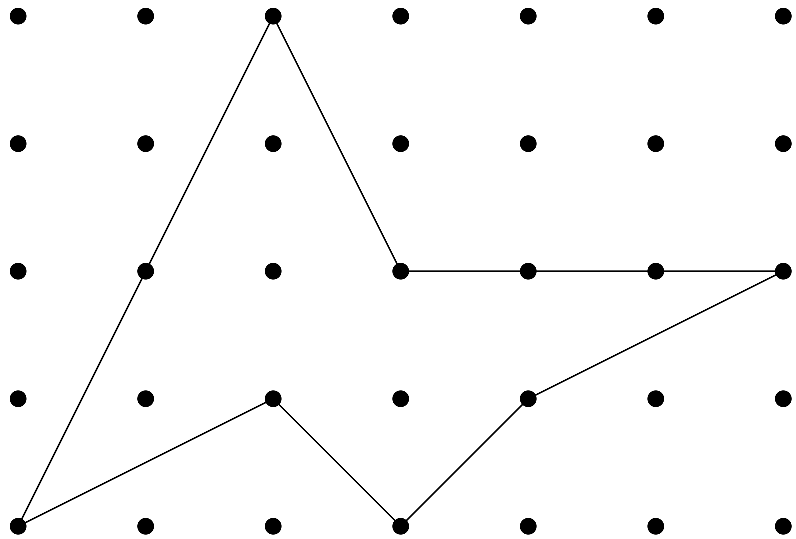
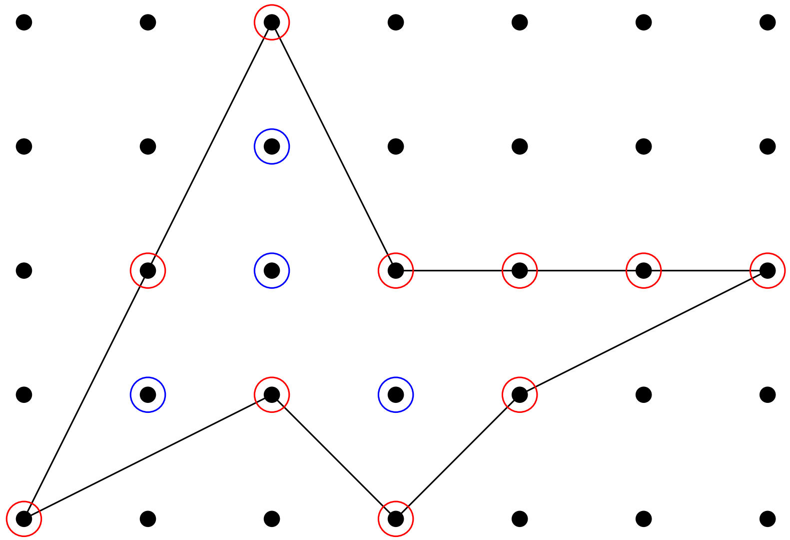
Lattice polygons

P: lattice polygon in $\mathbb{R}^2$
(vertices $\in \mathbb{Z}^2$, no self-intersections)



A, I, B



A $=$ area of P

I $=$ $\#$ interior points of P $(= 4)$

B $=$ $\#$boundary points of P $(= 10)$

Pick's theorem

Georg Alexander Pick (1859–1942)

$$\mathbf{A} = \frac{2\mathbf{I} + \mathbf{B} - 2}{2} = \frac{2 \cdot \mathbf{4} + \mathbf{10} - 2}{2} = 9$$

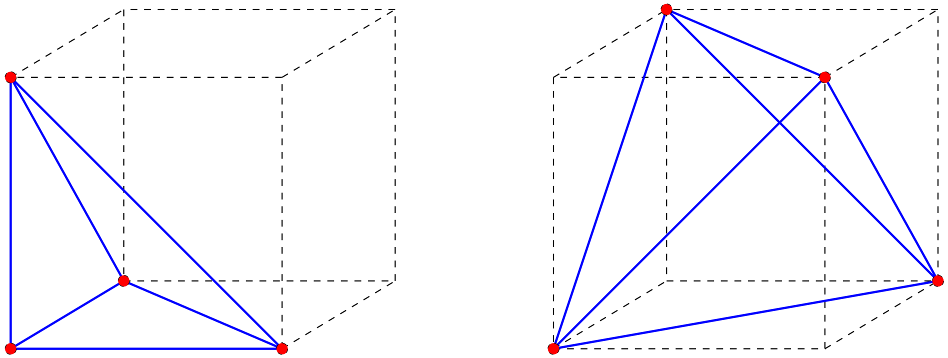
Higher dimensions?

Pick's theorem (seemingly) fails in higher dimensions.

