Some interesting polytopes

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Nonzero coefficients

Joint with Tewodros Amdeberhan

Let $\boldsymbol{f} = \boldsymbol{f(x_1, \ldots, x_n)} \in \mathbb{Q}[x_1, \ldots, x_n]$.

Define $\boldsymbol{N(f)}$ to be the number of nonzero coefficients of f.

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Define $\boldsymbol{N(f)}$ to be the number of nonzero coefficients of f.

Example. $N(x^2 - 5xy + 2x^3y^4) = 3$.

A second example

Example. $N\left(\prod_{1\leq i<j\leq n}(x_i - x_j)\right) = ?$

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Proof.

$$\prod_{1\leq i<j\leq n}(x_i - x_j) = \sum_{w\in\mathfrak{S}_n} \operatorname{sgn}(w) x_1^{w(1)} x_2^{w(2)} \cdots x_n^{w(n)}$$

A prototypical example

$$N\left(\prod_{1\le i<j\le n}(x_i+x_j)\right)=?$$

A prototypical example

$$N\left(\prod_{1\leq i<j\leq n}(x_i+x_j)\right)=\boldsymbol{f(n)},$$

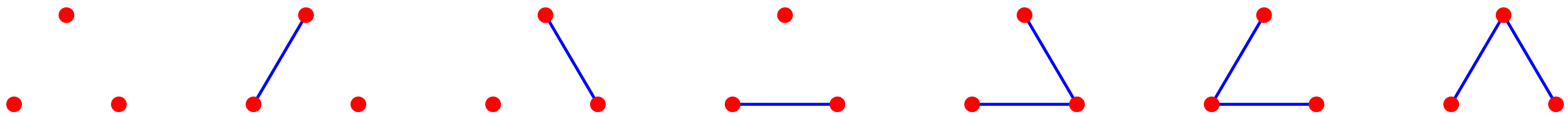
the number of **forests** on the vertex set $1,2,\ldots,n$.

An example

Example. $(x+y)(x+z)(y+z)$

$$= x^2y + xy^2 + x^z + xz^2 + y^2z + yz^2 + 2xyz,$$

so $f(3) = 7$.



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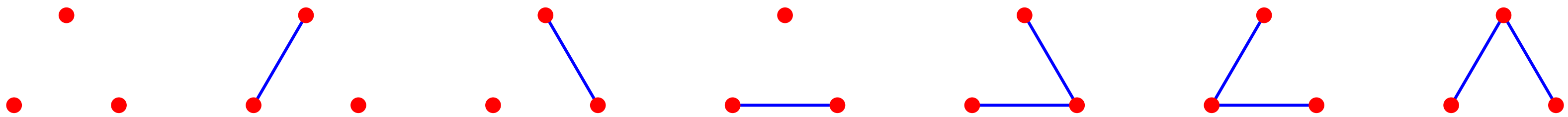

Note. $\sum_{n\geq 0} f(n)\frac{x^n}{n!} = \exp \sum_{j\geq 1} j^{j-2}\frac{x^j}{j!}$

Relation to polytopes

Recall: $Z(v_1, \ldots, v_k) = \{\sum \lambda_i v_i \,:\, 0 \le \lambda_i \le 1\}$, the **zonotope** generated by $v_1, \ldots, v_k$.

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$$\boldsymbol{P_n} := \prod_{1\le i<j\le n} (x_i+x_j) = x_2x_3^2\cdots x_n^{n-1} \prod_{1\le i<j\le n} (1+x_ix_j^{-1})$$

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Thus if x^α appears in P_n, then

$$\alpha-(0,1,\ldots,n-1) \in Z = Z(e_i-e_j \,:\, 1 \leq i < j \leq n).$$

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Thus if x^α appears in P_n, then

$$\alpha-(0,1,\ldots,n-1)\in Z=Z(e_i-e_j\,:\,1\le i<j\le n).$$

Lemma. *Every integer point in Z has this form.*

$\boldsymbol{N(P_n) = i(Z, 1)}$

Use zonotope techniques to determine $N(P_n)$.

