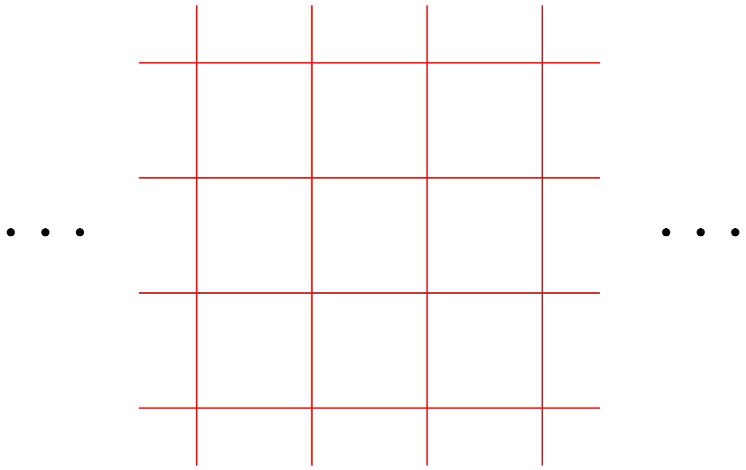
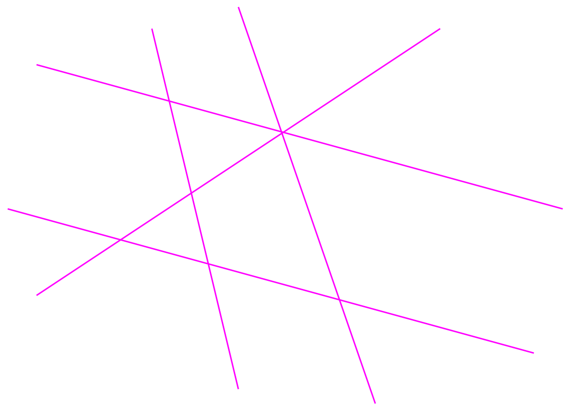
$\mathcal{A}$ = (discrete) hyperplane arrangement in $\mathbb{R}^n$





$$\mathcal{R} = \mathcal{R}_{\mathcal{A}} = \text{set of regions of } \mathcal{A}$$

If $\mathcal{R}_{\mathcal{A}}$ is finite, then let

$$r(\mathcal{A}) = \text{number of regions of } \mathcal{A}.$$

If $R, R' \in \mathcal{R}$ then let

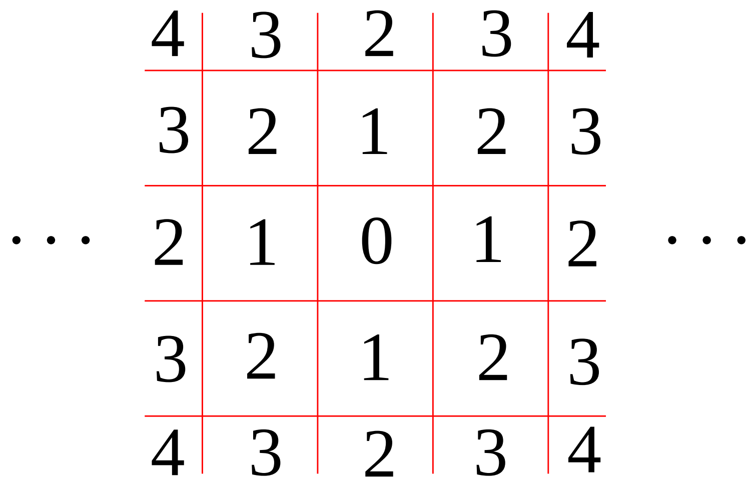
$$d(R, R') = \text{number of hyperplanes in } \mathcal{A}$$
$$\text{separating } R \text{ and } R',$$

the **distance** between R and R'.

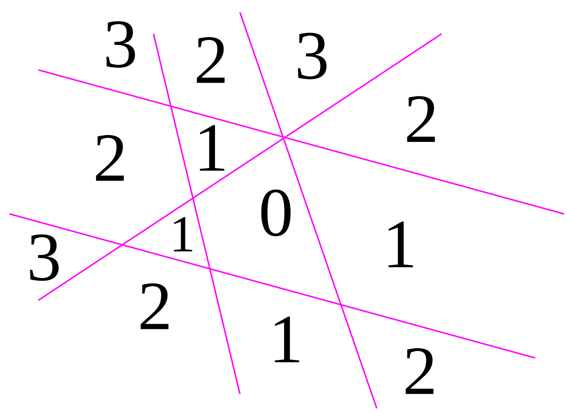
Fix a **base region** $R_0 \in \mathcal{R}$, and set $d(R) = d(R_0, R)$. Define the **distance enumerator** $D_{\mathcal{A}}(q)$ of $\mathcal{A}$ (with respect to R_0) by

$$D_{\mathcal{A}}(q) = \sum_{R \in \mathcal{R}} q^{d(R)}.$$

NOTE: $D_{\mathcal{A}}(1) = r(\mathcal{A})$ if $\mathcal{R}_{\mathcal{A}}$ is finite.



$$D(q) = 1 + 4q + 8q^2 + 12q^3 + \cdots = \frac{4q}{(1-q)^2}$$



$$D(q) = 1 + 4q + 5q^2 + 3q^3$$

Archetypal example:
braid arrangement.

