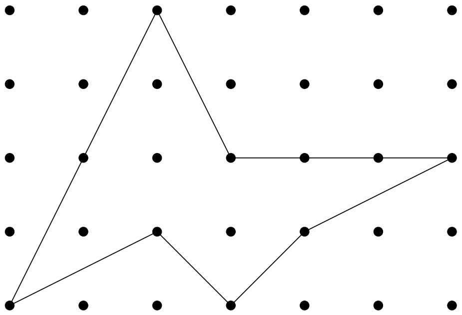
Lattice Points in Polytopes

Richard P. Stanley
U. Miami & M.I.T.

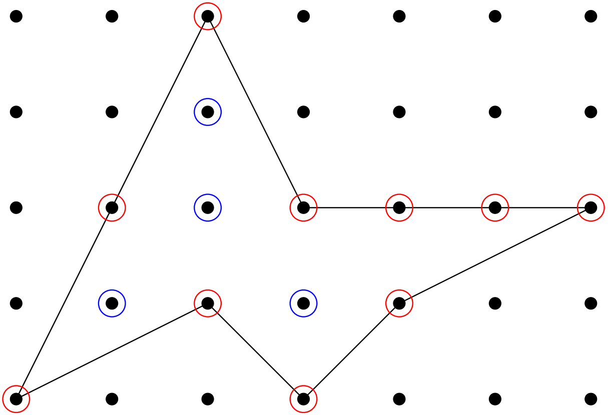
A lattice polygon

Georg Alexander Pick (1859–1942)

P: lattice polygon in $\mathbb{R}^2$
(vertices $\in \mathbb{Z}^2$, no self-intersections)



Boundary and interior lattice points



Pick's theorem

$$\begin{aligned} \boldsymbol{A} &= \text{area of } P \\ \boldsymbol{I} &= \#\text{ interior points of } P\ (=4) \\ \boldsymbol{B} &= \#\text{boundary points of } P\ (=10) \end{aligned}$$

Then

$$\boldsymbol{A} = \frac{2\boldsymbol{I} + \boldsymbol{B} - 2}{2}.$$

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Then

$$\mathbf{A} = \frac{2\mathbf{I} + \mathbf{B} - 2}{2}.$$

Example on previous slide:

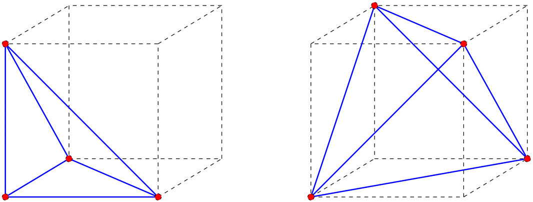
$$\frac{2 \cdot \mathbf{4} + \mathbf{10} - 2}{2} = 9.$$

Two tetrahedra

Pick's theorem (seemingly) fails in higher dimensions. For example, let T_1 and T_2 be the tetrahedra with vertices

$$\begin{aligned} v(T_1) &= \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\} \\ v(T_2) &= \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}. \end{aligned}$$

Failure of Pick's theorem in dim 3



Then

$$I(T_1) = I(T_2) = 0$$

$$B(T_1) = B(T_2) = 4$$

$$A(T_1) = 1/6, \quad A(T_2) = 1/3.$$

Polytope dilation

Let $\mathcal{P}$ be a convex polytope (convex hull of a finite set of points) in $\mathbb{R}^d$. For $n \geq 1$, let

$$n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}.$$

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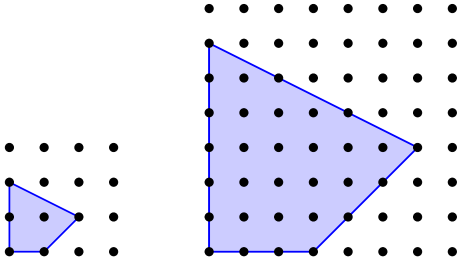

P **3P**

$i(\mathcal{P}, n)$

Let

$$\begin{aligned} i(\mathcal{P}, n) &= \#(n\mathcal{P} \cap \mathbb{Z}^d) \\ &= \#\{\alpha \in \mathcal{P} \,:\, n\alpha \in \mathbb{Z}^d\}, \end{aligned}$$

the number of lattice points in $n\mathcal{P}$.