Minkowski sums

For polytopes $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{R}^n$, define

$$\boldsymbol{\mathcal{P}} + \boldsymbol{\mathcal{Q}} = \{u + v \ : \ u \in \mathcal{P}, \ v \in \mathcal{Q}\}.$$

Let $\mathcal{P}$, $\mathcal{Q}$ be lattice polytopes in $\mathbb{R}^n$. Let

$$\begin{aligned} \boldsymbol{F(x)} &= \sum_{\alpha \in \mathcal{P} \cap \mathbb{Z}^n} c_\alpha x^\alpha, \ \ c_\alpha > 0 \\ \boldsymbol{G(x)} &= \sum_{\alpha \in \mathcal{Q} \cap \mathbb{Z}^n} d_\alpha x^\alpha, \ \ d_\alpha > 0. \end{aligned}$$

$F(x)G(x)$

For any polynomial $P(x) = \sum_\alpha b_\alpha x^\alpha$, let

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Clearly supp $F(x)G(x) \subseteq (\mathcal{P} + \mathcal{Q}) \cap \mathbb{Z}^n$.

When does equality hold? In this case

$$N(FG) = i(\mathcal{P} + \mathcal{Q}, 1).$$

Call the sum $\mathcal{P} + \mathcal{Q}$ **saturated**.

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A subtle question!

An cautionary example

Let $\boldsymbol{G}$ be a simple graph on $\{1, \ldots, n\}$ with edge set $\boldsymbol{E}$.

Let $\boldsymbol{F_G(x)} = \prod_{ij \in E} (1 + x_i x_j x_{n+1})$.

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Let $F_G(x) = \prod_{ij \in E} (1 + x_i x_j x_{n+1})$.

Note. The term x_{n+1} appears because we want to work in the lattice

$$L = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n :$$

$$a_1 + \cdots + a_n \equiv 0 \pmod 2\}.$$

Example (continued)

Theorem (Fulkerson-Hoffman-McAndrew, implicitly). Let $\boldsymbol{Z} = Z(e_i + e_j + e_{n+1} \,:\, ij \in E)$. The following two conditions are equivalent.

- $N(F_G) = i(Z, 1)$ (saturation)

Example (continued)

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- $N(F_G) = i(Z, 1)$ (saturation)
- Every induced subgraph of G has a most one connected component that is not bipartite.

The PS-polytope

For $t_i \geq 0$ define the **PS-polytope** $\boldsymbol{\Pi = \Pi(t_1, \ldots, t_n)} \subset \mathbb{R}^{n+1}$ by

$$\begin{aligned} x_i &\geq 0 \\ x_1 + \cdots + x_i &\leq t_1 + \cdots + t_i, \quad 1 \leq i \leq n \\ x_1 + x_2 + \cdots + x_{n+1} &= t_1 + t_2 + \cdots + t_n. \end{aligned}$$

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Equivalent to (and sometimes defined as) its projection

$$\begin{aligned} x_i &\geq 0 \\ x_1 + \cdots + x_i &\leq t_1 + \cdots + t_i, \quad 1 \leq i \leq n. \end{aligned}$$

Π as a Minkowski sum

$$\Pi = t_n \Delta_2 + t_{n-1} \Delta_3 + \cdots + t_1 \Delta_{n+1} \subset \mathbb{R}^{n+1}.$$