Example. Let T_1 and T_2 be the tetrahedra with vertices

$$\begin{aligned} v(T_1) &= \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\} \\ v(T_2) &= \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}. \end{aligned}$$

Two tetrahedra



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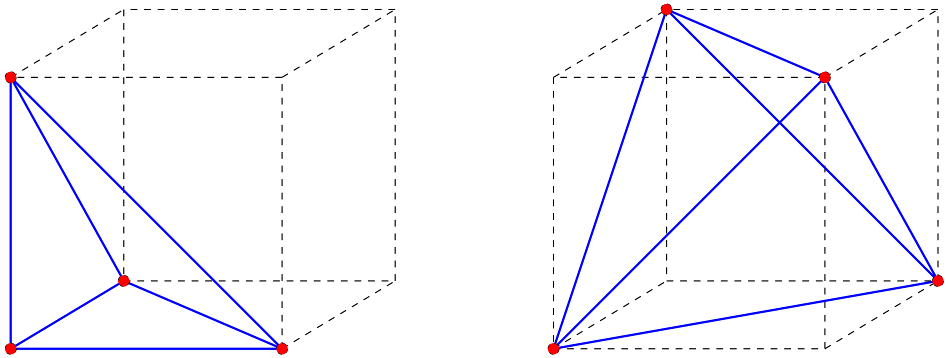

Then

$$I(T_1) = I(T_2) = 0$$

$$B(T_1) = B(T_2) = 4$$

$$\mathrm{vol}(T_1) = 1/6, \quad \mathrm{vol}(T_2) = 1/3.$$

Dilation

Let $\mathcal{P}$ be a convex polytope (convex hull of a finite set of points) in $\mathbb{R}^d$. For $n \geq 1$, let

$$\boldsymbol{n\mathcal{P}} = \{n\alpha \, : \, \alpha \in \mathcal{P}\}.$$

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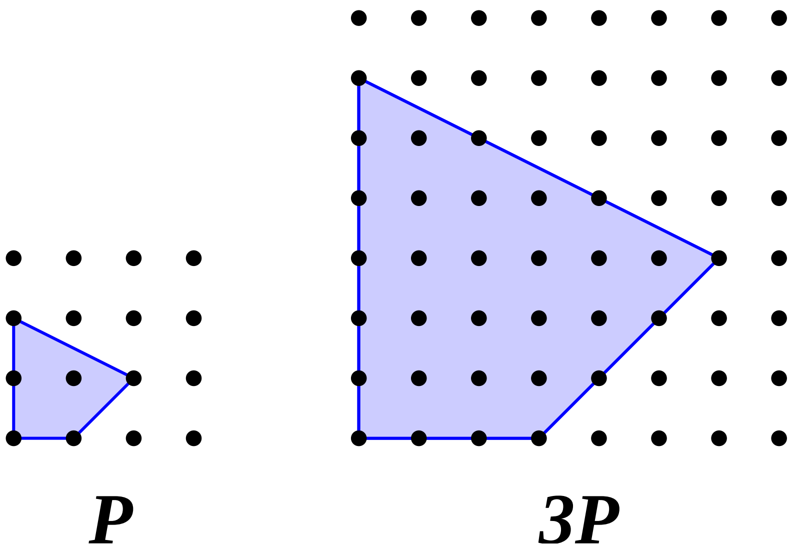


$i(\mathcal{P}, n)$

Let

$$\begin{aligned} i(\mathcal{P}, n) &= \#(n\mathcal{P} \cap \mathbb{Z}^d) \\ &= \#\{\alpha \in \mathcal{P} : n\alpha \in \mathbb{Z}^d\}, \end{aligned}$$

the number of lattice points in $n\mathcal{P}$.

$\bar{i}(\mathcal{P}, n)$

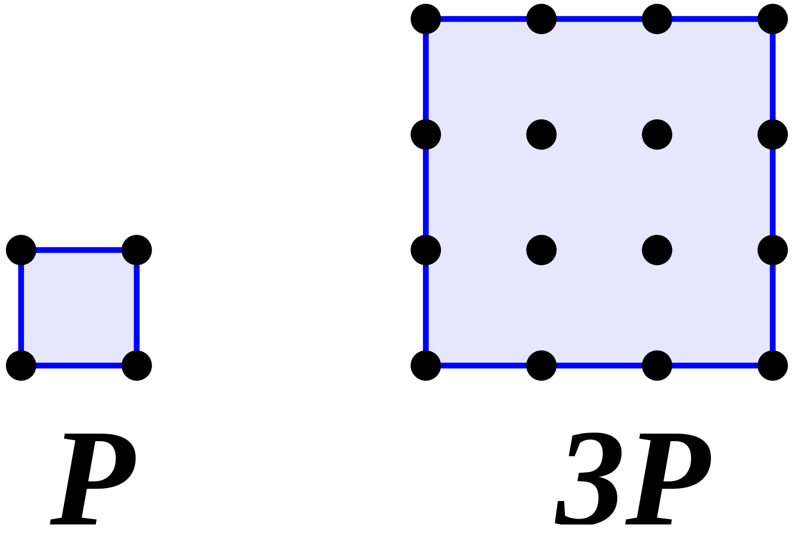
Similarly let

$$\mathcal{P}^\circ = \text{interior of } \mathcal{P} = \mathcal{P} - \partial\mathcal{P}$$

$$\begin{aligned} \bar{i}(\mathcal{P}, n) &= \#(n\mathcal{P}^\circ \cap \mathbb{Z}^d) \\ &= \#\{\alpha \in \mathcal{P}^\circ \ : \ n\alpha \in \mathbb{Z}^d\}, \end{aligned}$$

the number of lattice points in the **interior** of $n\mathcal{P}$.

