Alternating Permutations

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M.I.T.

Basic definitions

A sequence $a_1, a_2, \ldots, a_k$ of distinct integers is **alternating** if

$$a_1 > a_2 < a_3 > a_4 < \cdots ,$$

and **reverse alternating** if

$$a_1 < a_2 > a_3 < a_4 > \cdots .$$

Euler numbers

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E.g., $E_4 = 5 : \ 2143, 3142, 3241, 4132, 4231$

André's theorem

Theorem (**Désiré André**, 1879)

$$\boldsymbol{y} := \sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

$$= \mathbf{1} + \mathbf{1}x + \mathbf{1}\frac{x^2}{2!} + \mathbf{2}\frac{x^3}{3!} + \mathbf{5}\frac{x^4}{4!} + \mathbf{16}\frac{x^5}{5!}$$

$$+ \mathbf{61}\frac{x^6}{6!} + \mathbf{272}\frac{x^7}{7!} + \cdots$$

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E_{2n} is a **secant number**.

E_{2n+1} is a **tangent number**.

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Choose a reverse alternating permutation $\boldsymbol{v} = b_1 b_2 \cdots b_{n-k}$ of $[n] - S$.

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Let $\boldsymbol{w} = a_k \cdots a_2 a_1, \boldsymbol{n} + \mathbf{1}, b_1 b_2 \cdots b_{n-k}$.

Proof (continued)

$$w = a_k \cdots a_2 a_1, n+1, b_1 b_2 \cdots b_{n-k}$$

Given k, there are:

- $\binom{n}{k}$ choices for $\{a_1, a_2, \dots, a_k\}$
- E_k choices for $a_1 a_2 \cdots a_k$
- E_{n-k} choices for $b_1 b_2 \cdots b_{n-k}$.

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We obtain each alternating and reverse alternating $w \in \mathfrak{S}_{n+1}$ once each.

Completion of proof

$$\Rightarrow 2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}, \ n \geq 1$$

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Multiply by $x^{n+1}/(n+1)!$ and sum on $n \geq 0$:

$$2y' = 1 + y^2,\ \ y(0) = 1.$$

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Define

$$\begin{aligned} \tan x &= \sum_{n\geq 0} E_{2n+1} \frac{x^{2n+1}}{(2n+1)!} \\ \sec x &= \sum_{n\geq 0} E_{2n} \frac{x^{2n}}{(2n)!}. \end{aligned}$$

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$\Rightarrow$ **combinatorial trigonometry**

Exercises on combinatorial trig.

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EC2, Exercise 5.7

Boustrophedon

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From Greek *boustrophēdon* ($\beta o \upsilon \sigma \tau \rho o \varphi \eta \delta \acute{o} \nu$), turning like an ox while plowing: *bous*, ox + *strophē*, a turning (from *strephein*, to turn)

The boustrophedon array

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & \rightarrow & 1 & & & & & \\
 & & & & 1 & \leftarrow & 1 & \leftarrow & 0 & & & & \\
 & & & 0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 2 & & & \\
 & & 5 & \leftarrow & 5 & \leftarrow & 4 & \leftarrow & 2 & \leftarrow & 0 & & \\
 & 0 & \rightarrow & 5 & \rightarrow & 10 & \rightarrow & 14 & \rightarrow & 16 & \rightarrow & 16 & \\
61 & \leftarrow & 61 & \leftarrow & 56 & \leftarrow & 46 & \leftarrow & 32 & \leftarrow & 16 & \leftarrow & 0. \\
 & & & & & & \cdots & & & & & &
\end{array}$$

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 & & & & & & \cdots & & & & & &
\end{array}
$$

Boustrophedon entries

- last term in row n: E_{n-1}
- sum of terms in row n: E_n
- kth term in row n: number of alternating permutations in $\mathfrak{S}_n$ with first term k, the **Entringer number** $\boldsymbol{E_{n-1,k-1}}$.

