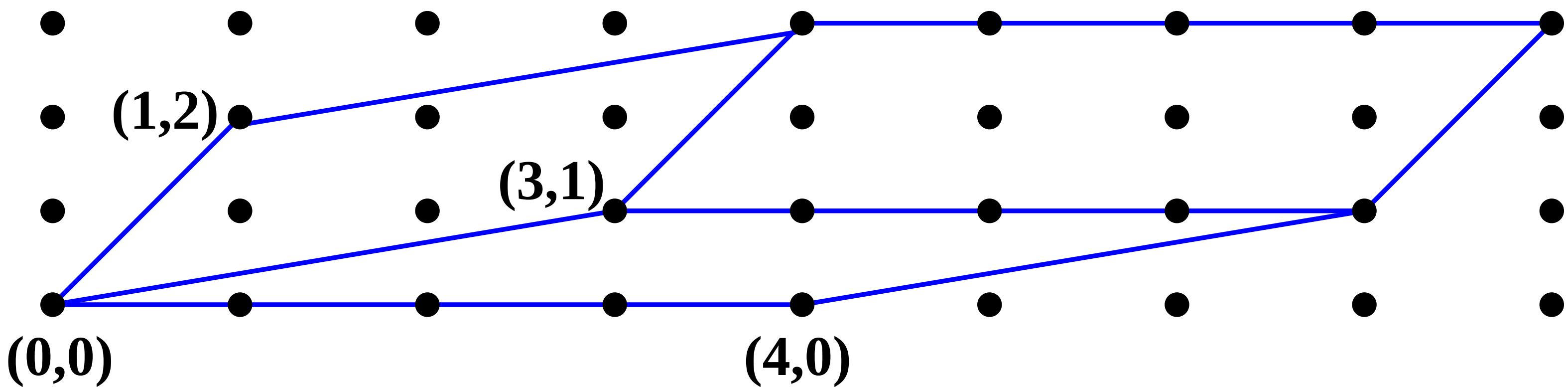

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$$
\begin{aligned}
i(Z,n) &= \left(\begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \right) n^2 \\
&\quad + (\gcd(4,0) + \gcd(3,1) \\
&\quad + \gcd(1,2))n + \det(\emptyset) \\
&= (4+8+5)n^2 + (4+1+1)n + 1 \\
&= 17n^2 + 6n + 1.
\end{aligned}
$$

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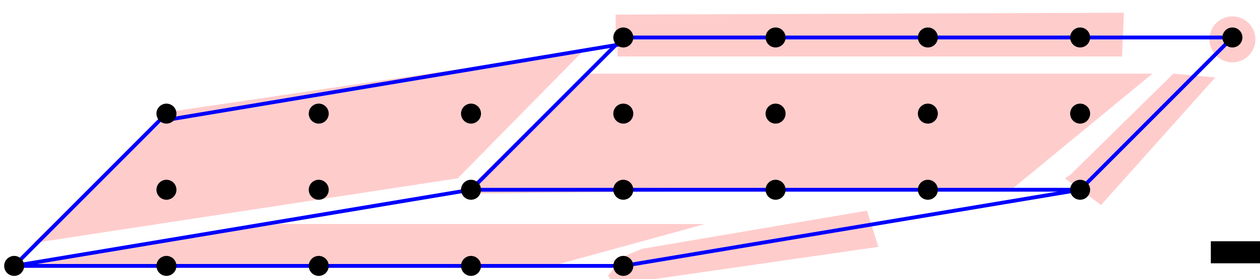

Corollaries

Corollary. *If Z is an integer zonotope generated by integer vectors, then the coefficients of $i(Z,n)$ are nonnegative integers.*

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Neither property is true for general integer polytopes. There are numerous conjectures concerning special cases.