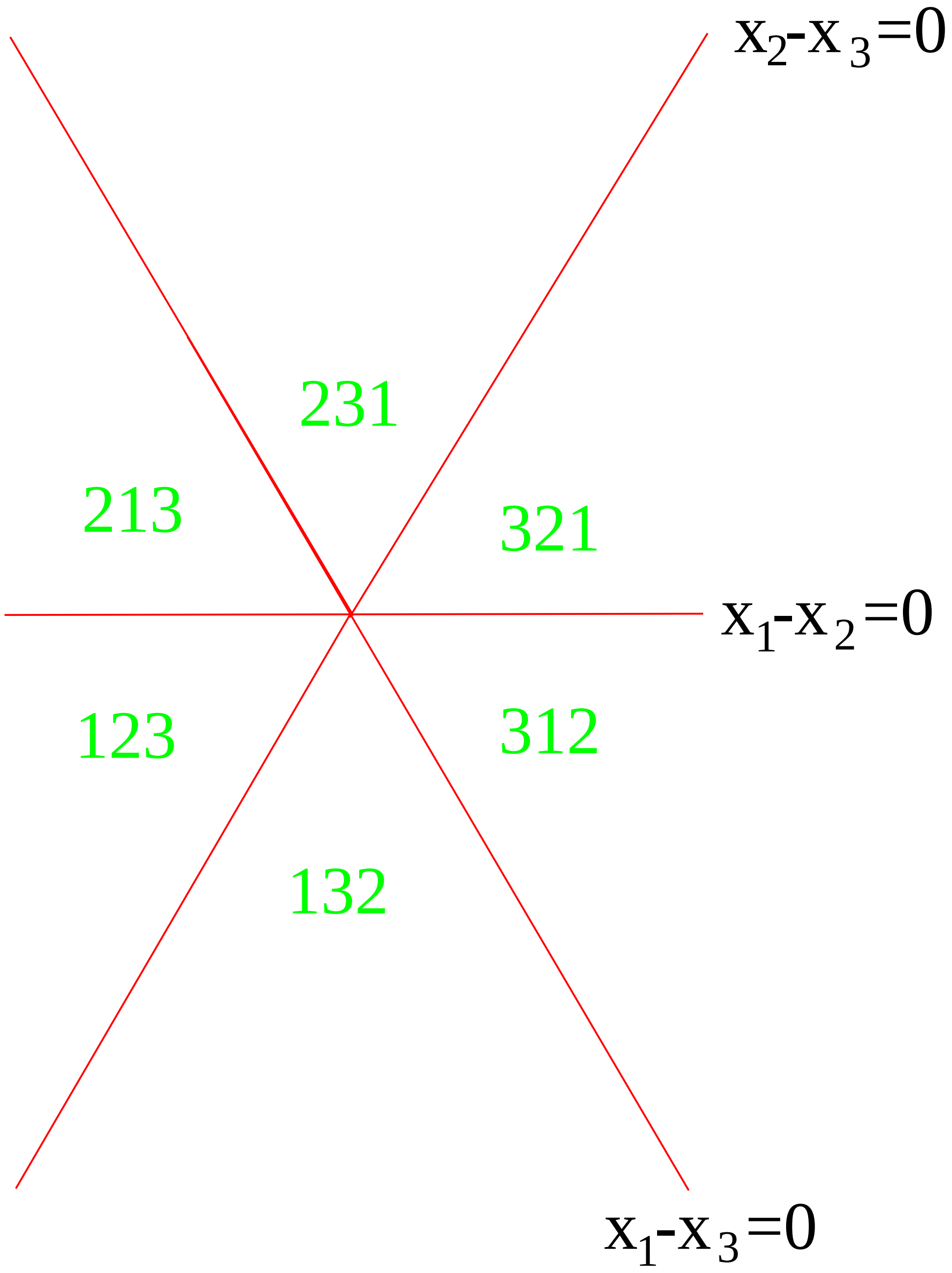
$\mathcal{B}_n\ :\ x_i - x_j = 0,\ 1 \leq i < j \leq n$ (in $\mathbb{R}^n$)

Let R_0 be defined by

$$x_1 > x_2 > \cdots > x_n.$$

The symmetric group $\mathfrak{S}_n$ acts regularly on $\mathcal{R}$, i.e., for each $R \in \mathcal{R}$ there is a unique $w = w(R) \in \mathfrak{S}_n$ such that

$$w \cdot R_0 = R.$$


x_2-x_3=0
231
213
321
x_1-x_2=0
123
312
132
x_1-x_3=0

.

Let $w = w(R)$ and $i < j$. Then $x_i - x_j = 0$ separates R from R_0 if and only if $w(i) > w(j)$, i.e., (i, j) is an **inversion** of w. Hence

$$d(R) = \ell(w),$$

the number of inversions (or **length**) of w. Thus

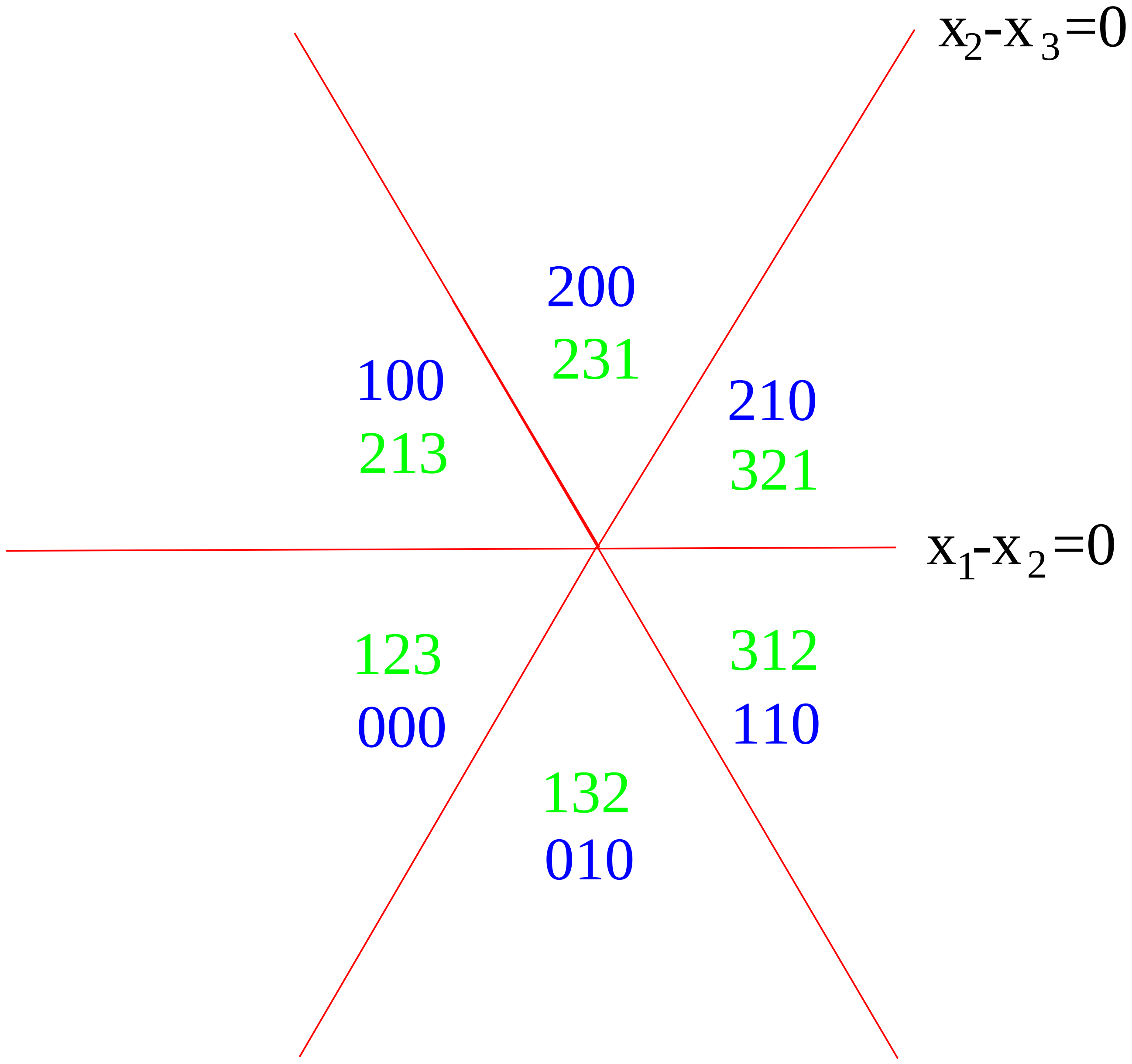
$$\begin{aligned} D_{\mathcal{B}_n}(q) &= \sum_{w \in \mathfrak{S}_n} q^{\ell(w)} \\ &= (1+q)(1+q+q^2) \\ &\qquad \cdots (1+q+q^2+\cdots+q^{n-1}). \end{aligned}$$