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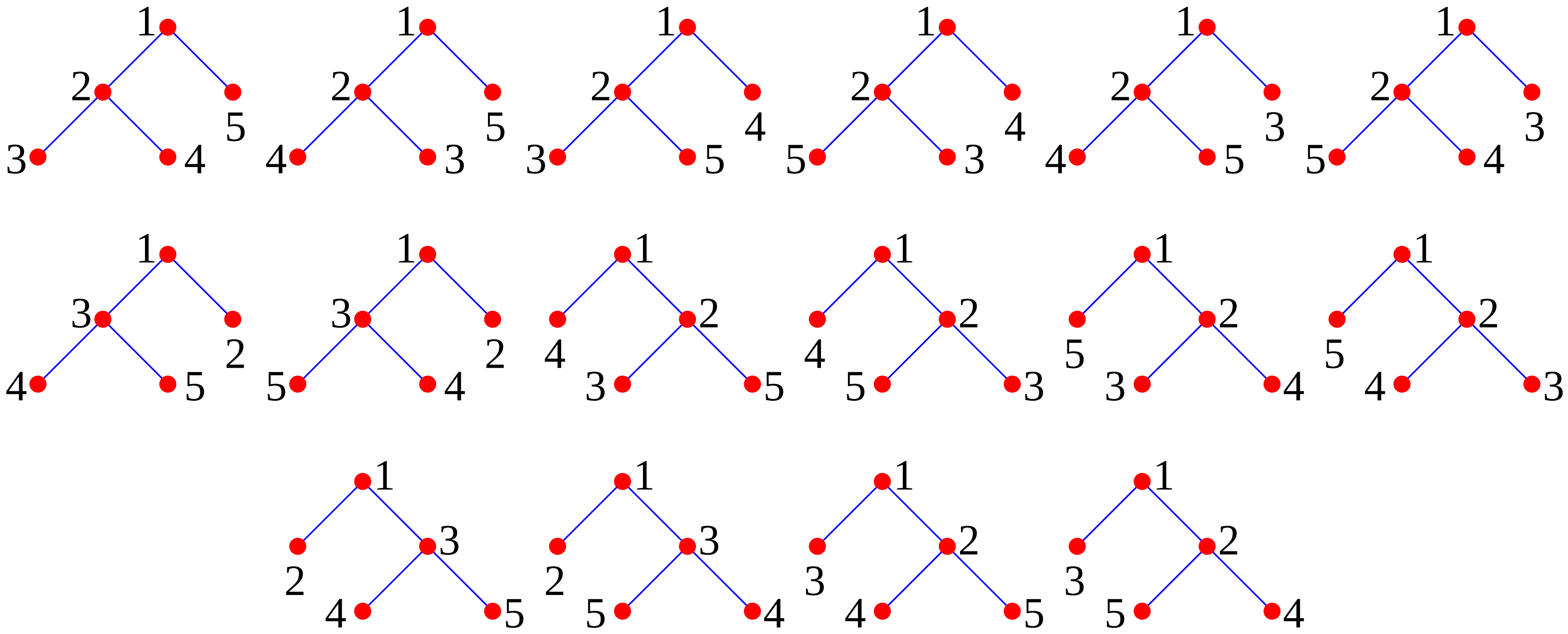
$$\sum_{m\geq 0}\sum_{n\geq 0} E_{m+n,[m,n]}\frac{x^m}{m!}\frac{y^n}{n!} = \frac{\cos x+\sin x}{\cos(x+y)},$$

$$[m,n] = \begin{cases} m, & m+n \text{ odd} \\ n, & m+n \text{ even}. \end{cases}$$