$\bar{i}(\mathcal{P}, n)$

Similarly let

$$\mathcal{P}^\circ = \text{interior of } \mathcal{P} = \mathcal{P} - \partial\mathcal{P}$$

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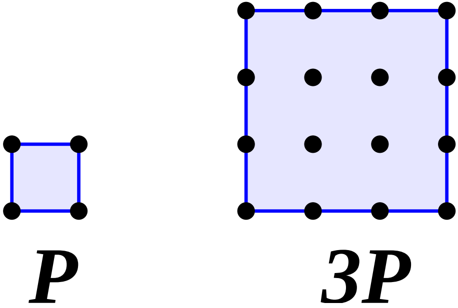
Note. Could use any lattice L instead of $\mathbb{Z}^d$.

An example



$$i(\mathcal{P}, n) = (n+1)^2$$

$$\bar{i}(\mathcal{P}, n) = (n-1)^2 = i(\mathcal{P}, -n).$$

The main result

Theorem (**Ehrhart** 1962, **Macdonald** 1963). *Let*

$$\mathcal{P} = \text{lattice polytope in } \mathbb{R}^N, \ \dim \mathcal{P} = \boldsymbol{d}.$$

Then $i(\mathcal{P}, n)$ is a polynomial (the **Ehrhart polynomial** *of $\mathcal{P}$) in n of degree d.*

Reciprocity and volume

Moreover,

$$\begin{aligned} i(\mathcal{P},0) &= 1 \\ \bar{i}(\mathcal{P},n) &= (-1)^d i(\mathcal{P},-n), \;\; n>0 \\ &\qquad\qquad\textbf{(reciprocity)}. \end{aligned}$$

Reciprocity and volume

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If $d = N$ then

$$i(\mathcal{P},n) = V(\mathcal{P})n^d + \text{ lower order terms},$$

where $\boldsymbol{V(\mathcal{P})}$ *is the volume of* $\mathcal{P}$.

Eugène Ehrhart

- April 29, 1906: born in Guebwiller, France
- 1932: begins teaching career in lycées
- 1959: Prize of French Sciences Academy
- 1963: begins work on Ph.D. thesis
- 1966: obtains Ph.D. thesis from Univ. of Strasbourg
- 1971: retires from teaching career
- January 17, 2000: dies

Photo of Ehrhart

Self-portrait

Generalized Pick's theorem

Corollary. *Let* $\mathcal{P} \subset \mathbb{R}^d$ *and* $\dim \mathcal{P} = d$. *Knowing any* d *of* $i(\mathcal{P}, n)$ *or* $\bar{i}(\mathcal{P}, n)$ *for* $n > 0$ *determines* $V(\mathcal{P})$.

Generalized Pick's theorem

Corollary. *Let $\mathcal{P} \subset \mathbb{R}^d$ and $\dim \mathcal{P} = d$. Knowing any d of $i(\mathcal{P}, n)$ or $\bar{i}(\mathcal{P}, n)$ for $n > 0$ determines $V(\mathcal{P})$.*

Proof. Together with $i(\mathcal{P}, 0) = 1$, this data determines $d + 1$ values of the polynomial $i(\mathcal{P}, n)$ of degree d. This uniquely determines $i(\mathcal{P}, n)$ and hence its leading coefficient $V(\mathcal{P})$. $\square$

Birkhoff polytope

Example. Let $\mathcal{B}_M \subset \mathbb{R}^{M\times M}$ be the **Birkhoff polytope** of all $M \times M$ **doubly-stochastic** matrices $A = (a_{ij})$, i.e.,

$$\begin{aligned} a_{ij} &\geq 0 \\ \sum_i a_{ij} &= 1 \text{ (column sums 1)} \\ \sum_j a_{ij} &= 1 \text{ (row sums 1)}. \end{aligned}$$

(Weak) magic squares

Note. $B = (b_{ij}) \in n\mathcal{B}_M \cap \mathbb{Z}^{M\times M}$ if and only if

$$\begin{aligned} b_{ij} &\in \mathbb{N} = \{0, 1, 2, \dots\} \\ \sum_i b_{ij} &= n \\ \sum_j b_{ij} &= n. \end{aligned}$$

Example of a magic square

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 3 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 4 & 0 \end{bmatrix} \quad (M = 4, \ \ n = 7)$$