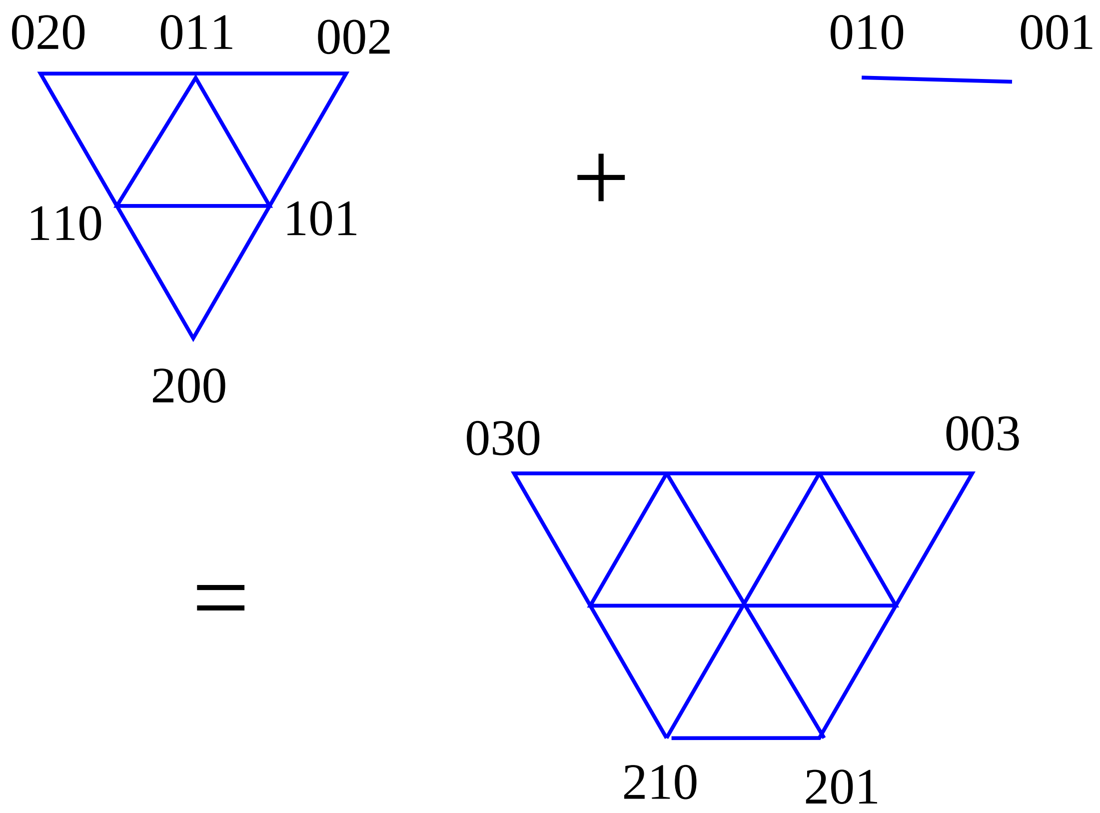
Here

$$\boldsymbol{\Delta_i} = \operatorname{conv}\{e_{n-i+1}, e_{n-i+2}, \ldots, e_n\},$$

where $\boldsymbol{e_j}$ is the jth standard unit vector in $\mathbb{R}^n$, so $\dim \Delta_i = i - 1$.

$\Pi(2,1)$

$$\Pi(2,1) = 2 \cdot \mathrm{conv}(e_1, e_2, e_3) + \mathrm{conv}(e_2, e_3)$$



Properties of the PS-polytope

- For $t_1, \ldots, t_n \in \mathbb{N}$, the sum $t_n\Delta_2 + \cdots + t_1\Delta_{n+1}$ is saturated, so

$$i(\Pi(t_1, \ldots, t_n), 1) = N\left(\prod_{j=1}^{n}(x_i + x_{i+1} + \cdots + x_{n+1})^{t_i}\right)$$

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Example. $t_1 = 2, t_2 = 1$:

$$N((x+y+z)^2(y+x)) =$$

$$N(\underbrace{x^2y + x^2z + 4xyz + \cdots + z^3}_{9 \text{ terms}}) = 9$$

Properties (continued)