An example



$$i(\mathcal{P}, n) \;=\; (n+1)^2$$

$$\bar{i}(\mathcal{P}, n) \;=\; (n-1)^2 = i(\mathcal{P}, -n)$$

Reeve's theorem

lattice polytope: polytope with integer vertices

Theorem (Reeve, 1957). *Let* $\mathcal{P}$ *be a three-dimensional lattice polytope. Then the volume* $V(\mathcal{P})$ *is a certain (explicit) function of* $i(\mathcal{P},1)$, $\bar{i}(\mathcal{P},1)$, *and* $i(\mathcal{P},2)$.

The Ehrhart-Macdonald theorem

Theorem (Ehrhart 1962, Macdonald 1963). *Let*

$$\mathcal{P} = \text{lattice polytope in } \mathbb{R}^N, \ \dim \mathcal{P} = d.$$

Then $i(\mathcal{P}, n)$ is a polynomial (the ***Ehrhart polynomial*** *of $\mathcal{P}$) in n of degree d.*

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Note. Eugène Ehrhart (1906–2000): taught at lycées in France, received Ph.D. in 1966.

Reciprocity, volume

Moreover,

$$
\begin{aligned}
i(\mathcal{P},0) &= 1 \\
\bar{i}(\mathcal{P},n) &= (-1)^d i(\mathcal{P},-n),\ n>0
\end{aligned}
$$

(reciprocity).

Reciprocity, volume

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(reciprocity).

If $d = N$ *then*

$$i(\mathcal{P},n) = V(\mathcal{P})n^d + \text{ lower order terms,}$$

where $\boldsymbol{V(\mathcal{P})}$ *is the volume of* $\mathcal{P}$*.*

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(“relative volume” for $d < N$)

Generalized Pick's theorem

Corollary (generalized Pick's theorem). *Let* $\mathcal{P} \subset \mathbb{R}^d$ *and* $\dim \mathcal{P} = d$. *Knowing any* d *of* $i(\mathcal{P}, n)$ *or* $\bar{i}(\mathcal{P}, n)$ *for* $n > 0$ *determines* $V(\mathcal{P})$.

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Proof. Together with $i(\mathcal{P}, 0) = 1$, this data determines $d + 1$ values of the polynomial $i(\mathcal{P}, n)$ of degree d. This uniquely determines $i(\mathcal{P}, n)$ and hence its leading coefficient $V(\mathcal{P})$. $\square$

Reeve's theorem redux

Example. When $d = 3$, $V(\mathcal{P})$ is determined by

$$\begin{aligned} i(\mathcal{P}, 1) &= \#(\mathcal{P} \cap \mathbb{Z}^3) \\ i(\mathcal{P}, 2) &= \#(2\mathcal{P} \cap \mathbb{Z}^3) \\ \bar{i}(\mathcal{P}, 1) &= \#(\mathcal{P}^\circ \cap \mathbb{Z}^3), \end{aligned}$$

which gives Reeve's theorem.

The Birkhoff polytope

Example (magic squares). Let $\mathcal{B}_M \subset \mathbb{R}^{M\times M}$ be the **Birkhoff polytope** of all $M \times M$ **doubly-stochastic** matrices $A = (a_{ij})$, i.e.,

$$a_{ij} \geq 0$$

$$\sum_i a_{ij} = 1 \text{ (column sums 1)}$$

$$\sum_j a_{ij} = 1 \text{ (row sums 1).}$$

Integer points in $\mathcal{B}_M$

Note. $B = (b_{ij}) \in n\mathcal{B}_M \cap \mathbb{Z}^{M\times M}$ if and only if

$$b_{ij} \in \mathbb{N} = \{0, 1, 2, \ldots\}$$

$$\sum_i b_{ij} = n \quad \sum_j b_{ij} = n.$$

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$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 3 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 4 & 0 \end{bmatrix} \quad (M = 4,\ n = 7)$$

$\boldsymbol{H_M(n)}$

$$\begin{aligned} \boldsymbol{H_M(n)} &:= \#\{M \times M\ \mathbb{N}\text{-matrices, line sums } n\} \\ &= i(\mathcal{B}_M, n). \end{aligned}$$

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$$\begin{bmatrix} a & n-a \\ n-a & a \end{bmatrix}, \quad 0 \le a \le n.$$

More examples

$$H_3(n) = \binom{n+2}{4} + \binom{n+3}{4} + \binom{n+4}{4}$$

(MacMahon)

$$H_M(0) = 1$$

$$H_M(1) = M! \text{ (permutation matrices)}$$

$$\sum_{M \geq 0} H_M(2) \frac{x^M}{M!^2} = \frac{e^{x/2}}{\sqrt{1-x}}$$

(Anand-Dumir-Gupta, 1966)