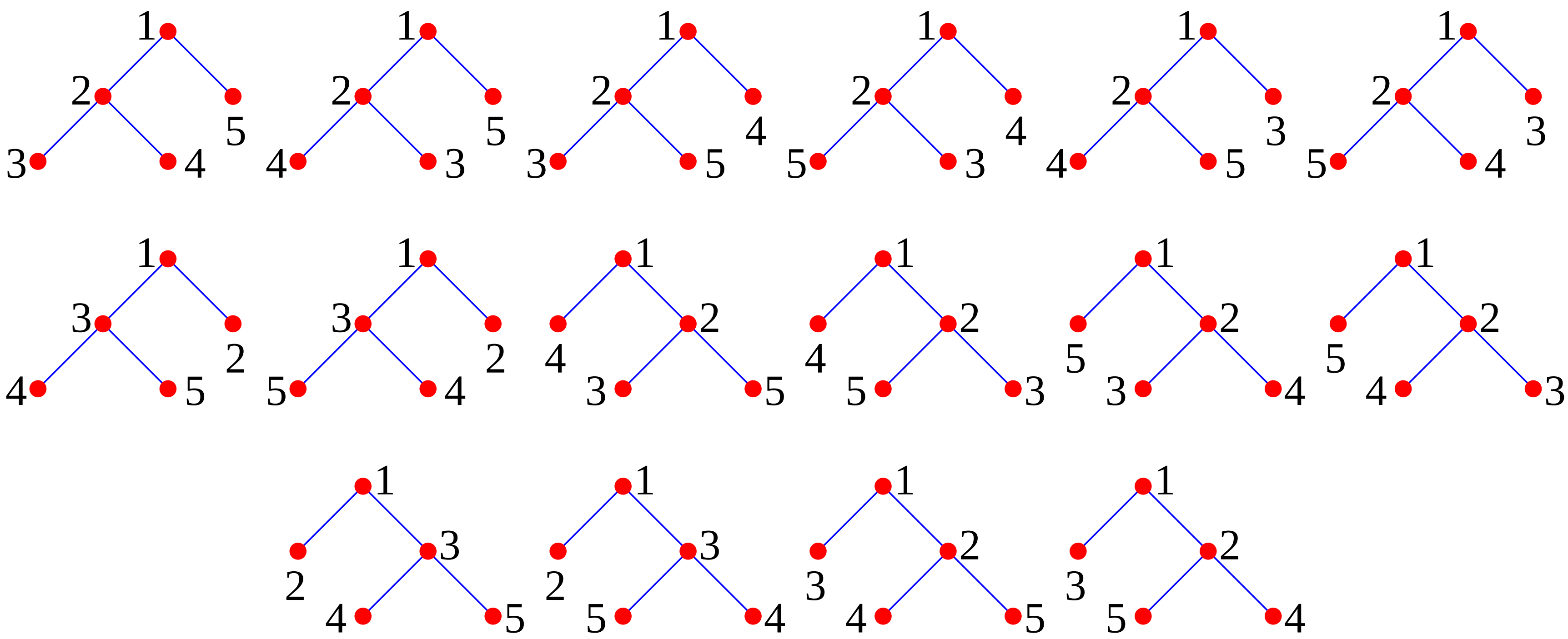
Some occurrences of Euler numbers

(1) E_{2n+1} is the number of complete increasing binary trees on the vertex set $[2n+1] = \{1, 2, \ldots, 2n+1\}$.