Alternative labelling rule:

- Set $\lambda(R_0) = (0, 0, \ldots, 0) \in \mathbb{Z}^n$.
- If R is labelled, R' is separated from R only by $x_i - x_j = 0$ $(i < j)$, and R' is unlabelled, then set

$$\lambda(R') = \lambda(R) + e_i,$$

where e_i is the ith unit coordinate vector.



x_2-x_3=0
200
231
100
213
210
321
x_1-x_2=0
123
000
312
110
132
010
x_1-x_3=0

Note. Let $w = w(R)$. Then

$$\lambda(R)_j = \#\{i \,:\, i < j,\ w(i) > w(j)\},$$

so $\lambda(R)$ is essentially the **inversion table** or **code** of w. A sequence $(a_1, \dots, a_n)$ is such a code if and only if $0 \le a_i \le n - i$. Moreover, if $\lambda(R) = (a_1, \dots, a_n)$ then

$$d(R) = a_1 + \cdots + a_n.$$

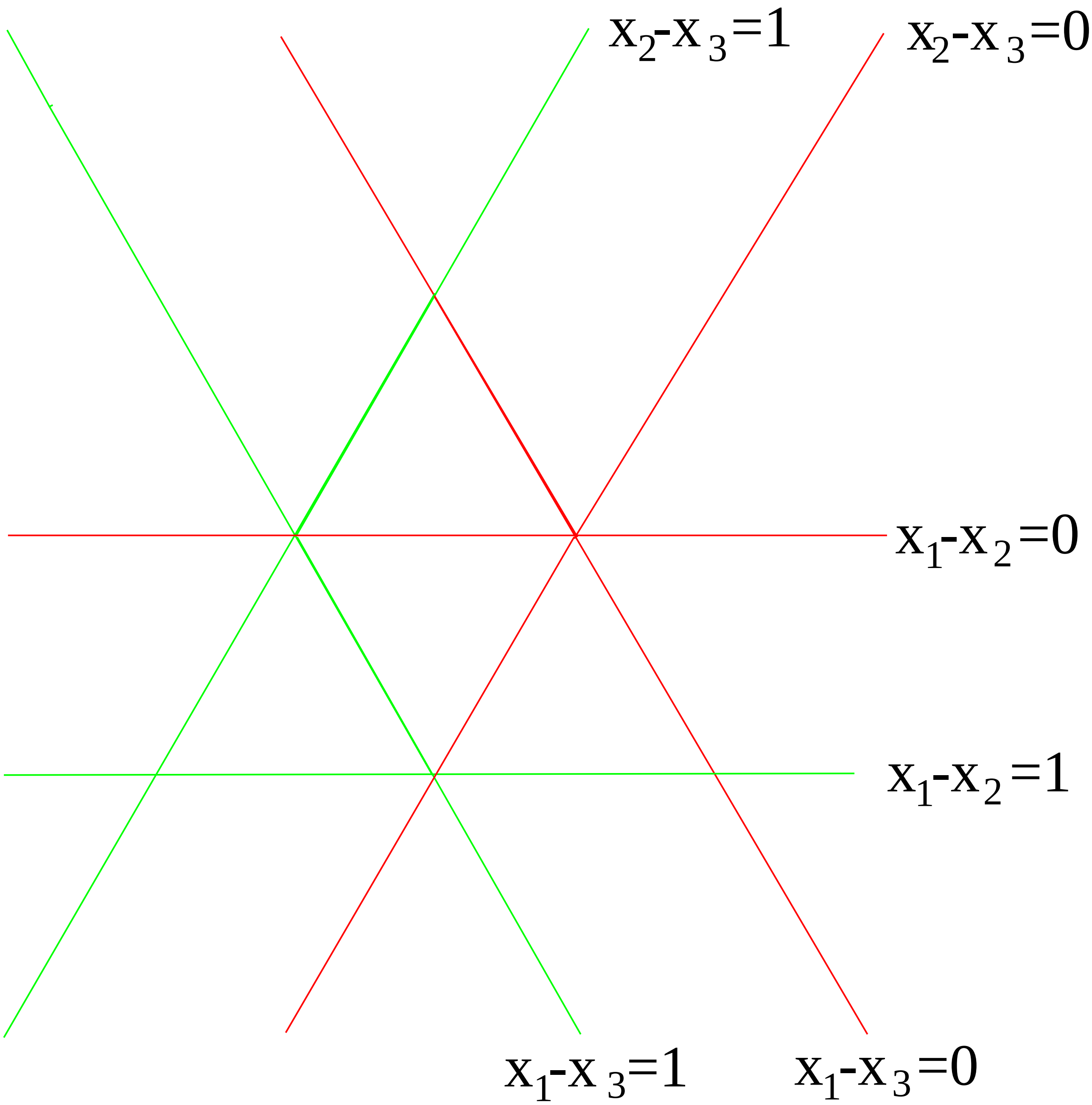
Hence

$$\begin{aligned} D_{\mathcal{B}_n}(q) &= \sum_{a_1=0}^{n-1} \cdots \sum_{a_n=0}^{0} q^{a_1+\cdots+a_n} \\ &= (1+q)\cdots(1+q+\cdots+q^{n-1}). \end{aligned}$$