Anand-Dumir-Gupta conjecture

Theorem (Birkhoff-von Neumann). *The vertices of* $\mathcal{B}_M$ *consist of the* $M!$ $M \times M$ *permutation matrices. Hence* $\mathcal{B}_M$ *is a lattice polytope.*

Anand-Dumir-Gupta conjecture

Theorem (Birkhoff-von Neumann). *The vertices of $\mathcal{B}_M$ consist of the $M!$ $M \times M$ permutation matrices. Hence $\mathcal{B}_M$ is a lattice polytope.*

Corollary (Anand-Dumir-Gupta conjecture). $H_M(n)$ *is a polynomial in* n *(of degree* $(M-1)^2$*).*

$H_4(n)$

Example. $H_4(n) = \frac{1}{11340}\left(11n^9 + 198n^8 + 1596n^7\right.$

$+7560n^6 + 23289n^5 + 48762n^5 + 70234n^4 + 68220n^2$

$\left.+40950n + 11340\right).$

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Open: positive coefficients

Positive magic squares

Reciprocity $\Rightarrow$

$$\pm H_M(-n) = \#\{M \times M \text{ matrices } B \text{ of}$$

$$\textbf{positive} \text{ integers, line sum } n\}.$$

But every such B can be obtained from an $M \times M$ matrix A of **nonnegative** integers by adding 1 to each entry.

Reciprocity for magic squares

Corollary.
$H_M(-1) = H_M(-2) = \cdots = H_M(-M+1) = 0$

$$H_M(-M-n) = (-1)^{M-1} H_M(n)$$

(greatly reduces computation)

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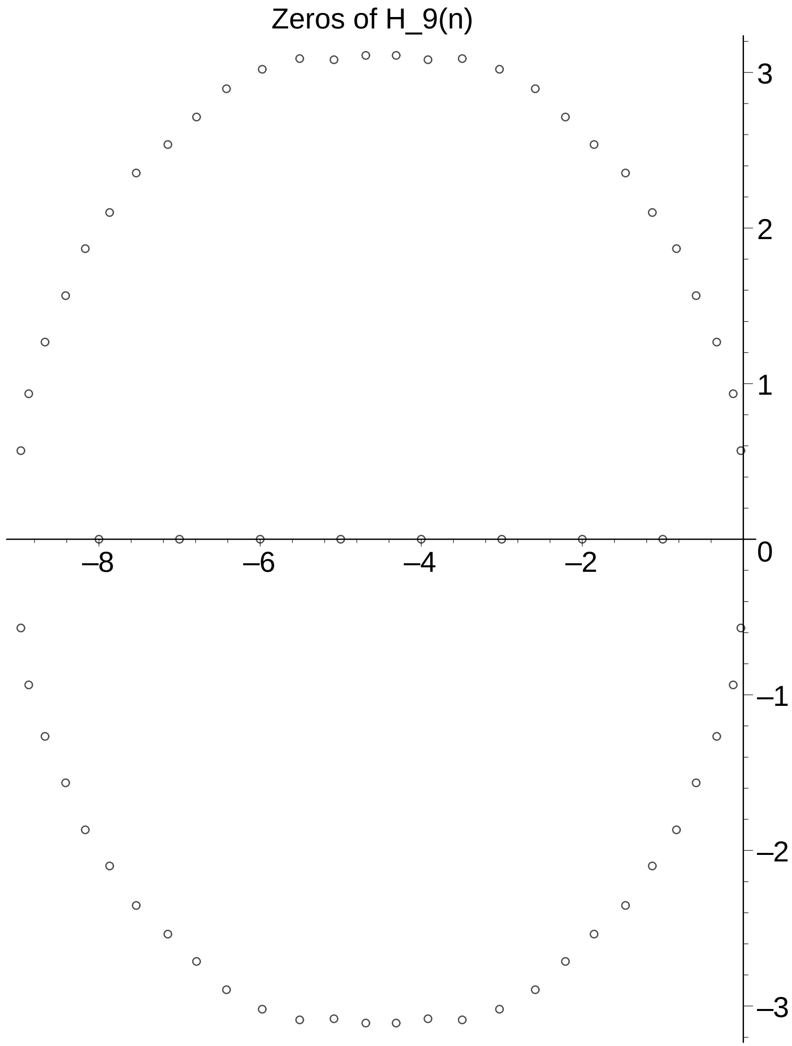
Applications e.g. to statistics (contingency tables) by **Diaconis**, et al.

Zeros (roots) of $H_9(n)$



Explicit calculation

For what polytopes $\mathcal{P}$ can $i(\mathcal{P}, n)$ be explicitly calculated or related to other interesting mathematics?

Main topic of subsequent talks.

