A Fibonacci Array

Richard P. Stanley

April 19, 2024

The diagram $\mathfrak{D}$

Define a diagram $\mathfrak{D}$ as follows.

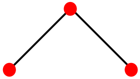
(P0) Single vertex (or point or node) T at the top.

(P1) Each vertex is connected by an edge to exactly two vertices in the row below.

(P2) The diagram is planar, i.e., edges cannot cross.

(P3) Given a vertex t and the two adjacent vertices u, v to t in the row below, complete this figure to a hexagon.

Step 1. Two vertices below T:



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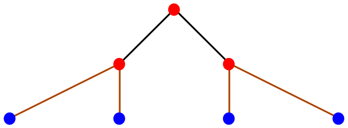
Step 2. Two vertices below each of these:



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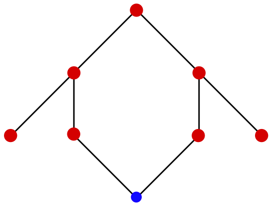
Step 3. Complete to a hexagon:



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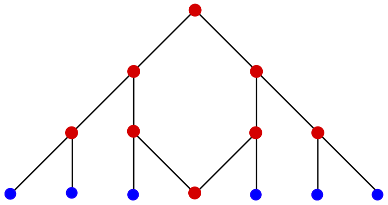
Step 4. Add remaining vertices on bottom row:



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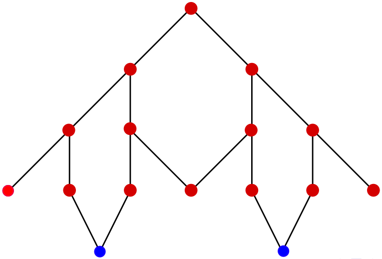
Step 5. Complete the two hexagons:



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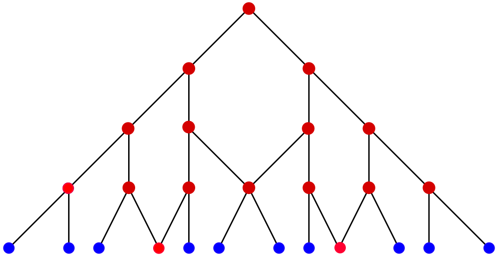
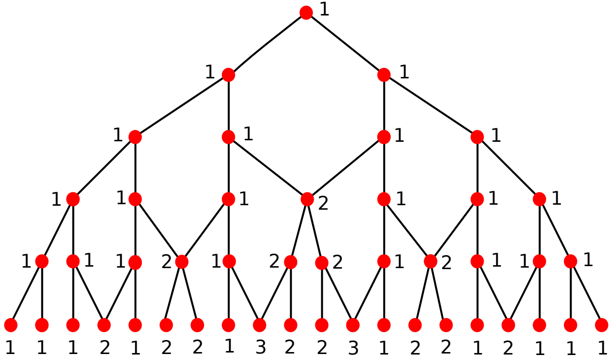
Step 6. Add remaining elements on bottom row:

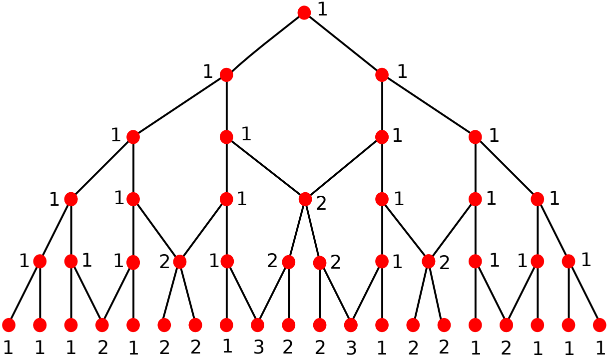


The Fibonacci array

Label each vertex with the number of chains from that vertex to T. Equivalently, T is labelled 1, and other vertices v are labelled by the sum of the vertex labels to which v is connected on the row above.



What are these numbers?

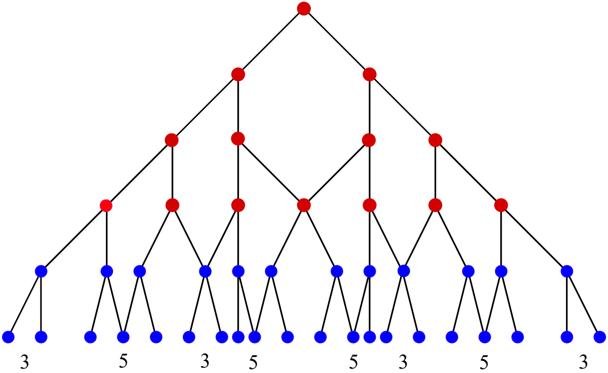


$$(1+x)(1+x^2)(1+x^3)(1+x^5)(1+x^8) =$$

$$1 + 1x + 1x^2 + 2x^3 + 1x^4 + 2x^5 + 2x^6 + 1x^7 + 3x^8 + 2x^9 + 2x^{10}$$

$$+3x^{11} + 1x^{12} + 2x^{13} + 2x^{14} + 1x^{15} + 2x^{16} + 1x^{17} + 1x^{18} + 1x^{19}$$

Two consecutive rows



What is this sequence $3, 5, 3, 5, 5, 3, 5, 3$?

Enter the golden mean

$$\phi = \frac{1+\sqrt{5}}{2} = 1.61803398\cdots, \text{ the golden mean}$$

Recall for rows 4-5 we got 3,5,3,5,5,3,5,3. Sequence is **symmetric** (or **palindromic**), so we need only describe first four terms c_1, c_2, c_3, c_4.

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$$\begin{aligned}
c_1 &= 1 + 2\lfloor \phi \rfloor &&= 3\\
c_2 &= 1 + 2\lfloor 2\phi \rfloor - 2\lfloor \phi \rfloor &&= 5\\
c_3 &= 1 + 2\lfloor 3\phi \rfloor - 2\lfloor 2\phi \rfloor &&= 3\\
c_4 &= 1 + 2\lfloor 4\phi \rfloor - 2\lfloor 3\phi \rfloor &&= 5\\
&\vdots\\
c_n &= 1 + 2\lfloor n\phi \rfloor - 2\lfloor (n-1)\phi \rfloor
\end{aligned}$$