The Shi arrangement

$\mathcal{S}_n: \; x_i - x_j = 0, 1, \quad 1 \le i < j \le n \; (\text{in } \mathbb{R}^n)$

(after J.-Y. Shi, 1986)

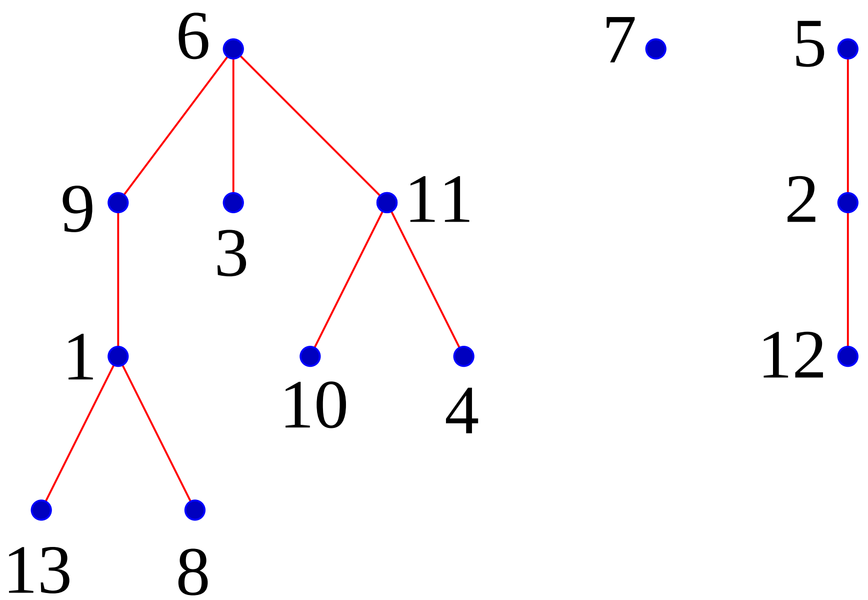

x_2-x_3=1
x_2-x_3=0
x_1-x_2=0
x_1-x_2=1
x_1-x_3=1
x_1-x_3=0

Theorem (Shi). $r(\mathcal{S}_n) = (n+1)^{n-1}$, the number of **rooted forests** on n (or unrooted trees on $n+1$ vertices).

Later proofs by Headley, Lewis, Pak-Stanley, Athanasiadis-Linusson, Postnikov, *et al.*

inversion of a forest:

$(i, j), \ i > j, \ i$ above j

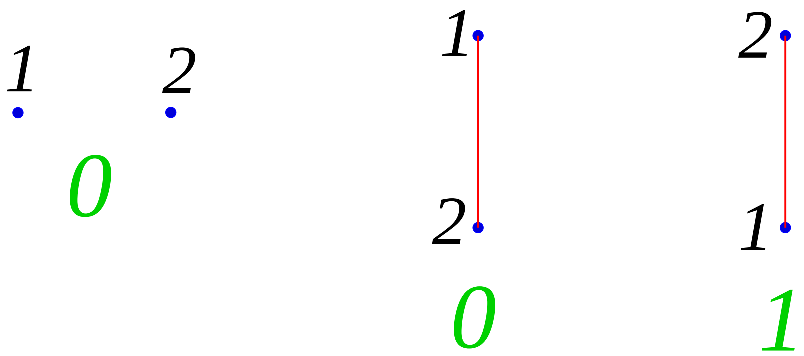


inversions: (6,1), (6,3), (6,4), (9,1), (9,8), (11,10), (11,4), (5,2)

inv(F) = 8

$$I_n(q) := \sum_{\substack{F=\text{rooted forest} \\ \text{on } 1,\ldots,n}} q^{\mathrm{inv}(F)}.$$

$$\begin{aligned} I_1(q) &= 1 \\ I_2(q) &= 2 + q \\ I_3(q) &= 6 + 6q + 3q^2 + q^3 \end{aligned}$$



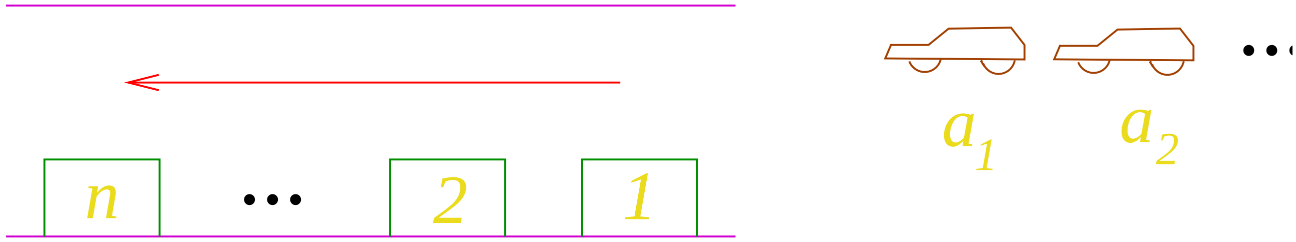
Theorem. (a) $I_n(1+q) =$

$$\sum_{\substack{\text{connected graphs} \\ \text{on } 1,2\ldots n}} q^{n+\#(\text{edges})}$$

(b) $I_n(q)(q-1)^n \dfrac{x^n}{n!} =$

$$\left(\sum_{n \geq 0} q^{\binom{n+1}{2}} \frac{x^n}{n!} \right) \Big/ \left(\sum_{n \geq 0} q^{\binom{n}{2}} \frac{x^n}{n!} \right)$$

Parking functions



Car C_i prefers space a_i. If a_i is occupied, then C_i takes the next available space. If all the cars can park, then $(a_1, \ldots, a_n)$ is a **parking function** (Konheim and Weiss, 1966).

Easy theorem. *Let* $b_1 \geq b_2 \geq \cdots \geq b_n$ *be the decreasing rearrangement of* $(a_1, \ldots, a_n) \in \mathbb{P}^n$. *Then* $(a_1, \ldots, a_n)$ *is a parking function if and only if* $b_i \leq n - i$.