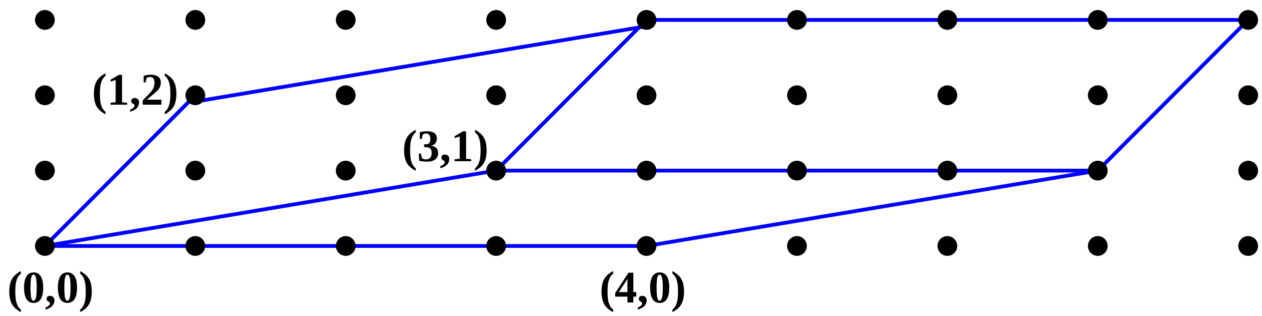
Zonotopes

Let $v_1, \ldots, v_k \in \mathbb{R}^d$. The **zonotope** $Z(v_1, \ldots, v_k)$ generated by $v_1, \ldots, v_k$:

$$\boldsymbol{Z(v_1, \ldots, v_k)} = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \ : \ 0 \leq \lambda_i \leq 1\}$$

An example

Example. $v_1 = (4,0)$, $v_2 = (3,1)$, $v_3 = (1,2)$



$i(Z, 1)$

Theorem. *Let*

$$Z = Z(v_1, \ldots, v_k) \subset \mathbb{R}^d,$$

where $v_i \in \mathbb{Z}^d$. *Then*

$$i(Z, 1) = \sum_X h(X),$$

where X *ranges over all linearly independent subsets of* $\{v_1, \ldots, v_k\}$, *and* $h(X)$ *is the gcd of all* $j \times j$ *minors* ($j = \#X$) *of the matrix whose rows are the elements of* X.

An example

Example. $v_1 = (4,0)$, $v_2 = (3,1)$, $v_3 = (1,2)$

$$
\begin{aligned}
i(Z,1) &= \begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \\
&\quad + \gcd(4,0) + \gcd(3,1) \\
&\quad + \gcd(1,2) + \det(\emptyset) \\
&= 4+8+5+4+1+1+1 \\
&= 24.
\end{aligned}
$$

Decomposition of a zonotope

$$\begin{aligned} i(Z,1) &= \begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \\ &\quad + \gcd(4,0) + \gcd(3,1) \\ &\quad + \gcd(1,2) + \det(\emptyset) \\ &= 4+8+5+4+1+1+1 \\ &= 24. \end{aligned}$$

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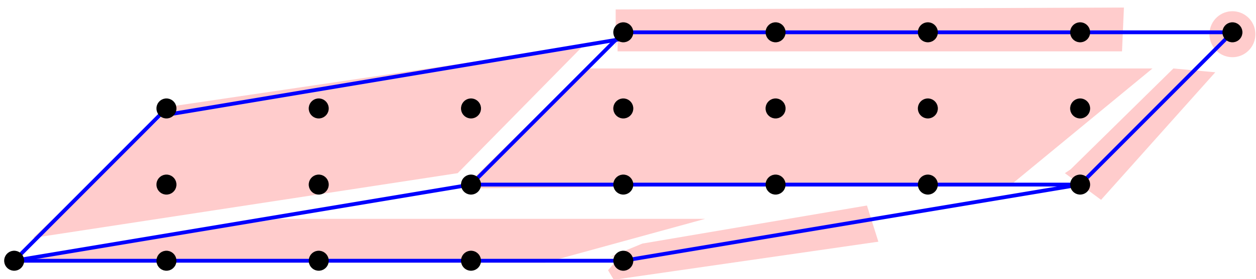

Ehrhart polynomial of a zonotope

Theorem. *Let* $Z = Z(v_1, \ldots, v_k) \subset \mathbb{R}^d$ *where* $v_i \in \mathbb{Z}^d$*. Then*

$$i(Z, n) = \sum_X h(X) n^{\#X},$$

where X *ranges over all linearly independent subsets of* $\{v_1, \ldots, v_k\}$*, and* $h(X)$ *is the gcd of all* $j \times j$ *minors (*$j = \#X$*) of the matrix whose rows are the elements of* X*.*

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Corollary. *The coefficients of* $i(Z, n)$ *are nonnegative integers.*

Ordered degree sequences

Let G be a graph (with no loops or multiple edges) on the vertex set $V(G) = \{1, 2, \ldots, n\}$. Let

