Hook Lengths and Contents

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M.I.T.

Standard Young tableau

standard Young tableau (SYT) of shape $\lambda = (4, 4, 3, 1)$:

$<$

$\wedge$

1	3	4	8
2	6	9	11
5	7	12	
10			

f_λ: number of SYT of shape λ

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$f_{3,2} = 5$:

$$\begin{matrix} 1\,2\,3 \\ 4\,5 \end{matrix} \qquad \begin{matrix} 1\,2\,4 \\ 3\,5 \end{matrix} \qquad \begin{matrix} 1\,2\,5 \\ 3\,4 \end{matrix}$$

$$\begin{matrix} 1\,3\,4 \\ 2\,5 \end{matrix} \qquad \begin{matrix} 1\,3\,5 \\ 2\,4 \end{matrix}$$

Hook length formula

For $u \in \lambda$, let h_u be the **hook length** at u, i.e., the number of squares directly below or to the right of u (counting u once)

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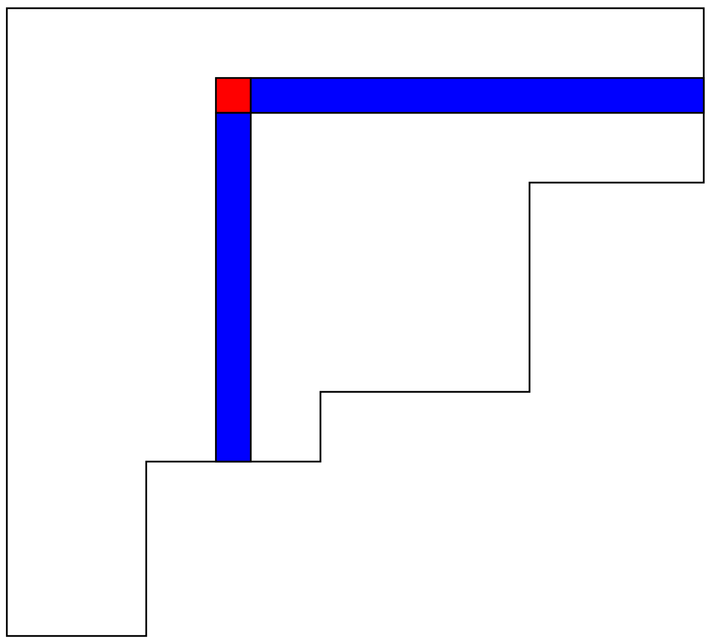

7	5	4	2
6	4	3	1
4	2	1	
1			

Theorem (Frame-Robinson-Thrall). *Let $\lambda \vdash n$. Then*

$$f_\lambda = \frac{n!}{\prod_{u \in \lambda} h_u}.$$

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$$f^{(4,4,3,1)} = \frac{12!}{7\cdot 6\cdot 5\cdot 4^3\cdot 3\cdot 2^2\cdot 1^3}$$

$$= 2970$$

Nekrasov-Okounkov identity

RSK algorithm (or representation theory) $\Rightarrow$

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!$$

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Nekrasov-Okounkov (2006), **G. Han** (2008):

$$\sum_{n \geq 0} \left(\sum_{\lambda \vdash n} f_\lambda^2 \prod_{u \in \lambda} (t + h_u^2) \right) \frac{x^n}{n!^2}$$

$$= \prod_{i \geq 1} (1 - x^i)^{-t-1}$$

A corollary

e_k : kth elementary SF

$$e_k(x_1, x_2, \dots) = \sum_{1 \le i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

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Corollary. *Let*

$$g_k(n) = \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 e_k(h_u^2 \,:\, u \in \lambda).$$

Then $g_k(n) \in \mathbb{Q}[n]$.

Conjecture of 韩国牛

Conjecture (Han). Let $j \in \mathbb{P}$. Then

$$\frac{1}{n!}\sum_{\lambda \vdash n} f_\lambda^2 \sum_{u \in \lambda} h_u^{2j} \in \mathbb{Q}[n].$$

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Stronger conjecture. For any symmetric function F,

$$\frac{1}{n!}\sum_{\lambda \vdash n} f_\lambda^2 F(h_u^2 : u \in \lambda) \in \mathbb{Q}[n].$$

Examples

Let $d_k(n) = \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \sum_{u \in \lambda} h_u^{2k}$.

$$\begin{aligned} d_1(n) &= \tfrac{1}{2} n(3n-1) \\ d_2(n) &= \tfrac{1}{24} n(n-1)(27n^2 - 67n + 74) \\ d_3(n) &= \tfrac{1}{48} n(n-1)(n-2) \\ &\quad (27n^3 - 174n^2 + 511n - 552). \end{aligned}$$

Open variants

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda} h_u \quad = \quad ?$$

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda \sum_{u\in\lambda} h_u \quad = \quad ?$$

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda \sum_{u\in\lambda} h_u^2 \quad = \quad ?$$

Contents

For $u \in \lambda$ let $c(u) = i - j$, the content of square $u = (i, j)$.