Theorem (H. Pollak). *Let*

$$G = \mathbb{Z}/(n+1)\mathbb{Z} = \{1, 2, \ldots, n+1\}.$$

Then each coset of the subgroup of G^n *generated by* $(1, 1, \ldots, 1)$ *contains a unique parking function.*

Corollary (Konheim-Weiss) *There are* $P(n) = (n+1)^{n-1}$ *parking functions of length* n*.*

Theorem (G. Kreweras). $q^{\binom{n}{2}} I_n(1/q) =$

$$\sum_{\substack{\text{parking functions} \\ (a_1,\ldots,a_n)}} q^{a_1+\cdots+a_n-n}$$

Labelling the Shi arrangement

(conjectured by I. Pak)

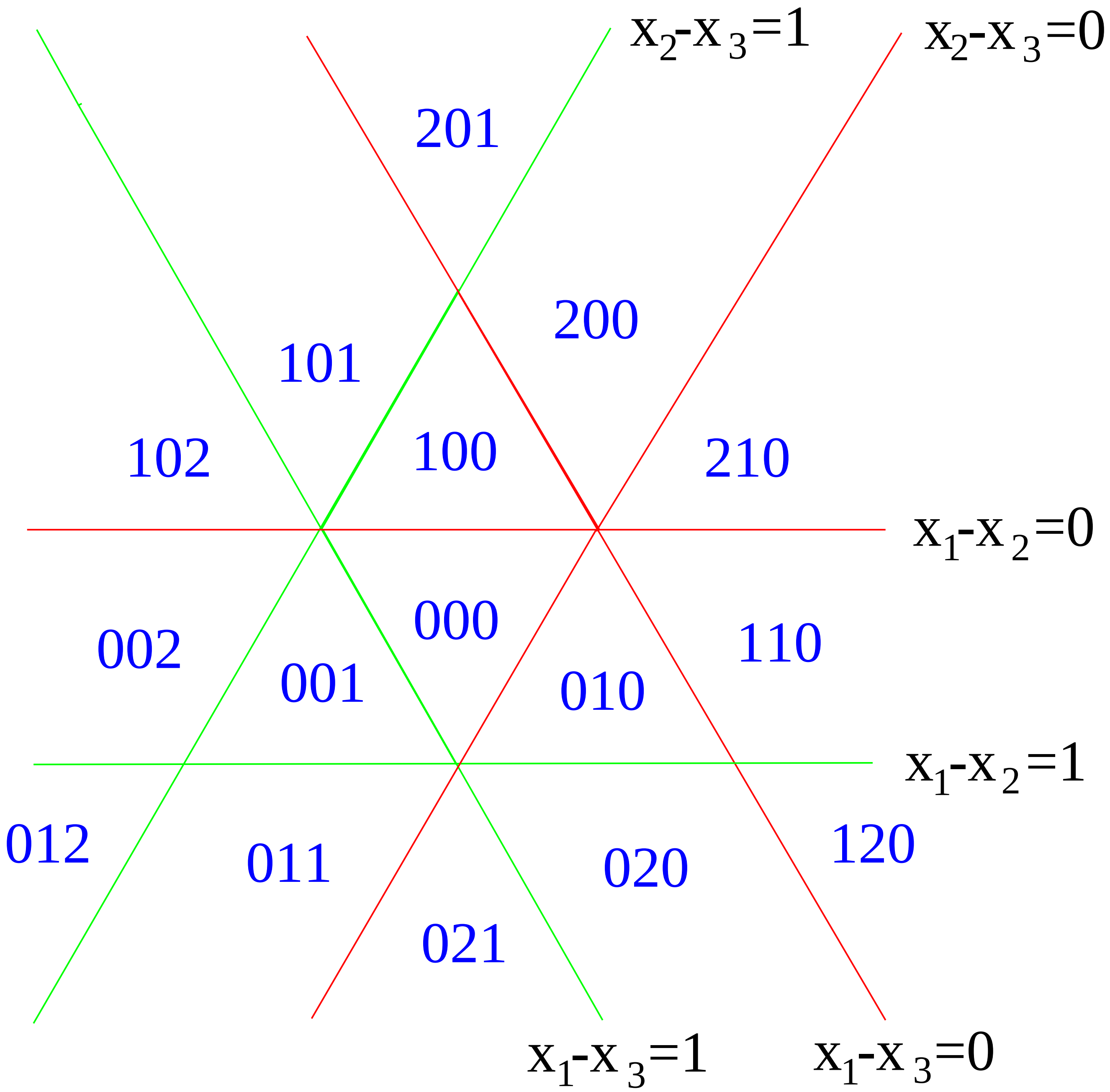
base region $R_0: \quad x_1 > \cdots > x_n$

- $\lambda(R_0) = (1, 1, \ldots, 1) \in \mathbb{Z}^n$
- If R is labelled, R' is separated from R only by $x_i - x_j = 0$ $(i < j)$, and R' is unlabelled, then set

$$\lambda(R') = \lambda(R) + e_i,$$

- If R is labelled, R' is separated from R only by $x_i - x_j = 1$ $(i < j)$, and R' is unlabelled, then set

$$\lambda(R') = \lambda(R) + e_j,$$



x2-x3=1
x2-x3=0
201
200
101
100
102
210
x1-x2=0
000
002
001
010
110
x1-x2=1
012
011
020
120
021
x1-x3=1
x1-x3=0

Theorem (R.S.) *The labels of the regions of $\mathcal{S}_n$ are just the parking functions of length n (each occurring once), with entries decreased by one.*