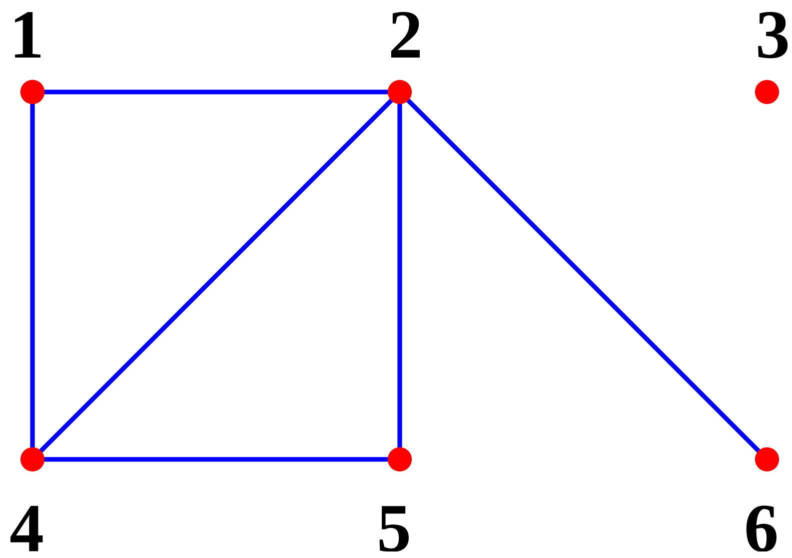
$$d_i = \text{degree (\# incident edges) of vertex } i.$$

Define the **ordered degree sequence** $d(G)$ of G by

$$d(G) = (d_1, \ldots, d_n).$$

An example

Example. $d(G) = (2,4,0,3,2,1)$



Number of distinct $d(G)$

Let $\boldsymbol{f(n)}$ be the number of distinct $d(G)$, where $V(G) = \{1, 2, \ldots, n\}$.

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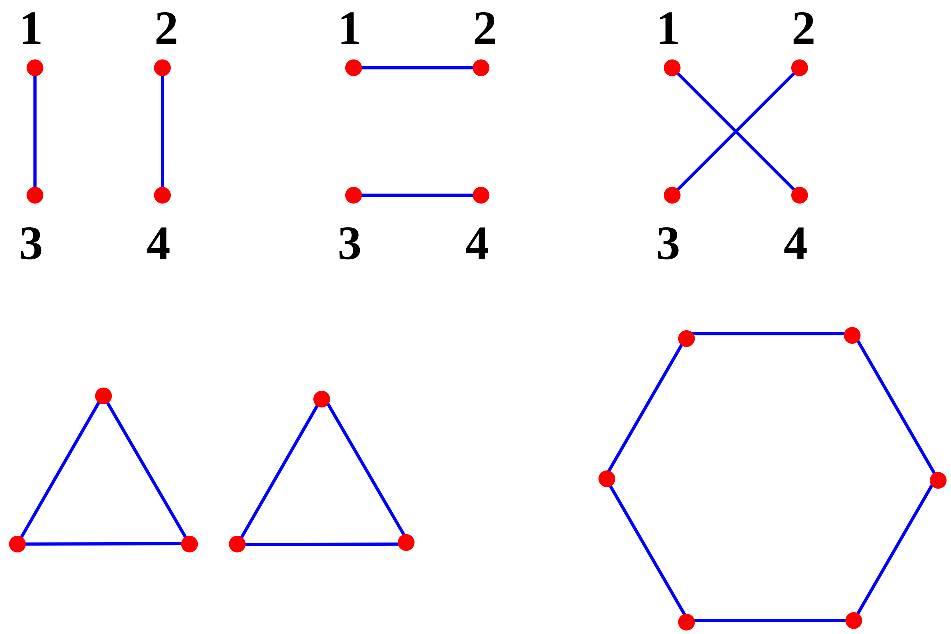
Example. If $n \leq 3$, all $d(G)$ are distinct, so

$$f(1) = 1, \; f(2) = 2^1 = 2, \; f(3) = 2^3 = 8.$$

$f(4)$

For $n \geq 4$ we can have $G \neq H$ but $d(G) = d(H)$, e.g.,



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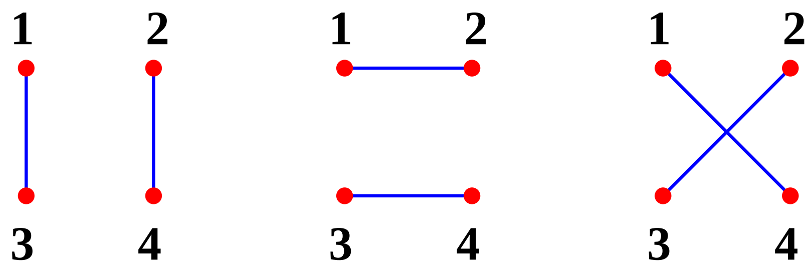

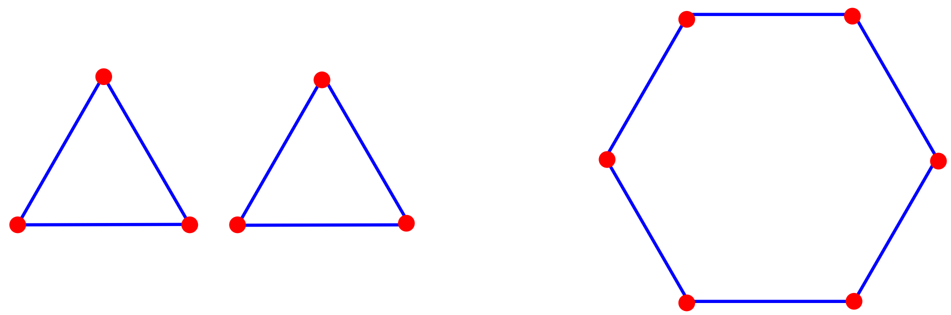

In fact, $f(4) = 54 < 2^6 = 64$.