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$$\in 7\mathcal{B}_4$$

$H_M(n)$

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$$\begin{aligned} H_1(n) &= 1 \\ H_2(n) &= n+1 \end{aligned}$$

$$\begin{bmatrix} a & n-a \\ n-a & a \end{bmatrix}, \quad 0 \le a \le n.$$

The case $M = 3$

$$H_3(n) = \binom{n+2}{4} + \binom{n+3}{4} + \binom{n+4}{4}$$

(MacMahon)

Values for small n

$$H_M(0) = ??$$

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Anand-Dumir-Gupta, 1966:

$$\sum_{M \geq 0} H_M(2) \frac{x^M}{M!^2} = ??$$

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Anand-Dumir-Gupta, 1966:

$$\sum_{M \geq 0} H_M(2) \frac{x^M}{M!^2} = \frac{e^{x/2}}{\sqrt{1-x}}$$

Anand-Dumir-Gupta conjecture

Theorem (**Birkhoff-von Neumann**). *The vertices of $\mathcal{B}_M$ consist of the $M!$ $M \times M$ permutation matrices. Hence $\mathcal{B}_M$ is a lattice polytope.*

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Corollary (Anand-Dumir-Gupta conjecture). *$H_M(n)$ is a polynomial in n (of degree $(M-1)^2$).*

$H_4(n)$

Example. $$H_4(n) = \frac{1}{11340}\big(11n^9 + 198n^8 + 1596n^7$$
$$+7560n^6 + 23289n^5 + 48762n^5 + 70234n^4 + 68220n^2$$
$$+40950n + 11340\big)\,.$$

Reciprocity for magic squares

Reciprocity $\Rightarrow \pm H_M(-n) =$

$$\#\{M \times M \text{ matrices } B \text{ of } \textbf{positive} \text{ integers, line sum } n\}.$$

But every such B can be obtained from an $M \times M$ matrix A of **nonnegative** integers by adding 1 to each entry.

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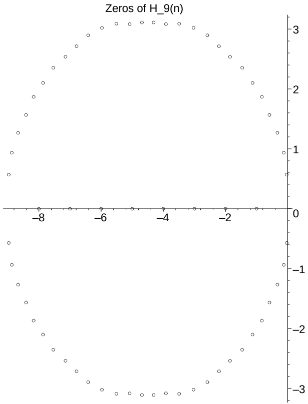
Corollary. $H_M(-1) = H_M(-2) = \cdots = H_M(-M+1) = 0$

$$H_M(-M-n) = (-1)^{M-1} H_M(n)$$

Two remarks

- Reciprocity greatly reduces computation.
- Applications of magic squares, e.g., to statistics (contingency tables).

Zeros of $H_9(n)$ in complex plane



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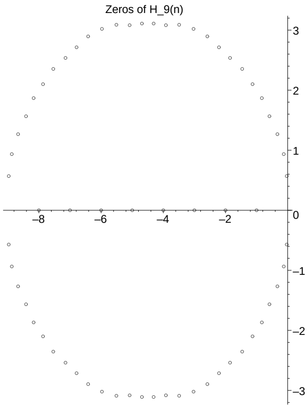

No explanation known.

Zonotopes

Let $\boldsymbol{v_1, \ldots, v_k} \in \mathbb{R}^d$. The **zonotope** $Z(v_1, \ldots, v_k)$ generated by $v_1, \ldots, v_k$:

$$\boldsymbol{Z(v_1, \ldots, v_k)} = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \;:\; 0 \leq \lambda_i \leq 1\}$$

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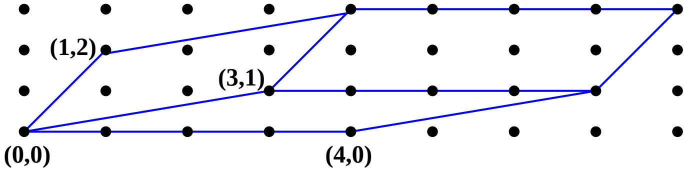
Example. $v_1 = (4,0)$, $v_2 = (3,1)$, $v_3 = (1,2)$



Lattice points in a zonotope

Theorem. *Let*

$$Z = Z(v_1, \ldots, v_k) \subset \mathbb{R}^d,$$

where $v_i \in \mathbb{Z}^d$. *Then*

$$i(Z, 1) = \sum_X h(X),$$

where X *ranges over all linearly independent subsets of* $\{v_1, \ldots, v_k\}$, *and* $h(X)$ *is the gcd of all* $j \times j$ *minors* ($j = \#X$) *of the matrix whose rows are the elements of* X.