Corollary. $D_{\mathcal{S}_n}(q) = q^{\binom{n}{2}} I_n(1/q)$

Generalizations of the Shi arrangement

Let $k \geq 1$. Define the **extended Shi arrangement** $\mathcal{S}_n^k$ by

$$x_i - x_j = -(k-1),\ -(k-2), \ldots, k,$$
$$1 \leq i < j \leq n,$$

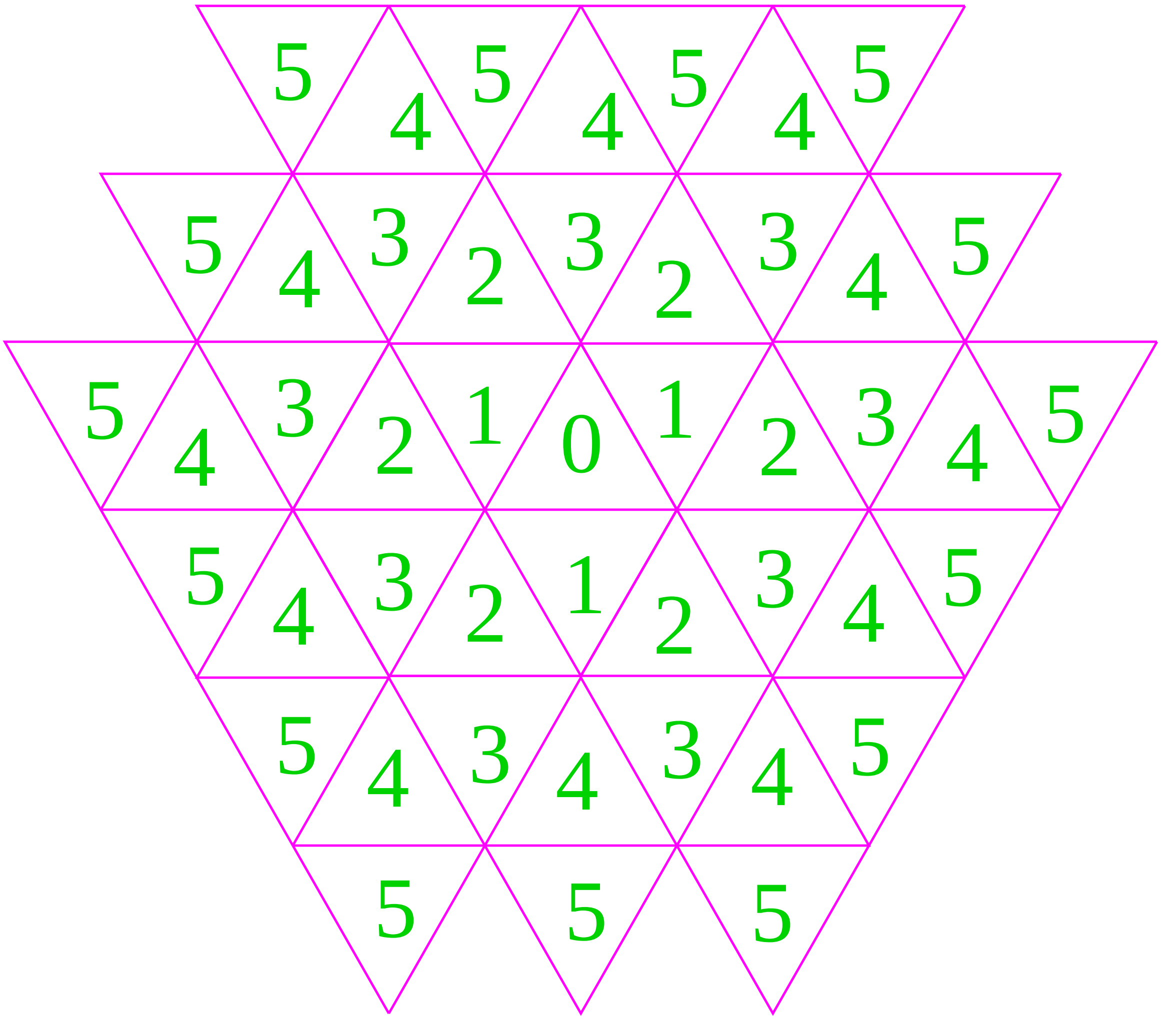
so $\mathcal{S}_n^1 = \mathcal{S}_n$.

All properties of $\mathcal{S}_n$ extend elegantly to $\mathcal{S}_n^k$:

- inversions of k-trees
- k-analogues of connected graphs
- k-parking functions
- labelling rule

When $k \to \infty$ we get the **affine braid arrangement**

$$\tilde{\mathcal{B}}_n : x_i - x_j = k \in \mathbb{Z}, \quad 1 \le i < j \le n.$$



$$D(q) = 1 + 3q + 6q^2 + 9q^3 + 12q^4 + 15q^5 + \cdots$$
$$= (1 + q + q^2)/(1 - q)^2$$

Labelling rule for $\tilde{\mathcal{B}}_n$:

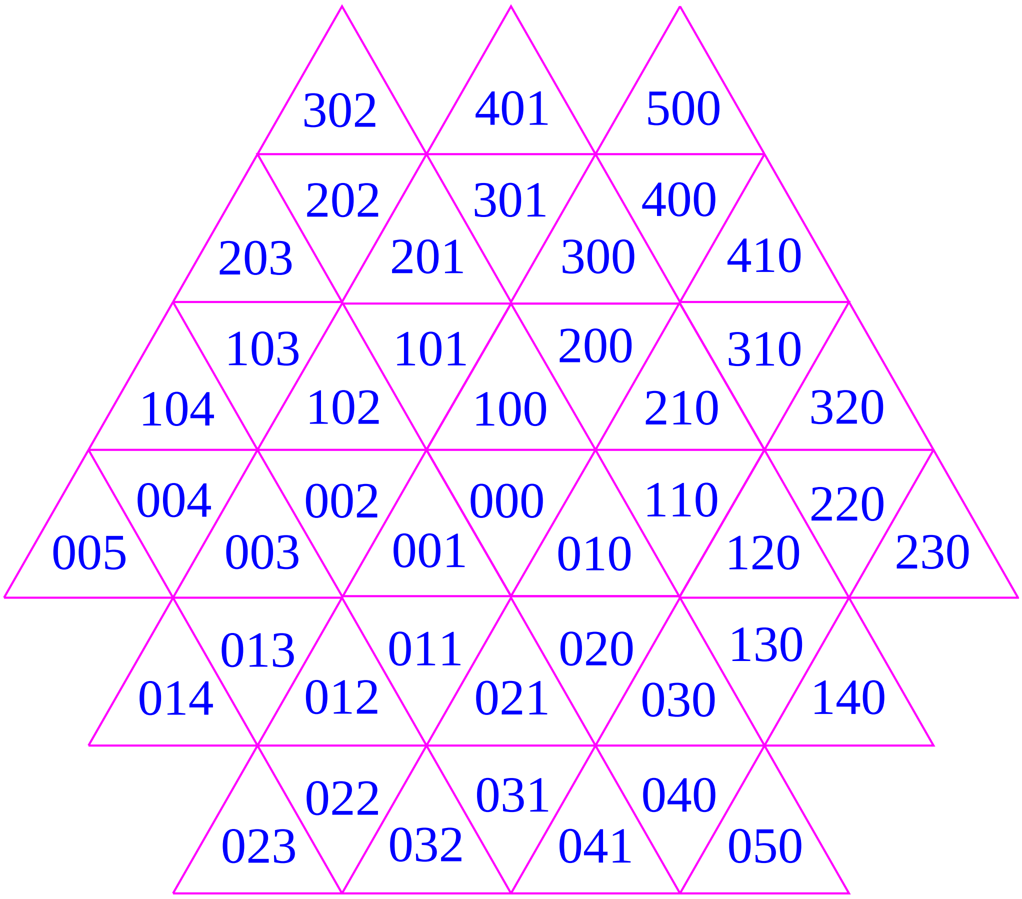
base region $R_0: \quad x_1 > \cdots > x_n$