The polytope of degree sequences

Let **conv** denote convex hull, and

$$\mathcal{D}_n = \mathrm{conv}\{d(G) : V(G) = \{1, \ldots, n\}\} \subset \mathbb{R}^n,$$

the **polytope of degree sequences (Perles, Koren)**.

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Easy fact. Let e_i be the ith unit coordinate vector in $\mathbb{R}^n$. E.g., if $n = 5$ then $e_2 = (0,1,0,0,0)$. Then

$$\mathcal{D}_n = Z(e_i + e_j \,:\, 1 \leq i < j \leq n).$$

The Erdős-Gallai theorem

Theorem (Erdős-Gallai). *Let* $\boldsymbol{\alpha} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. *Then* $\alpha = d(G)$ *for some* G *if and only if*

- $\alpha \in \mathcal{D}_n$
- $a_1 + a_2 + \cdots + a_n$ *is even.*

A generating function

"Fiddling around" leads to:

Theorem. *Let*

$$F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}$$

$$= \mathbf{1} + \mathbf{1}x + \mathbf{2}\frac{x^2}{2!} + \mathbf{8}\frac{x^3}{3!} + \mathbf{54}\frac{x^4}{4!} + \cdots .$$

Then

A nice formula

$$F(x) = \frac{1}{2}\left[\left(1 + 2\sum_{n\geq 1} n^n \frac{x^n}{n!}\right)^{1/2}\right.$$

$$\left.\times\left(1 - \sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}\right) + 1\right]$$

$$\times \exp \sum_{n\geq 1} n^{n-2}\frac{x^n}{n!}$$

A “bad” Ehrhart polynomial

Let $\mathcal{P}$ denote the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,13)$. Then

$$i(\mathcal{P},n) = \frac{13}{6}n^3 + n^2 - \frac{1}{6}n + 1.$$

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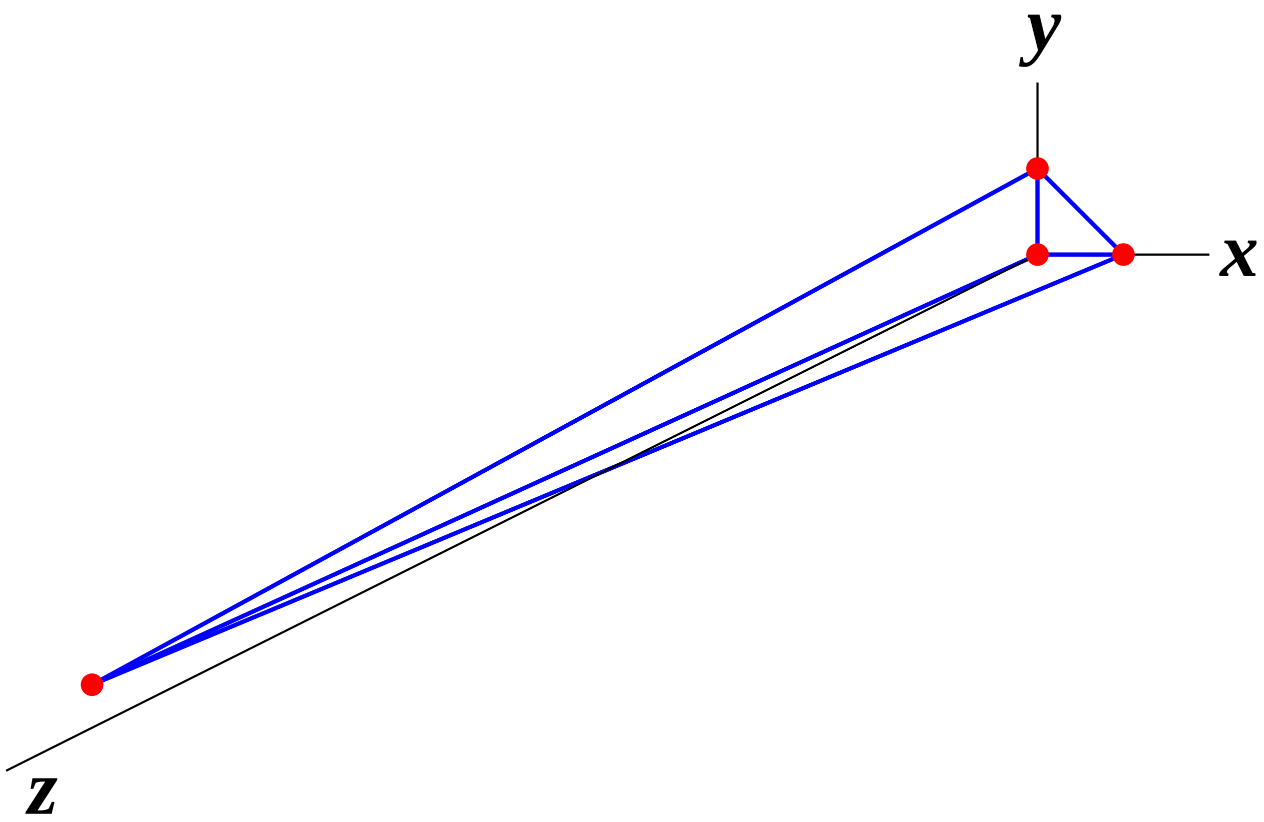
Thus in general the coefficients of Ehrhart polynomials are not “nice.” Is there a “better” basis?