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0	1	2	3
–1	0	1	2
–2	–1	0	
–3			

Semistandard tableaux

semistandard Young tableau (SSYT) of shape $\lambda = (4,4,3,1)$:

1	2	2	4
2	3	6	6
4	4	7	
7			

$$x^T = x_1\, x_2^3\, x_4^3\, x_5\, x_6^2\, x_7^2$$

Schur functions

$$s_\lambda = \sum_T x^T,$$

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$$s_3 = \sum_{i \le j \le k} x_i x_j x_k$$

$$s_{1,1,1} = \sum_{i<j<k} x_i x_j x_k = e_3.$$

Hook-content formula

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Theorem. $s_\lambda(1^t) = \prod_{u\in\lambda} \frac{t+c_u}{h_u}.$

A curiosity

Let $\kappa(w)$ denote the number of cycles of $w \in \mathfrak{S}_n$. Then

$$\frac{1}{n!} \sum_{u,v \in \mathfrak{S}_n} q^{\kappa(uvu^{-1}v^{-1})} = \sum_{\lambda \vdash n} \prod_{u \in \lambda} (q + c_u).$$

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Restatement:

$$n! \sum_{\lambda \vdash n} e_k(c_u \,:\, u \in \lambda)$$

$$= \#\{(u,v) \in \mathfrak{S}_n \times \mathfrak{S}_n \,:\, \kappa(uvu^{-1}v^{-1}) = n-k\}$$

Another curiosity

$$\sum_{w \in \mathfrak{S}_n} q^{\kappa(w^2)} = \sum_{\lambda \vdash n} f_\lambda \prod_{u \in \lambda} (q + c_u)$$

Han's conjecture for contents

Theorem. *For any symmetric function F,*

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2\, F(c_u\,:\,u\in\lambda)\in\mathbb{Q}[n].$$

Idea of proof. By linearity, suffices to take $F = e_\mu$.

Power sum symmetric functions

$$\begin{aligned} \boldsymbol{x} &= (x_1, x_2, \ldots) \\ \boldsymbol{p_m} = p_m(x) &= \sum_i x_i^m \\ \boldsymbol{p_\lambda} &= p_{\lambda_1} p_{\lambda_2} \cdots \end{aligned}$$

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Let $\boldsymbol{\ell(\lambda)}$ be the length (number of parts) of λ

$$\begin{aligned} p_m(1^t) &= t \\ p_\lambda(1^t) &= t^{\ell(\lambda)} \end{aligned}$$

Some notation

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$\boldsymbol{\rho(w)}$: **cycle type of $w \in \mathfrak{S}_n$, e.g.,**

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$$H_\lambda = \prod_{u \in \lambda} h_u$$

The fundamental tool

Theorem. $$\sum_{\lambda \vdash n} H_\lambda^{k-2} s_\lambda(x^{(1)}) \cdots s_\lambda(x^{(k)})$$

$$= \frac{1}{n!} \sum_{\substack{w_1 \cdots w_k = 1 \\ \text{in } \mathfrak{S}_n}} p_{\rho(w_1)}(x^{(1)}) \cdots p_{\rho(w_k)}(x^{(k)})$$

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Proof. Follows from representation theory: the relation between multiplying conjugacy class sums in $Z(\mathbb{C}G)$ (the center of the group algebra of the finite group G) and the irreducible characters of G.

Set $x^{(i)} = 1^{t_i}$**, so** $s_\lambda(x^{(i)}) \to \prod \frac{t_i + c_u}{h_u}$

$$p_{\rho(w_i)}(x^{(i)}) \to t_i^{\kappa(w_i)}.$$

Get

$$\sum_{\lambda \vdash n} H_\lambda^{-2} \prod_{u \in \lambda} (t_1 + c_u) \cdots (t_k + c_u) =$$

$$\frac{1}{n!} \sum_{\substack{w_1 \cdots w_k = 1 \\ \text{in } \mathfrak{S}_n}} t_1^{\kappa(w_1)} \cdots t_k^{\kappa(w_k)}.$$

Completion of proof

Take coefficient of $t_1^{n-\mu_1}\cdots t_k^{n-\mu_k}$:

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 e_\mu(c_u \,:\, u\in\lambda)$$

$$= \#\{(w_1,\ldots,w_k)\in\mathfrak{S}_n^k \,:\, w_1\cdots w_k = 1,$$

$$c(w_i) = n-\mu_i\}.$$

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$$\frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \, e_\mu(c_u \;:\; u \in \lambda)$$

$$= \#\{(w_1, \ldots, w_k) \in \mathfrak{S}_n^k \;:\; w_1 \cdots w_k = 1, \\ c(w_i) = n - \mu_i\}.$$

Elementary combinatorial argument shows this is a polynomial in n. $\Box$

Shifted parts

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \vdash n$.

partitions of 3: 300, 210, 111

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shifted parts of $(3, 1, 1, 0, 0)$: 7, 4, 3, 1, 0

Contents and shifted parts

Let $F(x; y)$ be symmetric in x and y variables **separately**, e.g.,

$$p_1(x)p_2(y) = (x_1 + x_2 + \cdots)(y_1^2 + y_2^2 + \cdots).$$

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$$p_1(x)p_2(y) = (x_1 + x_2 + \cdots)(y_1^2 + y_2^2 + \cdots).$$

Theorem. Let

$$r(n) = \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 F(c_u, u \in \lambda; \lambda_i + n - i, 1 \le i \le n$$

Then $r(n) \in \mathbb{Z}$ (n fixed), and $r(n) \in \mathbb{Q}[n]$.

