PROBLEMS AND CONJECTURES ON F-VECTORS

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$\mathbf{\Gamma}$ = "geometric object" made out of **faces** of well-defined dimensions.

E.g., simplicial complex, polyhedral complex, regular cell complex, ...

$$\mathbf{f_i} = \#(i\text{-dimensional faces})$$

$$\mathbf{d} = 1+\dim\Gamma = 1+\max\{\dim F \ : \ F \text{ is a face}\}$$

$$\mathbf{f(\Gamma)} = (f_0, \ldots, f_{d-1}),$$

the ***f*-vector** of Γ.

$$(f_{-1} = 1 \text{ unless } \Gamma = \emptyset)$$

Define the ***h*-vector** $\mathbf{h(\Gamma)} = (h_0, \ldots, h_d)$ by

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}.$$

Let $X = |\mathbf{\Gamma}|$ denote the **geometric realization** of Γ. Γ is **Cohen-Macaulay** (**C-M**) over K if $\forall p \in X$ and $i < \dim \Gamma = d - 1$,

$$\tilde{H}_i(X; K) = H_i(X, X - p; K) = 0.$$

Γ is **Gorenstein*** if in addition

$$\tilde{H}_{d-1}(X; K) = \tilde{H}_{d-1}(X, X - p; K) = K.$$

Dehn-Sommerville equations: *If Γ is a Gorenstein* simplicial complex, then $h_i = h_{d-i}$.*

Theorem (GLBT for simplicial polytopes). *If Γ is the boundary complex of a simplicial d-polytope (and hence Gorenstein*), then*

$$h_0 \le h_1 \le \cdots \le h_{\lfloor d/2 \rfloor}.$$

Problem 1. Does the GLBT hold for Gorenstein* simplicial complexes (or even complete simplicial fans, PL-spheres, or spheres)?

Known for the boundary of a $(d-1)$-ball that is a subcomplex of the boundary complex of a simplicial d-polytope.

Centrally-symmetric polytopes: abysmal ignorance (except lower bound theorems).

Problem 2. (a) What is the maximum number of facets (or i-dimensional faces) of a centrally-symmetric (simplicial) d-polytope with n vertices? Or of a Gorenstein* simplicial complex of dimension $d - 1$ and with n vertices and a free involution? Answers are **different**!

(b) **(Kalai)** Does every centrally-symmetric d-polytope (or Gorenstein* complex as above) have at least 3^d nonempty faces?

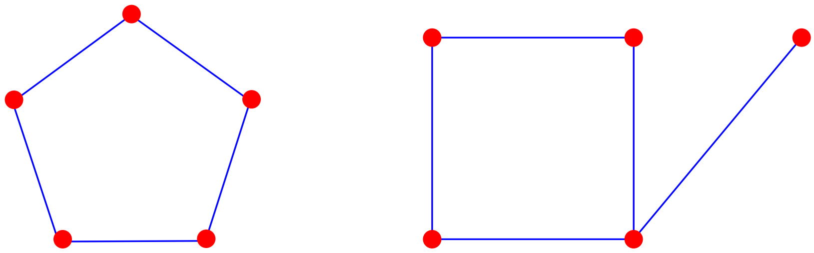
Flag complex: a simplicial complex for which every minimal nonface (or **missing face**) has two elements.

E.g., the order complex (set of chains) of a poset.

Same as clique complexes or independent set complexes of graphs.

Balanced complex: a $(d-1)$ dimensional simplicial complex whose vertices can be colored with d colors so that no edge is monochromatic. E.g., order complexes.

Problem 3. (a) **(Kalai)** Is the f-vector of a flag complex also the f-vector of a balanced complex?



(b) **(Charney-Davis)** Let Γ be a $(2e-1)$-dimensional Gorenstein* flag complex with $h(\Gamma) = (h_0, \dots, h_{2e})$. Is it true that

$$(-1)^e (h_0 - h_1 + h_2 - \cdots + h_{2e}) \geq 0?$$

True for face posets of convex polytopes.