Five vertices



Five vertices



Slightly more complicated for E_{2n}

Proof for $2n+1$

$\boldsymbol{b_1 b_2 \cdots b_m}$: sequence of distinct integers

$$\boldsymbol{b_i} = \min\{b_1, \ldots, b_m\}$$

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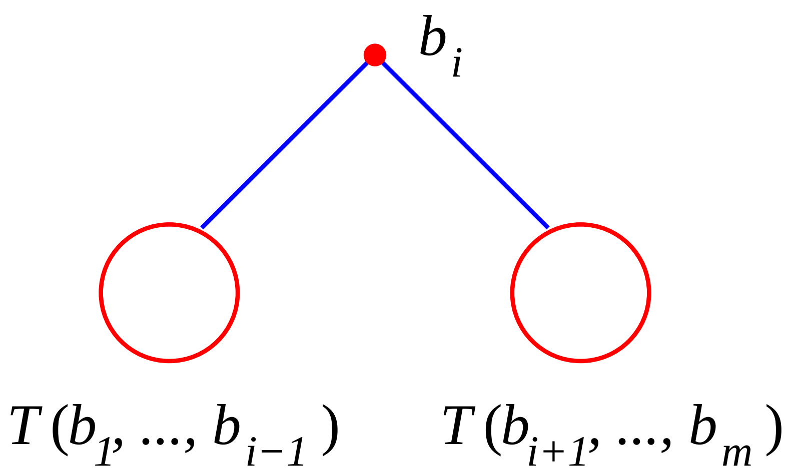
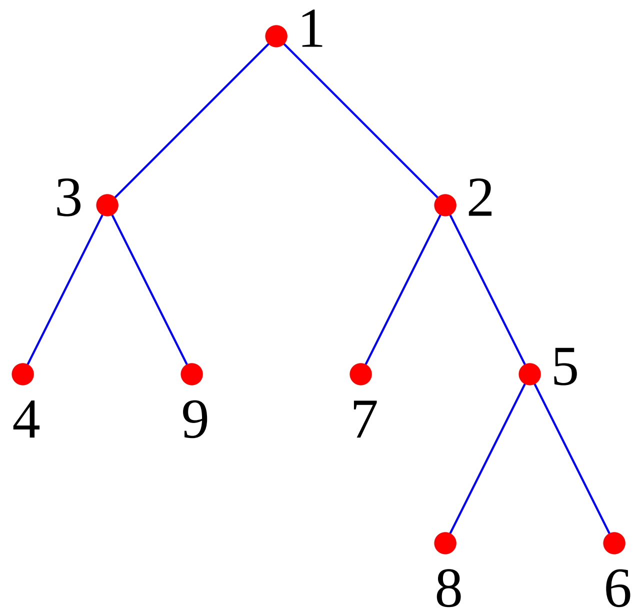
Define recursively a binary tree $\boldsymbol{T(b_1, \dots, b_m)}$ by



Completion of proof

Example. 439172856



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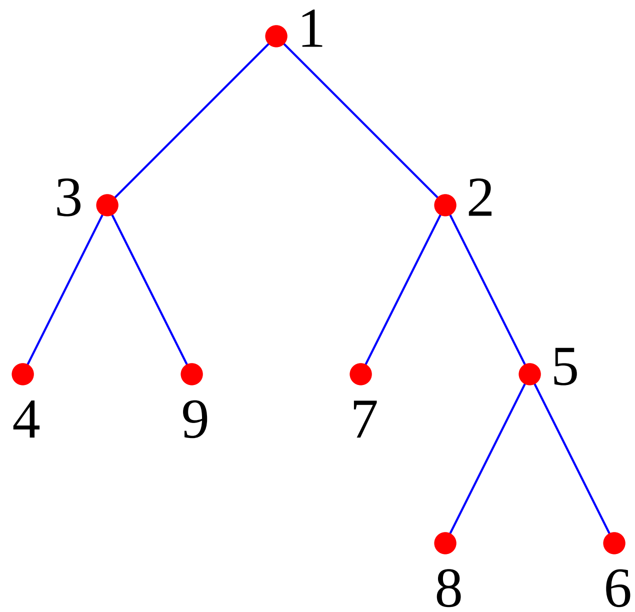

Let $\boldsymbol{w} \in \mathfrak{S}_{2n+1}$. Then $T(w)$ is complete if and only if w is alternating, and the map $w \mapsto T(w)$ gives the desired bijection.

Orbits of mergings

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$$1{-}2{-}3{-}4{-}5{-}6,\quad 12{-}3{-}4{-}5{-}6,\quad 12{-}34{-}5{-}6$$

$$125{-}34{-}6,\quad 125{-}346,\quad 123456$$

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Proof. Exercise. $\square$

Orbit representatives for $n = 5$

$12-3-4-5$	$123-4-5$	$1234-5$
$12-3-4-5$	$123-4-5$	$123-45$
$12-3-4-5$	$12-34-5$	$125-34$
$12-3-4-5$	$12-34-5$	$12-345$
$12-3-4-5$	$12-34-5$	$1234-5$

Volume of a polytope

(3) Let $\mathcal{E}_n$ be the convex polytope in $\mathbb{R}^n$ defined by

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Theorem. *The volume of* $\mathcal{E}_n$ *is* $E_n/n!$.