- $i(\Pi(t_1,\ldots,t_n),m) =$
$$\sum_{\boldsymbol{k}} \left(\!\!\binom{mt_1+1}{k_1}\!\!\right) \prod_{i=2}^{n} \left(\!\!\binom{mt_i}{k_i}\!\!\right),$$ where

$$\begin{aligned}\boldsymbol{k} \;=\; & \{(i_1,\ldots,i_n)\in\mathbb{P}^n \,:\, i_1+\cdots+i_j \geq j, \\ & i_1+\cdots+i_n = n\}\end{aligned}$$

$$\left(\!\!\binom{\boldsymbol{k}}{\boldsymbol{j}}\!\!\right) \;=\; \binom{k+j-1}{j}.$$

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$$\#\boldsymbol{k} = \boldsymbol{C_n} = \frac{1}{n+1}\binom{2n}{n}.$$

An example

$$
\begin{aligned}
i(\Pi(a,b,c),m) = \left(\!\!\binom{ma+1}{3}\!\!\right) + \left(\!\!\binom{ma+1}{2}\!\!\right)\left(\!\!\binom{mb}{1}\!\!\right) \\
+ \left(\!\!\binom{ma+1}{2}\!\!\right)\left(\!\!\binom{mc}{1}\!\!\right) \\
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\end{aligned}
$$

Positivity

Corollary. *Let* $t_1, \ldots, t_n \in \mathbb{N}$*. Then coefficients of* $i(\Pi(t_1, \ldots, t_n), m)$ *are nonnegative.*

Generalized permutohedra

Let $\boldsymbol{t_I} \geq 0$ for each $\boldsymbol{I} \subseteq [n+1]$, and $\boldsymbol{t} = (t_I \,:\, I \subseteq [n+1])$. Let

$$\boldsymbol{\Delta_I} = \mathrm{conv}(e_i \,:\, i \in I).$$

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Definition (**A. Postnikov**). Define the **generalized permutohedron**

$$P_n(t) = \sum_{I \subseteq [n-1]} t_I \Delta_I$$

(Minkowski sum).

Examples of gen. permutohedra

- $t_I = a_{\#I}$ (i.e., t_I depends only on $\#I$):

$$\boldsymbol{P_n}(\boldsymbol{t}) = \mathrm{conv}\{(a_{w(1)}, \dots, a_{w(n)}) \,:\, w \in \mathfrak{S}_n\},$$

the **permutohedron**.

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$$t_I = \begin{cases} 1, & I = \{i, i+1, i+2, \ldots, j\} \\ 0, & \text{otherwise}, \end{cases}$$

  the **associahedron** (realization of **Loday**).

Volume of $P_n(t)$

Theorem (**Postnikov**). For any $\boldsymbol{t}$,

$$\operatorname{vol} P_n(\boldsymbol{t}) = \frac{1}{n!} \sum_{(S_1,\ldots,S_n)} t_{S_1} \cdots t_{S_n},$$

where $S_1, \ldots, S_n \subset [n+1]$, such that for all $i_1 < \cdots < i_k$,

$$\#(S_{i_1} \cup \cdots \cup S_{i_k}) \geq k+1.$$

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Similar formula for Ehrhart polynomial.

The permutohedron

Recall: $t_I = a_{\#I}$ (i.e., t_I depends only on $\#I$):

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Theorem (**Postnikov**). Fix distinct $x_1, \ldots, x_n$, $\sum x_i = 0$. Then

$$\mathrm{vol}(P_{n-1}(\boldsymbol{t})) = \sum_{w \in \mathfrak{S}_n} \frac{(a_1 x_{w(1)} + \cdots + a_n x_{w(n)})^n}{\prod_{i=1}^{n-1}(x_{w(i)} - x_{w(i+1)})}.$$

Schur functions

partition λ of m of **length** $\leq n$:

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0, \quad \sum \lambda_i = m$$

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Schur function

$$s_\lambda(x_1, \ldots, x_n) = \sum_{w \in \mathfrak{S}_n} w \cdot \frac{x_1^{\lambda_1} \cdots x_n^{\lambda_n}}{\prod_{i<j}(1 - x_j/x_i)}.$$

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(= Weyl character formula for type A_n)

Lattice points in permutohedron

Let $s_\lambda = \sum_\alpha \boldsymbol{K_{\lambda\alpha}} x^\alpha$.