Matroid complex: a simplicial complex Γ on the vertex set V such that the restriction of Γ to any subset of V is pure (i.e., all maximal faces have same dimension).

Example. The linearly independent subsets of a subset of a vector space.

Multicomplex: a collection Λ of multisets such that $F \in \Lambda$ and $G \subset F \Rightarrow G \in \Lambda$. Λ is **pure** if all maximal elements have the same cardinality. Define $f(\Lambda)$ just like $f(\Gamma)$.

Problem 4. Is the h-vector of a matroid complex the f-vector of a **pure** multicomplex?

Note: A matroid complex is C-M (even shellable), and the h-vector of **any** C-M complex is the f-vector of a multicomplex.

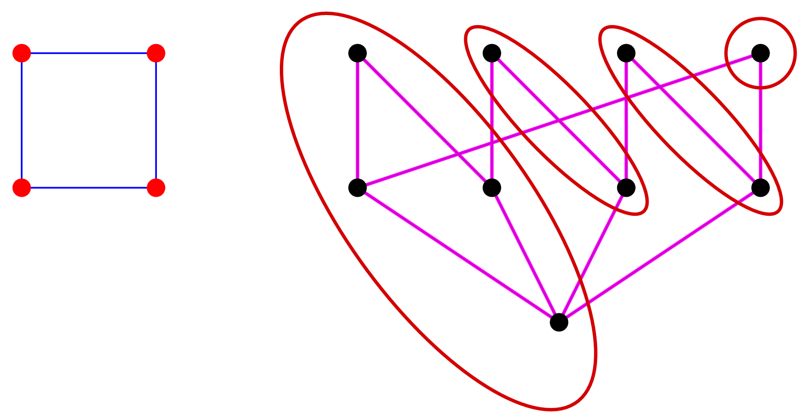
Example. $(1,3,1)$ is the f-vector of the multicomplex $\{\emptyset,1,2,3,11\}$ or $\{\emptyset,1,2,3,12\}$ but is not the f-vector of a **pure** multicomplex.

Partitioning. An **interval** of a simplicial complex Γ is a set

$$[F, F'] = \{G : F \subseteq G \subseteq F'\} \subseteq \Gamma$$

(so $F \subseteq F' \in \Gamma$).

An **interval partition** Π of Γ is a collection of nonempty pairwise disjoint intervals whose union is Γ.

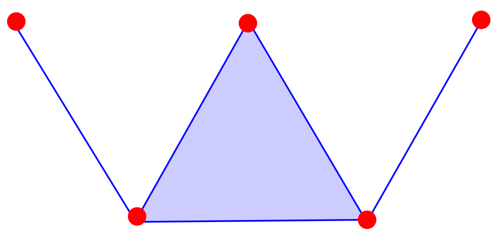


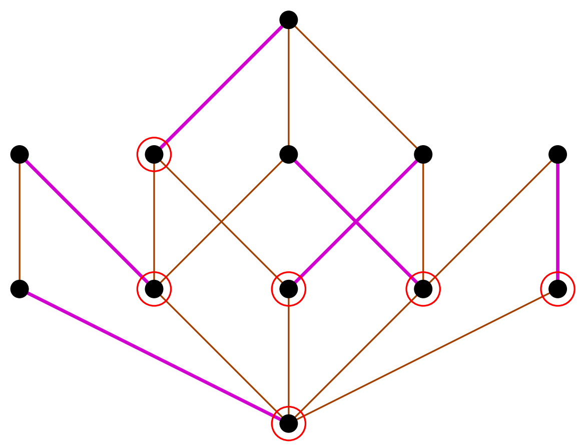
Theorem. *If* Γ *is* **acyclic** *(vanishing reduced homology), then there is a partition* Π *of* Γ *into two-element intervals* $[F, F']$ *such that*

$$\Delta := \{F \; : \; [F, F'] \in \Pi\}$$

is a subcomplex of Γ*.*

Corollary. $f(\Gamma) = f(C(\Delta))$, *where* $C(\Delta)$ *is the cone over* Δ*.*





Duval: generalization that gives a "partitioning proof" of the result of Björner-Kalai characterizing the f-vector of simplicial complexes with prescribed Betti numbers.