The permutohedron

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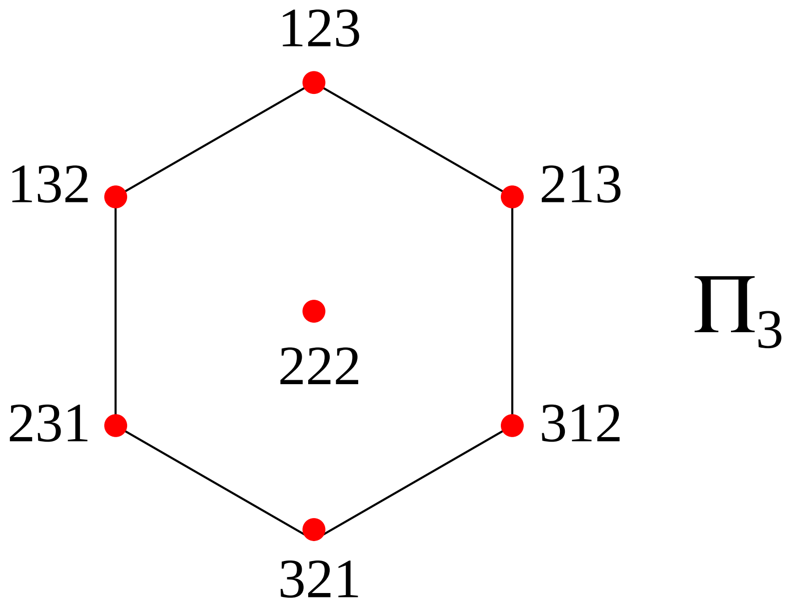
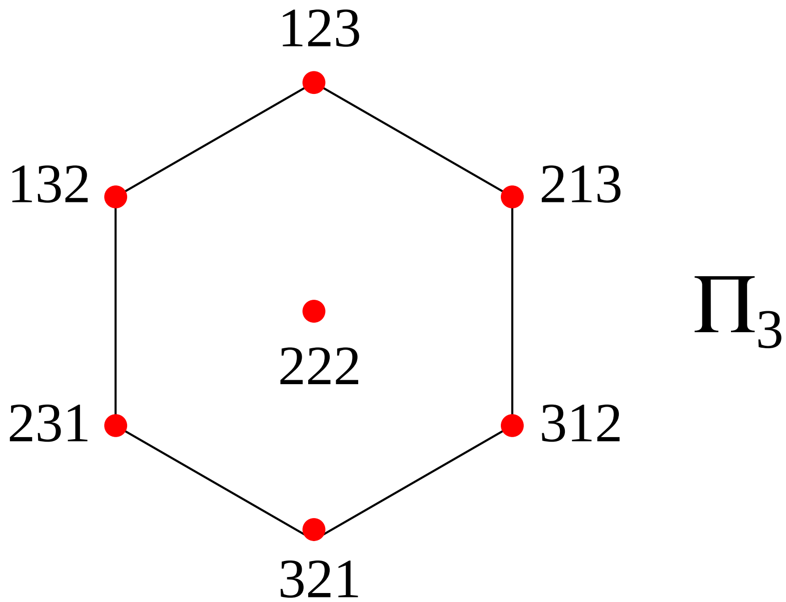
$$\Pi_d \approx Z(e_i - e_j : 1 \le i < j \le d)$$