An example

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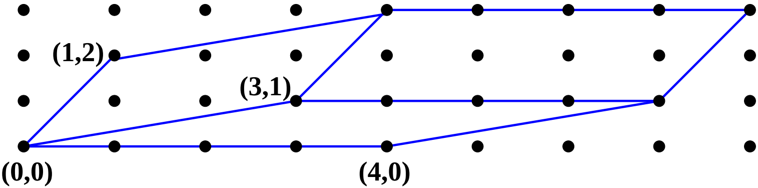

Computation of $i(Z,1)$

$$
\begin{aligned}
i(Z,1) &= \begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \\
&\quad + \gcd(4,0) + \gcd(3,1) \\
&\quad + \gcd(1,2) + \det(\varnothing) \\
&= 4 + 8 + 5 + 4 + 1 + 1 + 1 \\
&= 24.
\end{aligned}
$$

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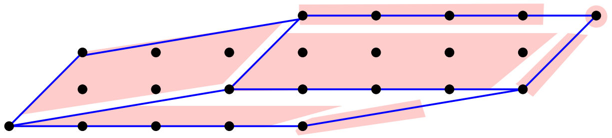

Corollaries

Corollary. *If Z is an integer zonotope generated by integer vectors, then the coefficients of $i(Z, n)$ are nonnegative integers.*

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Neither property (nonnegativity, integrality) is true for general integer polytopes. There are numerous conjectures concerning special cases.

The permutohedron

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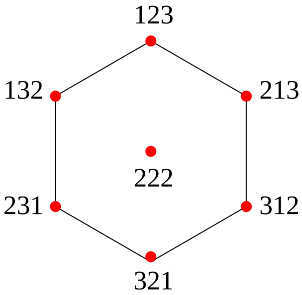
$$\Pi_d \approx Z(e_i - e_j \colon 1 \le i < j \le d)$$

Π_3



Π_3

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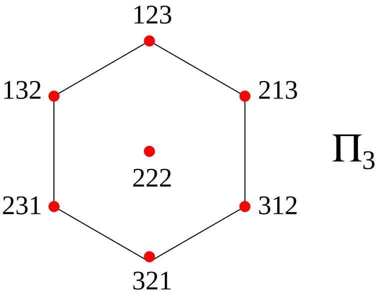
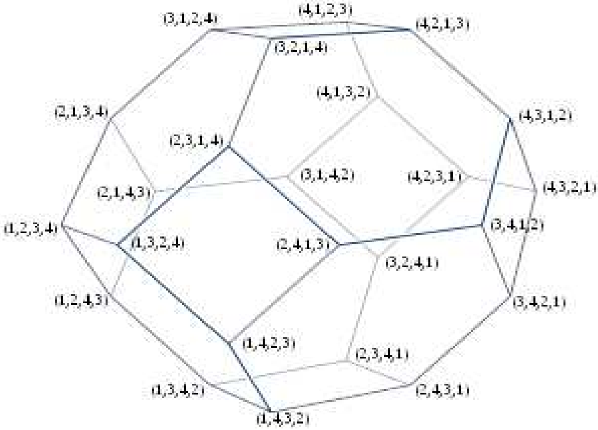


$$i(\Pi_3, n) \quad = \quad 3n^2 + 3n + 1$$

Π_4



(truncated octahedron)

$i(\Pi_d, n)$

Theorem. $i(\Pi_d, n) = \sum_{k=0}^{d-1} f_k(d) n^k$, *where*

$$f_k(d) = \#\{\text{forests with } k \text{ edges on vertices } 1, \ldots, d\}$$

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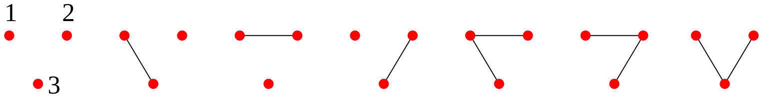

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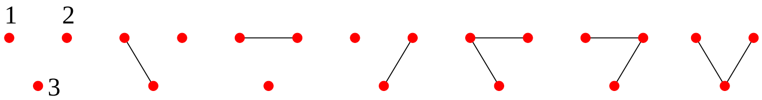


$$i(\Pi_3, n) = 3n^2 + 3n + 1$$

Can be greatly generalized (**Postnikov**, et al.).