Problem 5. (a) Suppose that Γ is acyclic, as well as $\mathrm{lk}(v)$ for every vertex v of Γ. Can Γ be partitioned into **four**-element intervals such that

$$\Delta := \{F \,:\, [F, F'] \in \Pi\}$$

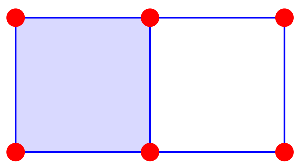
is a subcomplex of Γ?

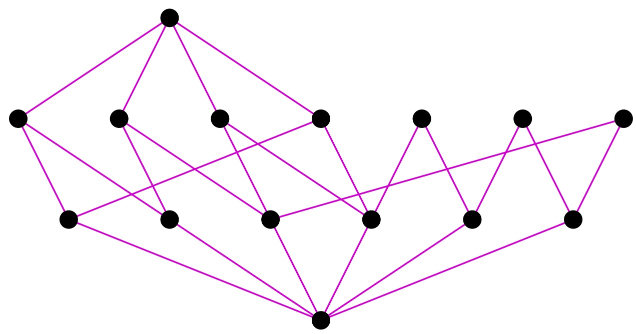
(b) Suppose that Γ is C-M. Can Γ be partitioned into intervals $[F, F']$ such that each F' is a facet (maximal face) of Γ?

Note: If Γ has a partition Π as in (b), then

$$\sum_i h_i x^i = \sum_{[F,F']\in\Pi} x^{\#F}.$$

Cubical complexes. Regard a (finite, abstract) cubical complex as a finite meet-semilattice such that every interval $[\hat{0}, t]$ is isomorphic to the face-lattice of a cube (whose dimension depends on t).





Let L be a **pure** cubical complex of rank d (or dimension $d-1$). Let s be a vertex (element covering $\hat{0}$). The subposet $\{t \in L : t \geq s\}$ is the face poset of a simplicial complex $\Gamma_s = \mathrm{lk}(s)$. Let

$$h(\Gamma_s, x) = \sum_{i=0}^{d-1} h_i(\Gamma_s) x^i.$$

Define $h_i(L)$ by the equation

$$\sum_{i=0}^{d} h_i(L)x^i = \frac{1}{1+x}\Big(2^{d-1}$$

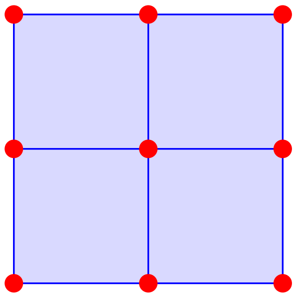
$$+x\sum_{s} h(\Gamma_s, x) + (-2)^{d-1}\,\widetilde{\chi}(L)x^{d+1}\Big),$$

where s ranges over all vertices of L and

$$\widetilde{\chi}(L) = \sum_{t\in L}(-1)^{\mathrm{rank}(t)-1},$$

the reduced Euler characteristic of L.

Easy: Right-hand side is a polynomial in x.



$$\sum_{i=0}^{3} h_i(L)x^i =$$

$$\frac{1}{1+x}\Big[2^2 + x(4 + 4(1+x)$$

$$+(1 + 2x + x^2)) + (-2)^2 \cdot 0 \cdot x^4\Big]$$

$$= \ 4 + 5x + x^2$$

$$\Rightarrow \ h(L) = (4, 5, 1, 0).$$

Problem 6. (a) Let L be a pure cubical complex of rank d. If L is a C-M poset (i.e., the order complex of L is C-M), then is $h_i(L) \geq 0$ for all i? (True if L is shellable.)