Diagram of the bad tetrahedron



The h-vector of $i(\mathcal{P}, n)$

Let $\mathcal{P}$ be a lattice polytope of dimension d. Since $i(\mathcal{P}, n)$ is a polynomial of degree d taking $\mathbb{Z} \to \mathbb{Z}$, $\exists$ $\boldsymbol{h_i} \in \mathbb{Z}$ such that

$$\sum_{n \geq 0} i(\mathcal{P}, n) x^n = \frac{h_0 + h_1 x + \cdots + h_d x^d}{(1-x)^{d+1}}.$$

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Definition. Define

$$\boldsymbol{h(\mathcal{P})} = (h_0, h_1, \ldots, h_d),$$

the **h-vector** of $\mathcal{P}$ (as an integral polytope).

An example

Example. Recall

$$i(\mathcal{B}_4, n) = \frac{1}{11340}(11n^9$$

$$+198n^8 + 1596n^7 + 7560n^6 + 23289n^5$$

$$+48762n^5 + 70234n^4 + 68220n^2$$

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Then

$$h(\mathcal{B}_4) = (1, 14, 87, 148, 87, 14, 1, 0, 0, 0).$$

Properties of $h(\mathcal{P})$

$$h(\mathcal{B}_4) = (1, 14, 87, 148, 87, 14, 1, 0, 0, 0).$$

- Always $h_0 = 1$.
- Trailing 0's $\Leftrightarrow$

$$i(\mathcal{B}_4, -1) = i(\mathcal{B}_4, -2) = i(\mathcal{B}_4, -3) = 0.$$

- Palindromic property

$$\Leftrightarrow i(\mathcal{B}_4, -n-4) = \pm i(\mathcal{B}_4, n).$$

Nonnegativity and monotonicity

Theorem A (nonnegativity), **McMullen, RS**:

$$h_i \geq 0$$

Nonnegativity and monotonicity

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Theorem B (monotonicity), **RS**: *If $\mathcal{P}$ and $\mathcal{Q}$ are lattice polytopes and $\mathcal{Q} \subseteq \mathcal{P}$, then for all i:*

$$h_i(\mathcal{Q}) \leq h_i(\mathcal{P}).$$

B $\Rightarrow$ A: take $\mathcal{Q} = \emptyset$.

Proofs

Both theorems can be proved geometrically.

There are also elegant algebraic proofs based on **commutative algebra**.

Related to **toric varieties**.

Fractional lattice polytopes

Example. Let $S_M(n)$ denote the number of **symmetric** $M \times M$ matrices of nonnegative integers, every row and column sum n. Then

$$
\begin{aligned}
S_3(n) &= \begin{cases} \frac{1}{8}(2n^3+9n^2+14n+8), & n \text{ even} \\ \frac{1}{8}(2n^3+9n^2+14n+7), & n \text{ odd} \end{cases} \\
&= \frac{1}{16}(4n^3+18n^2+28n+15+(-1)^n).
\end{aligned}
$$

Fractional lattice polytopes

Example. Let $S_M(n)$ denote the number of **symmetric** $M \times M$ matrices of nonnegative integers, every row and column sum n. Then

$$S_3(n) = \begin{cases} \frac{1}{8}(2n^3 + 9n^2 + 14n + 8), & n \text{ even} \\ \frac{1}{8}(2n^3 + 9n^2 + 14n + 7), & n \text{ odd} \end{cases}$$

$$= \frac{1}{16}(4n^3 + 18n^2 + 28n + 15 + (-1)^n).$$

Why a different polynomial depending on n modulo 2?

The symmetric Birkhoff polytope

$\mathcal{T}_M$: the polytope of all $M \times M$ **symmetric** doubly-stochastic matrices.

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Thus if v is a vertex of $\mathcal{T}_M$ then $\mathbf{2}v \in \mathbb{Z}^{M\times M}$.

$S_M(n)$ in general

Theorem. *There exist polynomials* $P_M(n)$ *and* $Q_M(n)$ *for which*

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Moreover, $\deg P_M(n) = \binom{M}{2}$.

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Difficult result (W. Dahmen and C. A. Micchelli, 1988):

$$\deg Q_M(n) = \begin{cases} \binom{M-1}{2} - 1, & M \text{ odd} \\ \binom{M-2}{2} - 1, & M \text{ even.} \end{cases}$$


THE END