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Define

$$\boldsymbol{\delta}(r) = \begin{cases} 1, & r \neq 0 \\ 0, & r = 0. \end{cases}$$

Then

$$\sum_\alpha \delta(K_{\lambda\alpha}) x^\alpha = \sum_{w \in \mathfrak{S}_n} w \cdot \frac{x_1^{\lambda_1} \cdots x_n^{\lambda_n}}{\prod_{i=1}^{n-1}(1 - x_{i+1}/x_i)}.$$

Example

Let $[\boldsymbol{a},\boldsymbol{b}] = 1 - a/b$.

$$s_{21}(x,y,z) = \frac{x^2y}{[y,x][z,y][z,x]} + \frac{xy^2}{[x,y][z,x][z,y]} + \cdots$$

$$= x^2y + xy^2 + y^2z + yz^2 + x^2z + xz^2 + \mathbf{2}xyz$$

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$$= x^2y + xy^2 + y^2z + yz^2 + x^2z + xz^2 + \mathbf{1}xyz$$

Number of terms

Fix m, n.

$$N(s_{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_m))$$

$$= \sum_{w \in \mathfrak{S}_n} w \cdot \frac{x_1^{\lambda_1} \cdots x_n^{\lambda_n}}{\prod_{i=1}^{n-1}(1 - x_{i+1}/x_i)}\Bigg|_{x_i=1}$$

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Let $x_i = q^{i-1}$ and $q \to 1$.

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Let $x_i = q^{i-1}$ and $q \to 1$.

E.g., $N(s_{(a,b,c)}(x,y,z)) =$
$1 + \frac{3}{2}(a-c) + \frac{1}{2}(a^2 + 2ab - 2b^2 + c^2 - 4ac + 2bc)$.

Brion's theorem

Example. Let $\mathcal{P}$ be the polytope $[2, 5]$ in $\mathbb{R}$, so $\mathcal{P}$ is defined by

$$\mathbf{(1)}\ x \geq 2, \qquad \mathbf{(2)}\ x \leq 5.$$

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Let

$$F_1(t) = \sum_{\substack{n\geq 2\\ n\in\mathbb{Z}}} t^n = \frac{t^2}{1-t}$$

$$F_2(t) = \sum_{\substack{n\leq 5\\ n\in\mathbb{Z}}} t^n = \frac{t^5}{1-\frac{1}{t}}.$$

$F_1(t) + F_2(t)$

$$\begin{aligned} F_1(t) + F_2(t) &= \frac{t^2}{1-t} + \frac{t^5}{1-\frac{1}{t}} \\ &= t^2 + t^3 + t^4 + t^5 \\ &= \sum_{m \in \mathcal{P} \cap \mathbb{Z}} t^m . \end{aligned}$$

Cone at a vertex

$\boldsymbol{\mathcal{P}}$: $\mathbb{Z}$-polytope in $\mathbb{R}^N$ with vertices $\boldsymbol{v_1}, \ldots, \boldsymbol{v_k}$