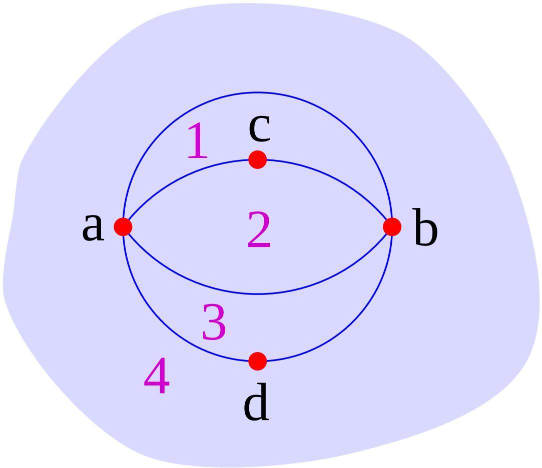
(b) **(Adin)** If L is in addition a Gorenstein* poset, then is it true that

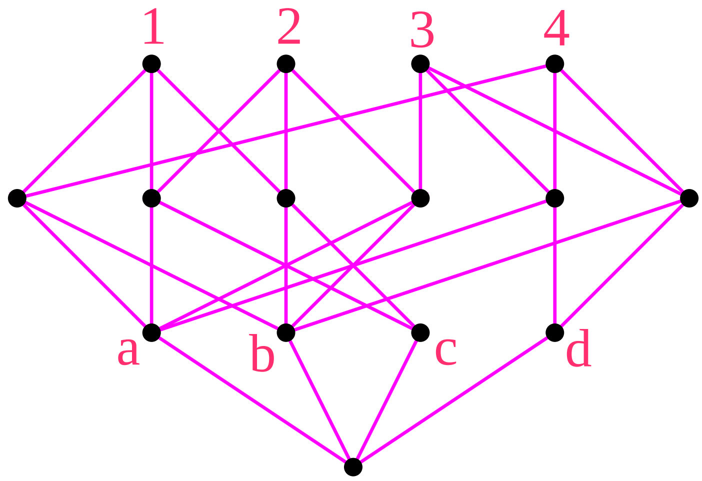
$$h_0(L) \leq h_1(L) \leq \cdots \leq h_{\lfloor d/2 \rfloor}(L)?$$

(**Adin**: $h_i = h_{d-i}$.)

Simplicial posets. Intuitively, a simplicial complex for which two faces can intersect in **any** subcomplex of both, not just a face.

Rigorously: a (finite) poset with $\hat{0}$ for which every interval is a boolean algebra.





$|P| \approx \mathbb{S}^2 \Rightarrow P - \hat{0}$ is **Gorenstein***

$$
\begin{aligned}
f(P) &= (4,6,4) \\
\sum_{i=0}^{3} h_i x^{3-i} &= (x-1)^3 + 4(x-1)^2 \\
&\quad +6(x-1)+4 \\
&= x^3 + x^2 + x + 1 \\
&\Rightarrow h(P) = (1,1,1,1).
\end{aligned}
$$

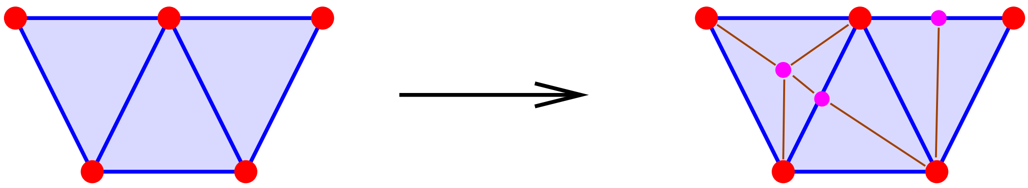
Theorem. (a) *If P is a C-M simplicial poset, then $h_0 = 1$ and $h_i \geq 0$ (complete characterization).*

(b) *If P is a Gorenstein* simplicial poset, then $h_0 = 1$, $h_i \geq 0$, and $h_i = h_{d-i}$ (complete characterization if f_{d-1} is* **even**, *e.g., d odd).*

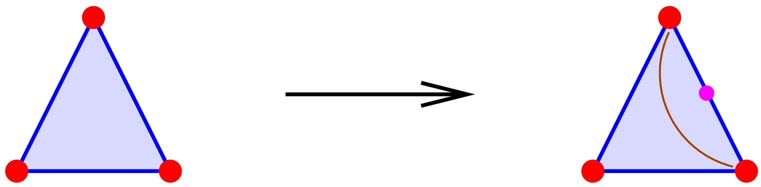
Problem 7. Characterize the h-vector of Gorenstein* simplicial posets when f_{d-1} is **odd**.