Application to graph theory

Let G be a graph (with no loops or multiple edges) on the vertex set $V(G) = \{1, 2, \ldots, n\}$. Let

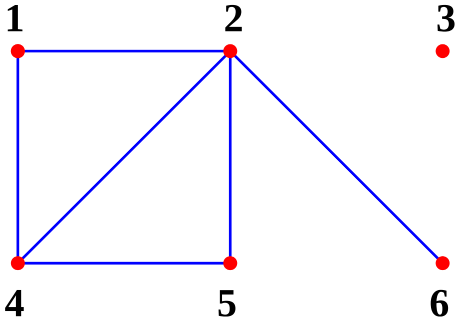
$$d_i = \text{degree } (\# \text{ incident edges}) \text{ of vertex } i.$$

Define the **ordered degree sequence** $d(G)$ of G by

$$d(G) = (d_1, \ldots, d_n).$$

Example of $d(G)$

Example. $d(G) = (2, 4, 0, 3, 2, 1)$

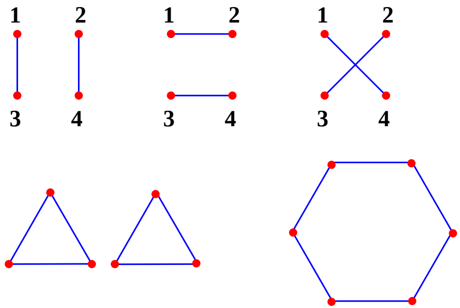


ordered degree sequences

Let $\boldsymbol{f(n)}$ be the number of distinct $d(G)$, where $V(G) = \{1, 2, \ldots, n\}$.

$f(n)$ for $n \le 4$

Example. If $n \le 3$, all $d(G)$ are distinct, so $f(1) = 1$, $f(2) = 2^1 = 2$, $f(3) = 2^3 = 8$. For $n \ge 4$ we can have $G \ne H$ but $d(G) = d(H)$, e.g.,



In fact, $f(4) = 54 < 2^6 = 64$.

The polytope of degree sequences

Let **conv** denote convex hull, and

$$\mathcal{D}_n = \mathrm{conv}\{d(G) : V(G) = \{1, \ldots, n\}\} \subset \mathbb{R}^n,$$

the **polytope of degree sequences** (**Perles, Koren**).

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Easy fact. Let e_i be the ith unit coordinate vector in $\mathbb{R}^n$. E.g., if $n = 5$ then $e_2 = (0, 1, 0, 0, 0)$. Then

$$\mathcal{D}_n = Z(e_i + e_j : 1 \le i < j \le n).$$

The Erdős-Gallai theorem

Theorem. *Let*

$$\boldsymbol{\alpha} = (a_1, \ldots, a_n) \in \mathbb{Z}^n.$$

Then $\alpha = d(G)$ *for some* G *if and only if*

- $\alpha \in \mathcal{D}_n$
- $a_1 + a_2 + \cdots + a_n$ *is even.*

A generating function

Enumerative techniques leads to:

Theorem. *Let*

$$\begin{aligned} F(x) &= \sum_{n \geq 0} f(n) \frac{x^n}{n!} \\ &= 1 + x + 2\frac{x^2}{2!} + 8\frac{x^3}{3!} + 54\frac{x^4}{4!} + \cdots . \end{aligned}$$

Then:

A formula for $F(x)$

$$F(x) = \frac{1}{2}\Bigg[\left(1 + 2\sum_{n\geq 1} n^n \frac{x^n}{n!}\right)^{1/2}$$

$$\times \left(1 - \sum_{n\geq 1} (n-1)^{n-1} \frac{x^n}{n!}\right) + 1\Bigg]$$

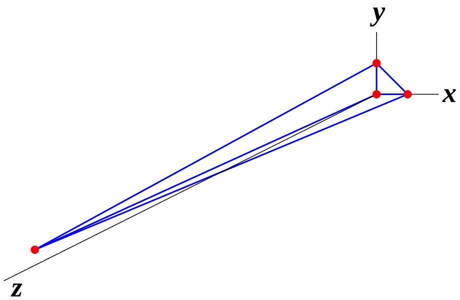
$$\times \exp \sum_{n\geq 1} n^{n-2} \frac{x^n}{n!} \qquad (0^0 = 1)$$

Coefficients of $i(\mathcal{P}, n)$

Let $\mathcal{P}$ denote the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 13)$. Then

$$i(\mathcal{P}, n) = \frac{13}{6} n^3 + n^2 - \frac{1}{6} n + 1.$$

The “bad” tetrahedron



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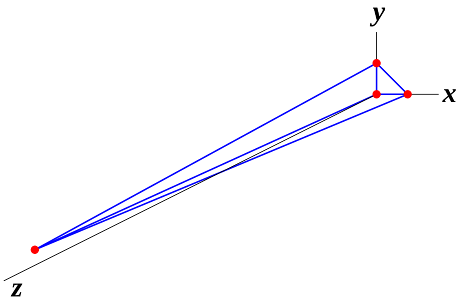

Thus in general the coefficients of Ehrhart polynomials are not “nice.” Is there a “better” basis?