$\boldsymbol{\mathcal{C}_i}$: cone at vertex i supporting $\mathcal{P}$

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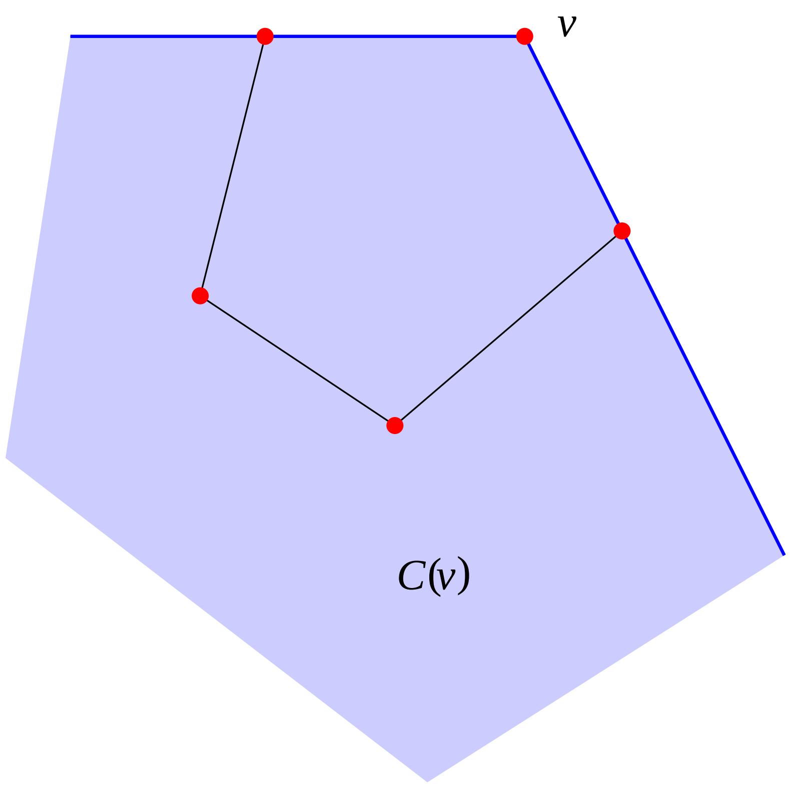


The general result

Let $\boldsymbol{F_i}(t_1, \ldots, t_N) = \sum_{(m_1,\ldots,m_N) \in \mathcal{C}_i \cap \mathbb{Z}^N} t_1^{m_1} \cdots t_N^{m_N}$.

The general result

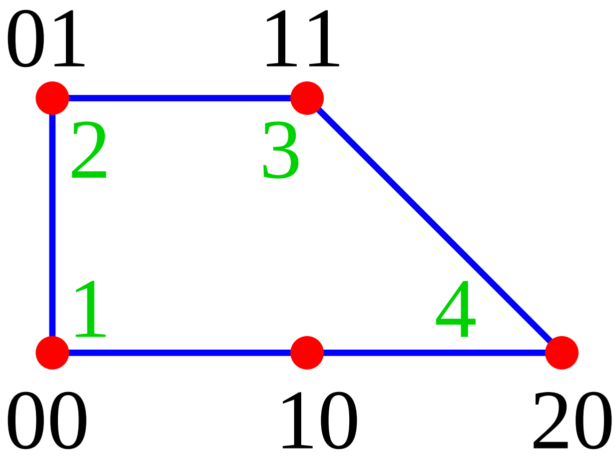
Let $\boldsymbol{F_i}(t_1, \ldots, t_N) = \sum_{(m_1,\ldots,m_N)\in\mathcal{C}_i\cap\mathbb{Z}^N} t_1^{m_1} \cdots t_N^{m_N}$.

Theorem (**Brion**). *Each F_i is a rational function of $t_1, \ldots, t_N$, and*

$$\sum_{i=1}^{k} F_i(t_1, \ldots, t_N) = \sum_{(m_1,\ldots,m_N)\in\mathcal{P}\cap\mathbb{Z}^N} t_1^{m_1} \cdots t_N^{m_N}$$

(as rational functions).