Naive proof

$$\mathrm{vol}(\mathcal{E}_n) = \int_{x_1=0}^{1} \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1\, dx_2 \cdots dx_n$$

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$$\boldsymbol{f_n(t)} := \int_{x_1=0}^{\boldsymbol{t}} \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1\, dx_2 \cdots dx_n$$

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$$\begin{aligned} f_n'(t) &= \int_{x_2=0}^{1-t}\int_{x_3=0}^{1-x_2}\cdots\int_{x_n=0}^{1-x_{n-1}} dx_2\, dx_3 \cdots dx_n \\ &= f_{n-1}(1-t). \end{aligned}$$

$F(y)$

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$$\begin{aligned} \boldsymbol{F(y)} &= \sum_{n \geq 0} f_n(t) y^n \\ \Rightarrow \frac{\partial^2}{\partial t^2} F(y) &= -y^2 F(y), \end{aligned}$$

etc.

Conclusion of proof

$$F(y) \;=\; (\sec y)(\cos(t-1)y + \sin ty)$$

$$\Rightarrow F(y)|_{t=1} \;=\; \sec y + \tan y.$$

Tridiagonal matrices

An $n \times n$ matrix $M = (m_{ij})$ is **tridiagonal** if $m_{ij} = 0$ whenever $|i - j| \geq 2$.

doubly-stochastic: $m_{ij} \geq 0$, row and column sums equal 1

$\mathcal{T}_n$: set of $n \times n$ tridiagonal doubly stochastic matrices

Polytope structure of $\mathcal{T}_n$

Easy fact: the map

$$\begin{aligned} \mathcal{T}_n &\to \mathbb{R}^{n-1} \\ M &\mapsto (m_{12}, m_{23}, \ldots, m_{n-1,n}) \end{aligned}$$

is a (linear) bijection from $\mathcal{T}$ to $\mathcal{E}_{n-1}$.

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Application (**Diaconis** et al.): random doubly stochastic tridiagonal matrices and random walks on $\mathcal{T}_n$

A modification

Let $\mathcal{F}_n$ be the convex polytope in $\mathbb{R}^n$ defined by

$$\begin{aligned} x_i &\geq 0, \quad 1 \leq i \leq n \\ x_i + x_{i+1} + x_{i+2} &\leq 1, \quad 1 \leq i \leq n-2. \end{aligned}$$

$$V_n = \mathrm{vol}(\mathcal{F}_n)$$

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$$V_n = \mathrm{vol}(\mathcal{F}_n)$$

n	1–3	4	5	6	7	8	9	10
$n!V_n$	1	2	5	14	47	182	786	3774

A "naive" recurrence

$$V_n = \boldsymbol{f_n}(1,1),$$

where

$$f_0(a,b) = 1, \quad f_n(0,b) = 0 \text{ for } n > 0$$

$$\frac{\partial}{\partial a} f_n(a,b) = f_{n-1}(b-a, 1-a).$$

$f_n(a,b)$ for $n \le 3$

$$\begin{aligned} f_1(a,b) &= a \\ f_2(a,b) &= \frac{1}{2}(2ab - a^2) \\ f_3(a,b) &= \frac{1}{6}(a^3 - 3a^2 - 3ab^2 + 6ab) \end{aligned}$$

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\begin{aligned}
f_1(a, b) &= a \\
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\end{aligned}
$$

Is there a “nice” generating function for $f_n(a, b)$ or $V_n = f_n(1, 1)$?

Distribution of $\mathrm{is}(w)$

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$$\mathrm{is}(48361572) = 3$$

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$$\mathrm{is}(\mathbf{4}83\mathbf{6}15\mathbf{7}2) = 3$$

Distribution of is(w)

is(w) $=$ length of longest increasing subsequence of $w \in \mathfrak{S}_n$

$$\mathrm{is}(\mathbf{4}83\mathbf{6}15\mathbf{7}2) = 3$$

Vershik-Kerov, Logan-Shepp:

$$\begin{aligned} E(n) &:= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{is}(w) \\ &\sim 2\sqrt{n} \end{aligned}$$

Limiting distribution of is(w)

Baik-Deift-Johansson:

For fixed $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathrm{Prob}\left(\frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \leq t\right) = F(t),$$

the **Tracy-Widom distribution**.