The h^*-vector of $i(\mathcal{P}, n)$

Let $\mathcal{P}$ be a lattice polytope of dimension d. Since $i(\mathcal{P}, n)$ is a polynomial of degree d, $\exists$ $\boldsymbol{h_i} \in \mathbb{Z}$ such that

$$\sum_{n \geq 0} i(\mathcal{P}, n) x^n = \frac{h_0 + h_1 x + \cdots + h_d x^d}{(1-x)^{d+1}}.$$

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Definition. Define

$$\boldsymbol{h^*(\mathcal{P})} = (h_0, h_1, \ldots, h_d),$$

the ***h^**-vector** of $\mathcal{P}$.

Example of an h^*-vector

Example. Recall

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Then

$$h^*(\mathcal{B}_4) = (1, 14, 87, 148, 87, 14, 1, 0, 0, 0).$$

Two terms of $h^*(\mathcal{P})$

- $h_0 = 1$
- $h_d = (-1)^{\dim \mathcal{P}} i(\mathcal{P}, -1) = I(\mathcal{P})$

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B $\Rightarrow$ A: take $\mathcal{Q} = \emptyset$.

Proofs: the Ehrhart ring

$\mathcal{P}$: (convex) lattice polytope in $\mathbb{R}^d$ with vertex set V

$\boldsymbol{x}^{\beta} = x^{\beta_1} \cdots x^{\beta_d}$, $\beta \in \mathbb{Z}^d$

Ehrhart ring (over $\mathbb{Q}$):

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This means (using finiteness of $R_{\mathcal{P}}$ over $\mathbb{Q}[V]$): if $\dim \mathcal{P} = m$ then there exist algebraically independent $\theta_1, \dots, \theta_m \in (R_{\mathcal{P}})_1$ such that $R_{\mathcal{P}}$ is a finitely-generated free $\mathbb{Q}[\theta_1, \dots, \theta_m]$-module.

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Corollary. $\sum_{n \geq 0} \underbrace{H(R_{\mathcal{P}}, n)}_{i(\mathcal{P},n)} x^n = \dfrac{x^{e_1} + \cdots + x^{e_r}}{(1-x)^m}$, so $h^*(\mathcal{P}) \geq 0$.

Monotonicity

The result $\mathcal{Q} \subseteq \mathcal{P} \Rightarrow h^*(\mathcal{Q}) \leq h^*(\mathcal{P})$ is proved similarly.

We have $R_{\mathcal{Q}} \subset R_{\mathcal{P}}$. The key fact is that we can find an h.s.o.p. $\theta_1, \ldots, \theta_k$ for $R_{\mathcal{Q}}$ that extends to an h.s.o.p. for $R_{\mathcal{P}}$.

The canonical module

Let $R = R_0 \oplus R_1 \oplus \cdots$ be a Cohen-Macaulay graded algebra over a field $K = R_0$, with Krull dimension m and Hilbert series

$$\sum_{n \geq 0} (\dim_K R_n) x^n = \frac{\sum_{j=1}^{r} x^{e_j}}{(1 - x^{d_1}) \cdots (1 - x^{d_m})}.$$

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canonical module: $\boldsymbol{\Omega(R)} = \mathrm{Ext}_A^{t-m}(R, A)$, a graded R-module.

Reciprocity redux

Basic result in commutative/homological algebra:

$$\sum_{n\geq 0}(\dim_K \Omega(R)_n)x^n = \frac{x^c \sum_{j=1}^r x^{-e_j}}{(1-x^{d_1})\cdots(1-x^{d_m})}.$$

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$\Omega(R_{\mathcal{P}}) = \mathrm{span}_{\mathbb{Q}}\{x^\beta y^n : \beta \in \mathbb{Z}^d,\ n \in \mathbb{P},\ \frac{\beta}{n} \in \mathrm{interior}(\mathcal{P})\}$

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Corollary. $\bar{i}(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, n)$.