E.g, $(1, 0, 1, 0, 1)$ is **not** the h-vector of a Gorenstein* simplicial poset.

Subdivisions.



geometric
subdivision



topological
subdivision

Δ' is a **quasigeometric subdivision** of Δ if the vertices of an i-dimensional face of Δ' do not all lie on a face of Δ of smaller dimension.

geometric $\Rightarrow$ quasigeometric $\Rightarrow$ topological

Theorem. *If Δ' is a* **quasigeometric** *subdivision of the C-M (or even Buchsbaum) complex, then*

$$\forall i \;\; h_i(\Delta') \geq h_i(\Delta).$$

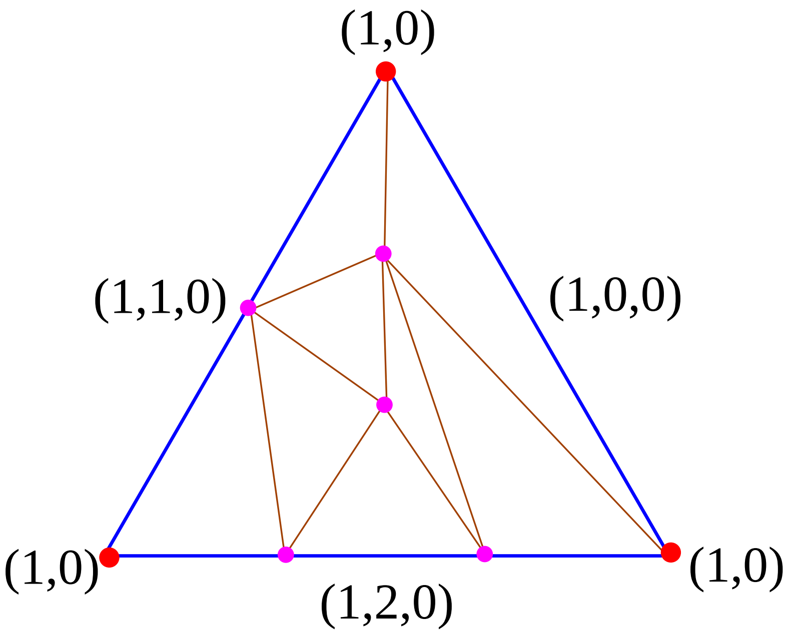
Problem 8. Does the above result hold for **topological** subdivisions?

Local h-vectors. Let $\#V = d$, and let Γ be a (simplicial) subdivision (geometric or quasigeometric) of the simplex 2^V. Let

$$\mathbf{\Gamma_W} = \text{restriction of } \Gamma \text{ to } W \subseteq V$$

$$\mathbf{h}(\mathbf{\Gamma_W}, \mathbf{x}) = \sum_i h_i(\Gamma_W) x^i.$$

$$\ell_{\mathbf{V}}(\mathbf{\Gamma}, \mathbf{x}) = \sum_{W \subseteq V} (-1)^{\#(V-W)} h(\Gamma_W, x).$$



h(Γ) = (1,5,2,0)

$$\begin{aligned}\ell_V(\Gamma, x) &= -1 + 3 \cdot 1 - [1 + (1 + x) + (1 + 2x)] \\ &\quad + (1 + 5x + 2x^2) \\ &= 2x + 2x^2.\end{aligned}$$

Problem 9. When does $\ell_V(\Gamma, x) = 0$?