Alternating analogues

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The first is much easier!

Longest alternating subsequences

$\mathbf{as}(\boldsymbol{w})$= length of longest alt. subseq. of w

$$w = \mathbf{5}6\mathbf{2}1\mathbf{8}\mathbf{3}\mathbf{4}7 \Rightarrow \mathrm{as}(w) = 5$$

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$$\boldsymbol{D(n)} = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\mathrm{as}(w) \sim \ ?$$

Definitions of $a_k(n)$ and $b_k(n)$

$$\begin{aligned} a_k(n) &= \#\{w \in \mathfrak{S}_n \,:\, \mathrm{as}(w) = k\} \\ b_k(n) &= a_1(n) + a_2(n) + \cdots + a_k(n) \\ &= \#\{w \in \mathfrak{S}_n \,:\, \mathrm{as}(w) \le k\} \end{aligned}$$

The case $n = 3$

w	$\mathrm{as}(w)$
$\mathbf{1}23$	1
$1\mathbf{32}$	2
$\mathbf{213}$	3
$\mathbf{2}3\mathbf{1}$	2
$\mathbf{312}$	3
$\mathbf{3}2\mathbf{1}$	2

The case $n = 3$

w	$\text{as}(w)$
123	1
132	2
213	3
231	2
312	3
321	2

$$a_1(3) = 1,\ a_2(3) = 3,\ a_3(3) = 2$$

$$b_1(3) = 1,\ b_2(3) = 4,\ b_3(3) = 6$$

The main lemma

Lemma. $\forall\, w \in \mathfrak{S}_n\ \exists$ *alternating subsequence of maximal length that contains* n.

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Corollary.

$$\Rightarrow a_k(n) = \sum_{j=1}^{n} \binom{n-1}{j-1}$$

$$\sum_{2r+s=k-1} \left(a_{2r}(j-1) + a_{2r+1}(j-1)\right) a_s(n-j)$$

The main generating function

$$B(x,t) = \sum_{k,n\geq 0} b_k(n) t^k \frac{x^n}{n!}$$

Theorem.

$$B(x,t) = \frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho},$$

where $\rho = \sqrt{1-t^2}$.

Formulas for $b_k(n)$

Corollary.

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no such formulas for longest **increasing** subsequences

Mean (expectation) of $\mathrm{as}(w)$

$$\boldsymbol{D(n)} = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{as}(w) = \frac{1}{n!} \sum_{k=1}^{n} k \cdot a_k(n),$$

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Let

$$\begin{aligned} A(x,t) &= \sum_{k,n\geq 0} a_k(n)t^k\frac{x^n}{n!} = (1-t)B(x,t) \\ &= (1-t)\left(\frac{2/\rho}{1-\frac{1-\rho}{t}e^{\rho x}}-\frac{1}{\rho}\right). \end{aligned}$$

Formula for $D(n)$

$$\begin{aligned}
\sum_{n\geq 0} D(n)x^n &= \frac{\partial}{\partial t}A(x,1) \\
&= \frac{6x-3x^2+x^3}{6(1-x)^2} \\
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Compare $E(n) \sim \mathbf{2\sqrt{n}}$.

Variance of $\mathrm{as}(w)$

$$\boldsymbol{V(n)} = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \left(\mathrm{as}(w) - \frac{4n+1}{6} \right)^2, \quad n \geq 2$$

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similar results for higher moments

A new distribution?

$$\boldsymbol{P(t)} = \lim_{n\to\infty} \operatorname{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\operatorname{as}(w) - 2n/3}{\sqrt{n}} \le t\right)$$

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Stanley distribution?