Further properties: I. Brion's theorem

Example. Let $\mathcal{P}$ be the polytope $[2,5]$ in $\mathbb{R}$, so $\mathcal{P}$ is defined by

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Let

$$\begin{aligned} F_1(t) &= \sum_{\substack{n\geq 2\\ n\in\mathbb{Z}}} t^n = \frac{t^2}{1-t} \\ F_2(t) &= \sum_{\substack{n\leq 5\\ n\in\mathbb{Z}}} t^n = \frac{t^5}{1-\frac{1}{t}}. \end{aligned}$$

$F_1(t) + F_2(t)$

$$\begin{aligned} F_1(t) + F_2(t) &= \frac{t^2}{1-t} + \frac{t^5}{1-\frac{1}{t}} \\ &= t^2 + t^3 + t^4 + t^5 \\ &= \sum_{m \in \mathcal{P} \cap \mathbb{Z}} t^m . \end{aligned}$$

Cone at a vertex

$\mathcal{P}$: $\mathbb{Z}$-polytope in $\mathbb{R}^N$ with vertices $\boldsymbol{v_1}, \ldots, \boldsymbol{v_k}$

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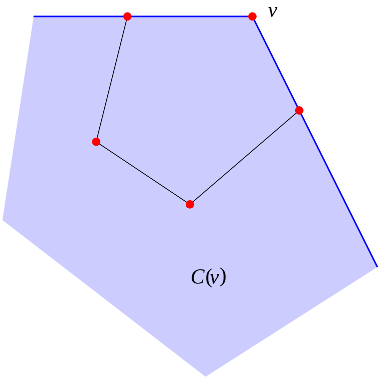

The general result

Let $\boldsymbol{F_i}(t_1, \ldots, t_N) = \sum_{(m_1,\ldots,m_N)\in\mathcal{C}_i\cap\mathbb{Z}^N} t_1^{m_1}\cdots t_N^{m_N}$.

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Theorem (**Brion**). *Each F_i is a rational function of $t_1, \ldots, t_N$, and*

$$\sum_{i=1}^{k} F_i(t_1, \ldots, t_N) = \sum_{(m_1,\ldots,m_N)\in\mathcal{P}\cap\mathbb{Z}^N} t_1^{m_1}\cdots t_N^{m_N}$$

(as rational functions).

II. Complexity

Computing $i(\mathcal{P}, n)$, or even $i(\mathcal{P}, 1)$ is **#*P*-complete**. Thus an "efficient" (polynomial time) algorithm is extremely unlikely. However:

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Theorem (A. Barvinok, 1994). *For* **fixed** $\dim \mathcal{P}$, $\exists$ *polynomial-time algorithm for computing* $i(\mathcal{P}, n)$.

III. Fractional lattice polytopes

Example. Let $S_M(n)$ denote the number of symmetric $M \times M$ matrices of nonnegative integers, every row and column sum n. Then

$$\begin{aligned} S_3(n) &= \begin{cases} \frac{1}{8}(2n^3+9n^2+14n+8), & n \text{ even} \\ \frac{1}{8}(2n^3+9n^2+14n+7), & n \text{ odd} \end{cases} \\ &= \frac{1}{16}(4n^3+18n^2+28n+15+(-1)^n). \end{aligned}$$

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Why a different polynomial depending on n modulo 2?

The symmetric Birkhoff polytope

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Thus if v is a vertex of $\mathcal{T}_M$ then $\mathbf{2}v \in \mathbb{Z}^{M\times M}$.

$S_M(n)$ in general

Theorem. *There exist polynomials $P_M(n)$ and $Q_M(n)$ for which*

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Difficult result (**Dahmen** and **Micchelli**, 1988):

$$\deg Q_M(n) = \begin{cases} \binom{M-1}{2} - 1, & M \text{ odd} \\ \binom{M-2}{2} - 1, & M \text{ even.} \end{cases}$$

IV. Some curious triangles

For $\alpha > 0$ let $\boldsymbol{T_\alpha}$ be the triangle in $\mathbb{R}^2$ with vertices $(0,0), (0,\alpha), (1/\alpha, 0)$, so $\mathrm{area}(T_\alpha) = \frac{1}{2}$. Can define

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- $\alpha = \frac{1}{2}(3+\sqrt{5})$

The last slide

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