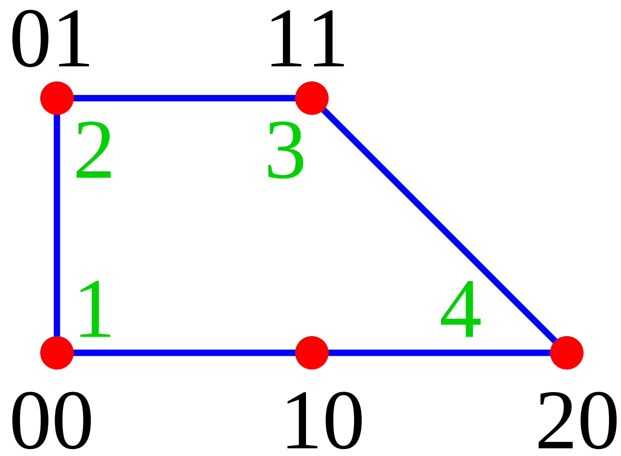
Another example



$$F_1 = \sum_{a,b \geq 0} x^a y^b = \frac{1}{(1-x)(1-y)}$$

$$F_2 = \sum_{\substack{a \geq 0 \\ b \leq 1}} x^a y^b = \frac{y}{(1-x)(1-y^{-1})}$$

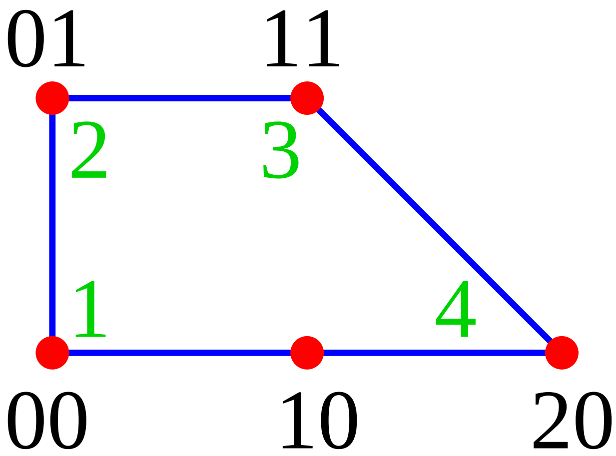
Example (continued)



$$F_3 = \sum_{\substack{b \leq 1 \\ a+b \leq 1}} x^a y^b = \frac{xy}{(1-y^{-1})(1-xy^{-1})}$$

$$F_4 = \sum_{\substack{b \geq 0 \\ a+b \leq 1}} x^a y^b = \frac{x^2}{(1-x^{-1})(1-x^{-1}y)}$$

Example (concluded)



$$F_1 + F_2 + F_3 + F_4 = 1 + x + y + xy + x^2$$

A further variation

Lattice path matroid polytope: a variation of PS-polytopes investigated by **Bonin-Mier-Noy** and **Bidkhori**.

Descent polytopes

Denis Chebikin, Ph.D. thesis, M.I.T., 2008, and **Richard Ehrenborg**

$\boldsymbol{S} \subseteq [n-1] = \{1, 2, \ldots, n-1\}$

Descent polytope DP$_{\boldsymbol{S}}$ $\subset \mathbb{R}^n$:

$$0 \leq x_i \leq 1$$

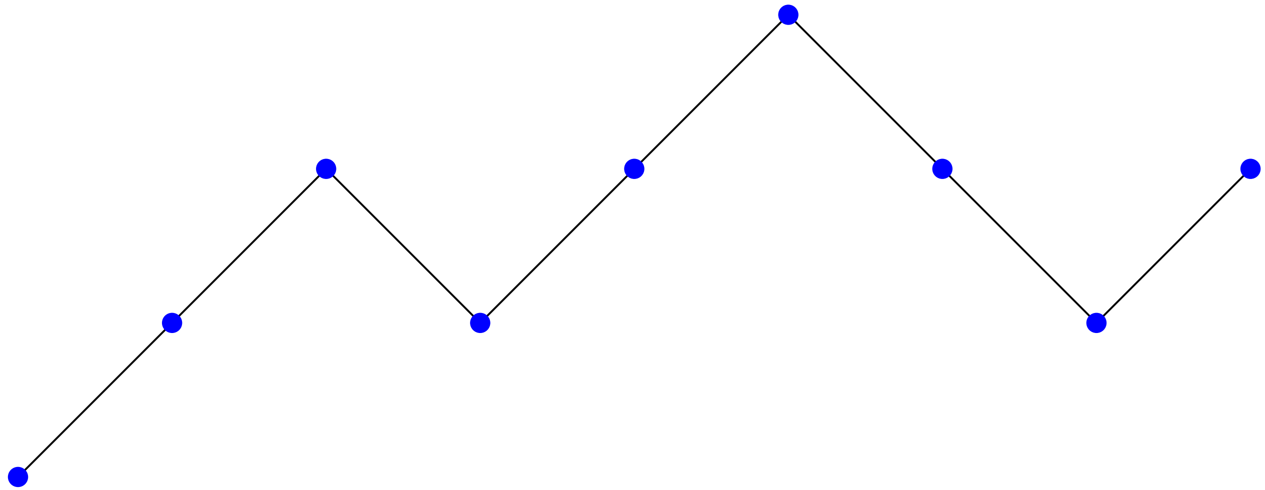
$$x_i \geq x_{i+1} \ \text{ if } i \in S$$