Π_3

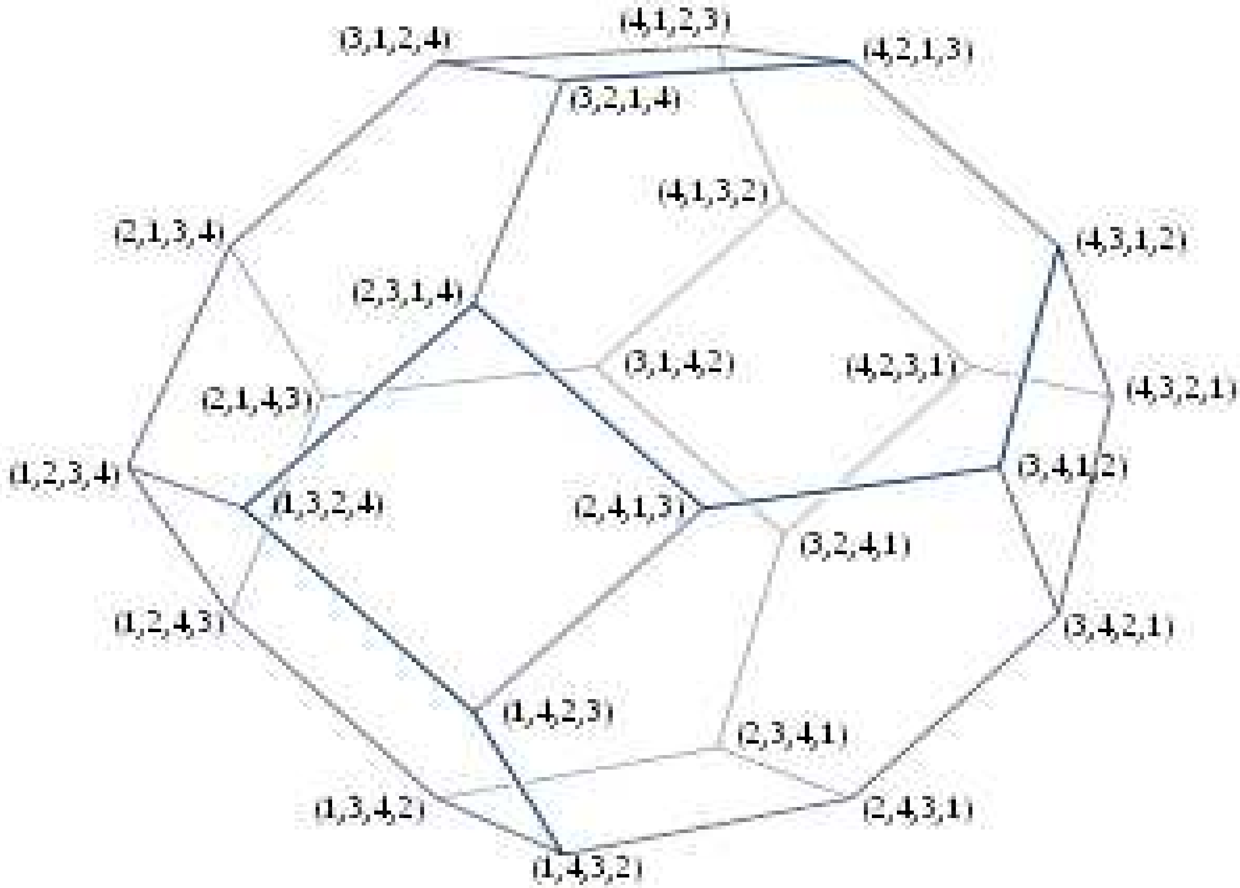


Π_3



$$i(\Pi_3, n) \;=\; 3n^2 + 3n + 1$$

Π_4



(truncated octahedron)

$i(\Pi_d, n)$

Theorem. $i(\Pi_d, n) = \sum_{k=0}^{d-1} f_k(d) x^k$, where

$\boldsymbol{f_k(d)} = \#\{$forests with k edges on vertices $1, \ldots, d\}$

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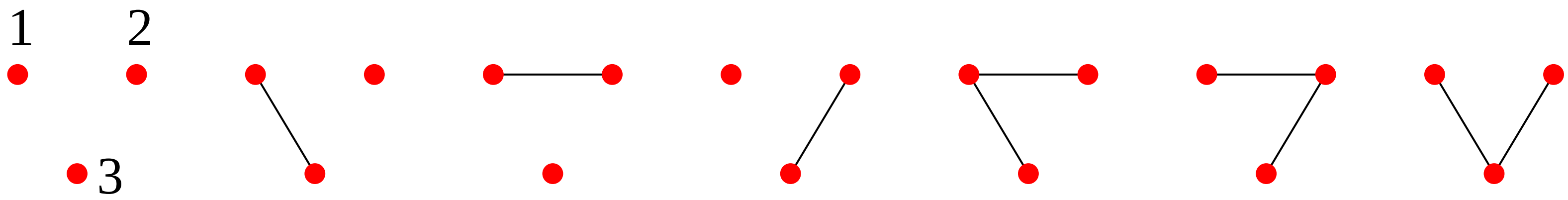

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Application to graph theory

Let G be a graph (with no loops or multiple edges) on the vertex set $V(G) = \{1, 2, \ldots, n\}$. Let

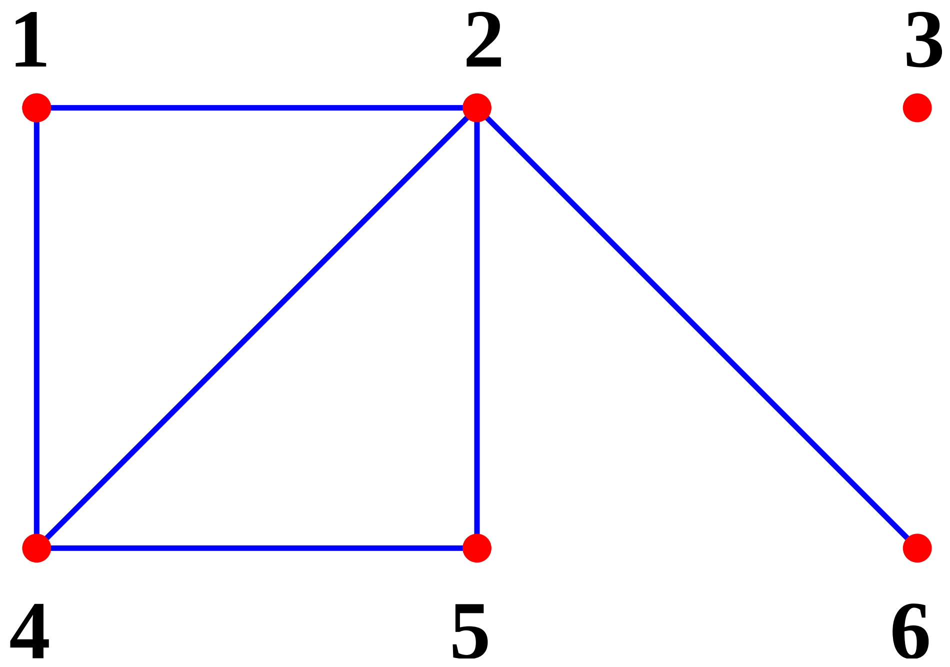
$$d_i = \text{degree (\# incident edges) of vertex } i.$$