Limiting distribution

Theorem (Pemantle, Widom, (Wilf)).

$$\lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}(w)-2n/3}{\sqrt{n}} \le t\right)$$

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Umbral enumeration

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Example.

$$\begin{aligned}(1+E^2)^3 &= 1+3E^2+3E^4+E^6\\ &= 1+3E_2+3E_4+E_6\\ &= 1+3\cdot 1+3\cdot 5+61\\ &= 80\end{aligned}$$

Another example

$$
\begin{aligned}
(1+t)^E &= 1 + Et + \binom{E}{2} t^2 + \binom{E}{3} t^3 + \cdots \\
&= 1 + Et + \frac{1}{2} E(E-1) t^2 + \cdots \\
&= 1 + E_1 t + \frac{1}{2}(E_2 - E_1)) t^2 + \cdots \\
&= 1 + t + \frac{1}{2}(1-1) t^2 + \cdots \\
&= 1 + t + O(t^3).
\end{aligned}
$$

An umbral quiz

Let B be the Bell number umbra. Then

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Alt. fixed-point free involutions

fixed point free involution $w \in \mathfrak{S}_{2n}$: all cycles of length two (number $= 1 \cdot 3 \cdot 5 \cdots (2n-1)$)

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$$\begin{aligned} n = 3: \quad 214365 &= (1,2)(3,4)(5,6) \\ 645231 &= (1,6)(2,4)(3,5) \end{aligned}$$

$$f(3) = 2$$

An umbral theorem

Theorem.

$$F(x) = \sum_{n \geq 0} f(n) x^n$$

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$$= \left(\frac{1+x}{1-x}\right)^{(E^2+1)/4}$$

Proof idea

Proof. Uses representation theory of the symmetric group $\mathfrak{S}_n$.

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Now use known results on combinatorial properties of characters of $\mathfrak{S}_n$.

Ramanujan’s Second Notebook

Theorem (Ramanujan, Berndt, implicitly) *As* $x \to 0+$,

$$2\sum_{n\geq 0}(-1)^n\left(\frac{1-x}{1+x}\right)^{n(n+1)} \sim \sum_{k\geq 0} f(k)x^k = F(x),$$

an **analytic** (non-formal) identity.

A formal identity

Corollary (via Ramanujan, Andrews).

$$F(x) = 2\sum_{n\geq 0} q^n \frac{\prod_{j=1}^{n}(1-q^{2j-1})}{\prod_{j=1}^{2n+1}(1+q^j)},$$

where $q = \left(\frac{1-x}{1+x}\right)^{2/3}$*, a* ***formal*** *identity.*

Simple result, hard proof

Recall: number of n-cycles in $\mathfrak{S}_n$ is $(n-1)!$.

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Theorem. *Let* $\boldsymbol{b(n)}$ *be the number of* ***alternating*** n*-cycles in* $\mathfrak{S}_n$*. Then if* n *is odd,*

$$b(n) = \frac{1}{n} \sum_{d|n} \mu(d)(-1)^{(d-1)/2} E_{n/d}.$$

Special case

Corollary. *Let* p *be an odd prime. Then*

$$b(p) = \frac{1}{p}\left(E_p - (-1)^{(p-1)/2}\right).$$

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Combinatorial proof?

Inc. subsequences of alt. perms.

Recall: $\mathbf{is}(\boldsymbol{w})$ $=$ length of longest increasing subsequence of $w \in \mathfrak{S}_n$. Define

$$\boldsymbol{C(n)} = \frac{1}{E_n} \sum_{w} \mathrm{is}(w),$$

where w ranges over all E_n alternating permutations in $\mathfrak{S}_n$.

β

Crude estimate: what is

$$\beta = \lim_{n\to\infty} \frac{\log C(n)}{\log n}?$$

I.e., $C(n) = n^{\beta + o(1)}$.

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Easy: $\frac{1}{2} \leq \beta \leq 1$

Limiting distribution?

What is the (suitably scaled) limiting distribution of $\mathrm{is}(w)$, where w ranges over all alternating permutations in $\mathfrak{S}_n$?

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$\cdots\cdots$


Darn!
That's
the
end...
H