$$x_i \leq x_{i+1} \ \text{ if } i \notin S$$

Same as **order polytope** $\mathcal{O}(Z_S)$ of **zigzag poset** Z_S.

Example of zigzag poset

$n = 9, \quad S = \{3, 6, 7\}$



$\mathcal{O}(Z_S) = \{$order-preserving maps $f : Z_S \to [0, 1]\}$

Combinatorics of DP_S

Volume and Ehrhart polynomial of DP_S follows from theory of P-partitions. In particular, let $w = a_1 \cdots a_n \in \mathfrak{S}_n$ and define

$$\boldsymbol{D(w)} = \{i \,:\, a_i > a_{i+1}\} \subseteq [n-1],$$

the **descent set** of w. Define

$$\beta_n(S) = \#\{w \in \mathfrak{S}_n \,:\, D(w) = S\}.$$

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Theorem. $\mathrm{vol}(\mathrm{DP}_S) = \dfrac{\beta_n(S)}{n!}$

The f-vector of DP_S

$(f_0, f_1, \ldots, f_{n-1})$: f-vector of DP_S, i.e., f_i is the number of i-dimensional faces. Set $f_n = 1$.

Define the **f-polynomial** $F_S(t) = \sum_{i=0}^{n} f_i t^i$.

x, y: noncommuting variables

For $S \subseteq [n-1]$ define $v_S = v_1 \cdots v_{n-1}$, where

$$v_i = \begin{cases} x, & \text{if } i \notin S \\ y, & \text{if } i \in S. \end{cases}$$

A generating function

$$\boldsymbol{\Phi_n(x,y)} := \sum_{S\subseteq[n-1]} F_S(t)v_S$$

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$$\begin{aligned}\boldsymbol{\Phi(x,y)} &= \sum_{n\geq 1}\Phi_n(x,y)\\ &= (2+t)+(3+3t+t^2)(x+y)+\cdots.\end{aligned}$$

E.g., $n=2$, $S=\emptyset$: $0\leq x_1\leq x_2\leq 1$, a triangle, so coefficient of x is $3+3t+t^2$.

Chebikin-Ehrenborg theorem

Theorem. $\Phi(x,y) =$

$$\left(1 + \frac{t+1}{1-(t+1)\left((1-y)^{-1}x + (1-x)^{-1}y\right)}\right)$$

$$\cdot \frac{1}{1-x-y}.$$

The flag f-vector of DP_S

Let $\boldsymbol{T} = \{a_0 < \cdots < a_k\} \subseteq [0, n-1]$. Define

$$\boldsymbol{\alpha_S}(T) = \#\{F_0 \subset F_1 \subset \cdots \subset F_k \ : \ \dim F_i = a_i\}.$$

Call α_S the **flag f-vector** of DP_S.

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Open. Is there a "nice" generating function for $\alpha_S(T)$'s (or equivalently, the flag h-vector of cd-index) generalizing Chebikin's theorem?


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