Similar results are false

Note. Let

$$P(n) = \frac{1}{n!}\sum_{\lambda \vdash n} f_\lambda^2 \left(\sum_i \lambda_i^2\right).$$

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Then

$$P(3) = \frac{16}{3} \notin \mathbb{Z}$$

$$P(n) \notin \mathbb{Q}[n]$$

$$\Lambda_{\mathbb{Q}} = \{\text{symmetric functions over } \mathbb{Q}\}$$

Define a linear transformation

$$\varphi : \Lambda_{\mathbb{Q}} \longrightarrow \mathbb{Q}[t]$$

by

$$\varphi(s_\lambda) = \frac{\prod_{i=1}^{n}(t + \lambda_i + n - i)}{H_\lambda}.$$

Key lemma for shifted parts

Lemma. Let $\mu \vdash n$, $\ell = \ell(\mu)$. Then

$$\varphi(p_\mu) =$$

$$(-1)^{n-\ell} \sum_{i=0}^{m} \binom{m}{i} t(t+1)\cdots(t+i-1),$$

where $m = m_1(\mu)$, the number of parts of μ equal to 1.

Proof of main result

Theorem. For any symmetric function F,

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Proof based on the multiset identity

$$\{h_u : u \in \lambda\} \cup \{\lambda_i - \lambda_j - i + j : 1 \le i < j \le n\}$$

$$= \{n + c_u : u \in \lambda\} \cup \{1^{n-1}, 2^{n-2}, \ldots, n-1\}.$$

Example. $\lambda = (3, 1, 0, 0)$,

$\lambda - (1, 2, 3, 4) = (2, -1, -3, -4)$:

$$\{4, 2, 1, 1\} \cup \{3, 5, 6, 2, 3, 1\}$$

$$= \{3, 4, 5, 6\} \cup \{1, 1, 1, 2, 2, 3\}$$

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$$= \{n + c_u : u \in \lambda\} \cup \{1^{n-1}, 2^{n-2}, \ldots, n-1\}.$$

Note.

$$\lambda_i - \lambda_j - i + j = (\lambda_i + n - i) - (\lambda_j + n - j)$$

Not symmetric in $\lambda_i + n - i$ and $\lambda_j + n - j$, but $(\lambda_i - \lambda_j - i + j)^2$ is symmetric.

Further results

Recall:

Nekrasov-Okounkov (2006), G. Han (2008):

$$\sum_{n\geq 0}\left(\sum_{\lambda\vdash n} f_\lambda^2 \prod_{u\in\lambda}(t+h_u^2)\right)\frac{x^n}{n!^2}$$

$$=\prod_{i\geq 1}(1-x^i)^{-t-1}$$

Content analogue

"Content Nekrasov-Okounkov identity"

$$\sum_{n\geq 0}\left(\sum_{\lambda\vdash n} f_\lambda^2 \prod_{u\in\lambda}(t+c_u^2)\right)\frac{x^n}{n!^2}$$

$$=(1-x)^{-t}.$$

Proof: simple consequence of "dual Cauchy identity" and the hook-content formula.

Fujii et al. variant

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda}\prod_{i=0}^{k-1}(c_u^2 - i^2)$$

$$= \frac{(2k)!}{(k+1)!^2}(n)_{k+1}.$$

where $(n)_{k+1} = n(n-1)\cdots(n-k)$.

Okada's conjecture

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda}\prod_{i=1}^{k}(h_u^2 - i^2)$$

$$= \frac{1}{2(k+1)^2}\binom{2k}{k}\binom{2k+2}{k+1}(n)_{k+1}$$

Proof of Okada's conjecture

Proof (Greta Panova).

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- **Both sides are polynomials in n.**
- **The two sides agree for $1 \leq n \leq k+2$.**
- **Key step: both sides have degree $k+1$.**

Central factorial numbers

$$x^n = \sum_k T(n,k)\, x(x-1^2)(x-2^2)\cdots(x-k^2)$$

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$n \backslash k$	1	2	3	4	5
1	1				
2	1	1			
3	1	5	1		
4	1	21	14	1	
5	1	85	147	30	1

Okada-Panova redux

Corollary.

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda} h_u^{2k}$$

$$= \sum_{j=1}^{k+1} T(k+1,j)\frac{1}{2j^2}\binom{2(j-1)}{j-1}\binom{2j}{j}(n)_j.$$