Define the **ordered degree sequence** $d(G)$ of G by

$$d(G) = (d_1, \ldots, d_n).$$

Example of $d(G)$

Example. $d(G) = (2, 4, 0, 3, 2, 1)$

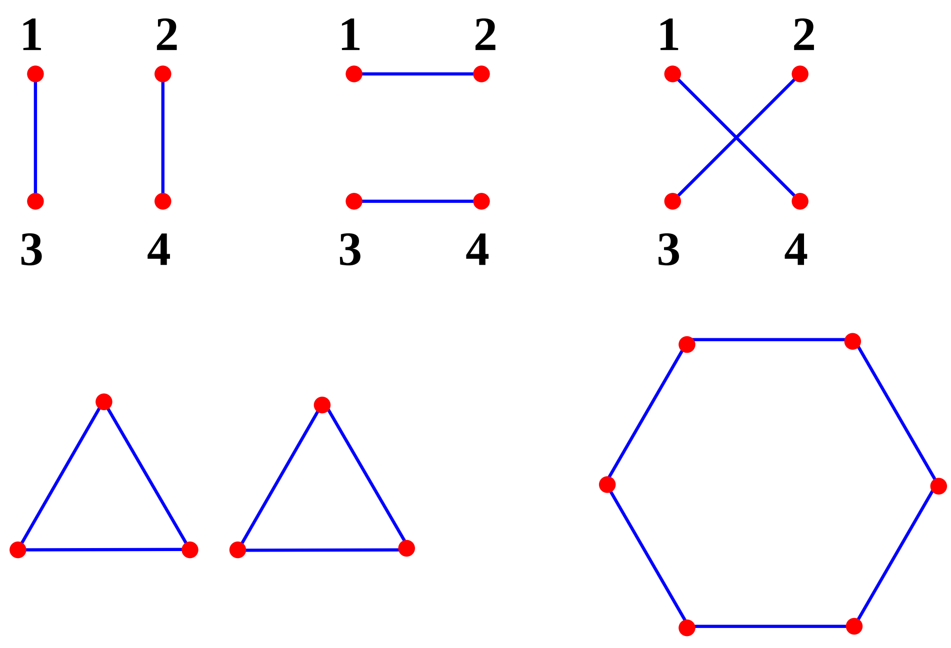


of ordered degree sequences

Let $\boldsymbol{f(n)}$ be the number of distinct $d(G)$, where $V(G) = \{1, 2, \ldots, n\}$.

$f(n)$ for $n \leq 4$

Example. If $n \leq 3$, all $d(G)$ are distinct, so $f(1) = 1$, $f(2) = 2^1 = 2$, $f(3) = 2^3 = 8$. For $n \geq 4$ we can have $G \neq H$ but $d(G) = d(H)$, e.g.,



In fact, $f(4) = 54 < 2^6 = 64$.

The polytope of degree sequences

Let **conv** denote convex hull, and

$$\mathcal{D}_n = \mathrm{conv}\{d(G) : V(G) = \{1, \ldots, n\}\} \subset \mathbb{R}^n,$$

the **polytope of degree sequences** (**Perles, Koren**).

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Easy fact. Let e_i be the ith unit coordinate vector in $\mathbb{R}^n$. E.g., if $n = 5$ then $e_2 = (0, 1, 0, 0, 0)$. Then

$$\mathcal{D}_n = Z(e_i + e_j : 1 \leq i < j \leq n).$$

The Erdős-Gallai theorem

Theorem. *Let*

$$\boldsymbol{\alpha} = (a_1, \ldots, a_n) \in \mathbb{Z}^n.$$

Then $\alpha = d(G)$ *for some* G *if and only if*

- $\alpha \in \mathcal{D}_n$
- $a_1 + a_2 + \cdots + a_n$ *is even.*

A generating function

Enumerative techniques leads to:

Theorem. *Let*

$$F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}$$
$$= 1 + x + 2\frac{x^2}{2!} + 8\frac{x^3}{3!} + 54\frac{x^4}{4!} + \cdots .$$

Then:

A formula for $F(x)$

$$F(x) = \frac{1}{2}\left[\left(1 + 2\sum_{n\geq 1} n^n \frac{x^n}{n!}\right)^{1/2}\right.$$

$$\left.\times\left(1 - \sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}\right) + 1\right]$$

$$\times \exp \sum_{n\geq 1} n^{n-2}\frac{x^n}{n!} \qquad (0^0 = 1)$$

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Two references

M. Beck and S. Robins, *Computing the Continuous Discretely*, Springer, 2010.

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THE END