- $\lambda(R_0) = (1, 1, \ldots, 1) \in \mathbb{Z}^n$
- If R is labelled, R' is separated from R only by $x_i - x_j = k \leq 0$ $(i < j)$, and R' is unlabelled, then set

$$\lambda(R') = \lambda(R) + e_i,$$

- If R is labelled, R' is separated from R only by $x_i - x_j = k > 0$ $(i < j)$, and R' is unlabelled, then set

$$\lambda(R') = \lambda(R) + e_j,$$


302
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320
004
002
000
110
220
005
003
001
010
120
230
013
011
020
130
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012
021
030
140
022
031
040
023
032
041
050

Theorem. *The labels of* $\mathcal{R}_{\tilde{\mathcal{B}}_n}$ *are the sequences* $(a_1, \ldots, a_n) \in \mathbb{N}^n$ *with at least one zero. Each label appears exactly once.*

Corollary (Bott, 1956)

$$D_{\tilde{\mathcal{B}}_n}(q) = \frac{1}{(1-q)^n} - \frac{q^n}{(1-q)^n}$$
$$= \frac{1+q+\cdots+q^{n-1}}{(1-q)^{n-1}}$$

Other arrangements

Catalan arrangement

$$\mathcal{C}_n : \quad x_i = x_j = 0, \pm 1 \, (1 \leq i < j \leq n)$$

$$r(\mathcal{C}_n) = n! \, C_n,$$

where

$$C_n = \frac{1}{n+1}\binom{2n}{n} \quad \text{(Catalan number)}.$$

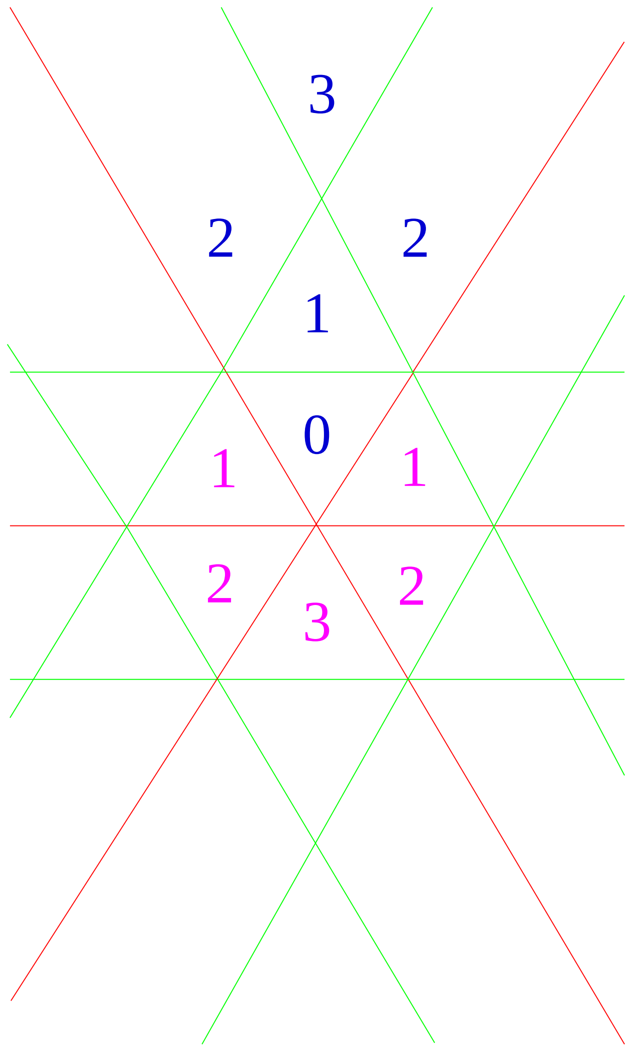
$$D_{\mathcal{C}_n}(q) = (n!)_q \, C_n(q),$$

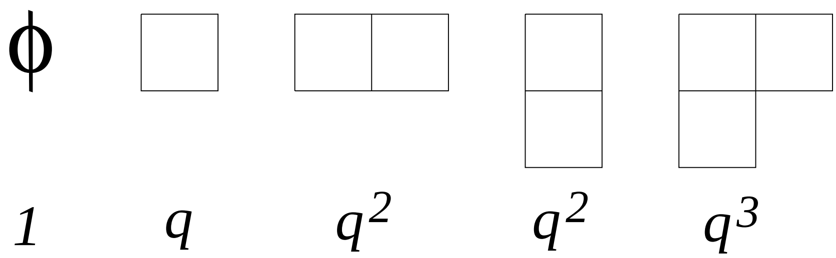
where

$$(n!)_q = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$$

$$C_n(q) = \sum_{\lambda} q^{|\lambda|},$$

where $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition with $\lambda_i \le n - i$.


3
2
2
1
0
1
1
2
2
3


ϕ
1
q
q2
q2
q3

Threshold arrangement

$$\mathcal{T}_n : \quad x_i + x_j = 0 \ (1 \leq i < j \leq n)$$

$$\sum_{n \geq 0} r(\mathcal{T}_n) \frac{x^n}{n!} = \frac{e^x(1-x)}{2-e^x}$$

$$D_{\mathcal{T}_n(q)} = ??$$

NOTE: Somewhat nicer is the **augmented threshold arrangement**:

$$\mathcal{T}_n^0 : \mathcal{T}_n \text{ and } x_i = 0 \ (1 \leq i \leq n)$$

$$\sum_{n \geq 0}^{r} (\mathcal{T}_n^0) \frac{x^n}{n!} = \frac{e^x}{2-e^x}$$

Linial arrangement:

$$\mathcal{L}_n : \quad x_i - x_j = 1 \ (1 \leq i < j \leq n)$$

$r(\mathcal{L}_n) = \#$**alternating trees** on $\{0, 1, \ldots, n\}$, i.e., every vertex is $<$ all its neighbors or $>$ all its neighbors.

$$y = \sum_{n \geq 0} r(\mathcal{L}_n) \frac{x^n}{n!}$$

$$\Rightarrow y = \exp\left(\frac{x}{2}(y+1)\right).$$

$$D_{\mathcal{L}_n}(q) = ??$$

B_n-Shi arrangement:

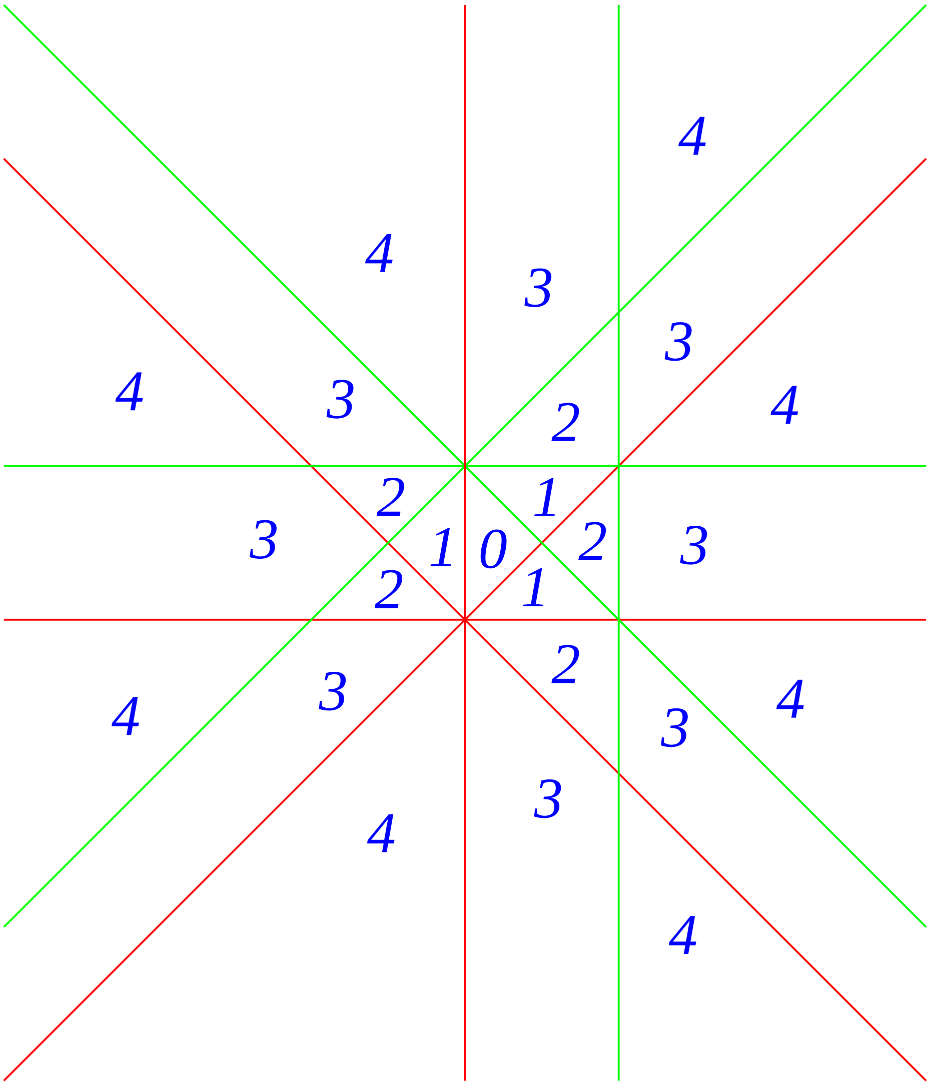
$$\mathcal{S}_n^B : \quad \begin{aligned} x_i - x_j &= 0, 1, \quad 1 \le i < j \le n \\ x_i + x_j &= 0, 1, \quad 1 \le i < j \le n \\ 2x_i &= 0, 1, \quad 1 \le i \le n \end{aligned}$$

$$\chi(\mathcal{S}_n^B, x) = (x - 2n)^n$$

$$r(\mathcal{S}_n^B) = (2n + 1)^n$$

$$D_{\mathcal{S}_n^B}(q) = ??$$

Similarly for C_n-Shi, BC_n-Shi, and D_n-Shi arrangements.



$$1 + 3q + 5q^2 + 8q^3 + 8q^4$$

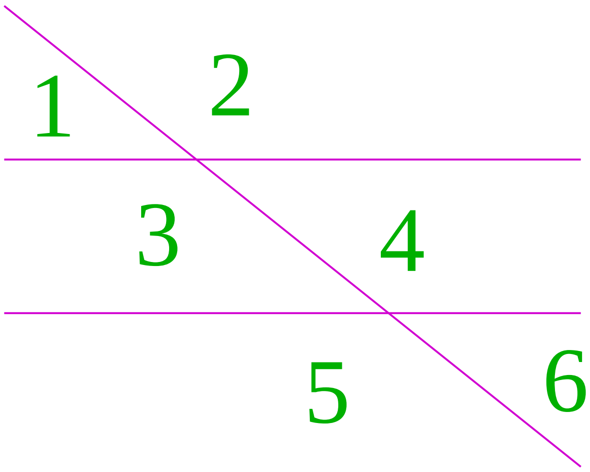
B_2 - Shi

Archilochus (died *c.* 652 BC):

"The fox knows many things, but the hedgehog knows one big thing."

Distance matrix $M = M_{\mathcal{A}}$ of $\mathcal{A}$:
For $R, R' \in \mathcal{R}$,

$$\mathbf{M}_{RR'} = q^{d(R,R')}.$$



	1	2	3	4	5	6
1	1	q	q	q^2	q^2	q^3
2	q	1	q	q^2	q^3	q^2
3	q	q^2	1	q	q	q^2
4	q^2	q	q	1	q^2	q
5	q^2	q^3	q	q^2	1	q
6	q^3	q^2	q^2	q	q	1

Theorem (Varchenko).

$$\det M = \prod_{i=1}^{n-1} \left(q^{i(i+1)} - 1\right)^{c_i}.$$

Recall that the **Smith normal form** (SNF) of M is a canonical form for AMB, where $A, B \in \mathrm{GL}(n, \mathbb{Z})$, $\det A = \pm 1$, $\det B = \pm 1$. It has the form

$$\mathrm{diag}(p_1(q), \ldots, p_n(q)),$$

where $p_i | p_{i+1}$. Note

$$p_1(q) \cdots p_n(q) = \pm \det M.$$

SNF of $M_{\mathcal{A}}$ not known in general, even for the braid arrangement. However:

Theorem (Denham-Hanlon). *Let a_i be the number of diagonal entries of the SNF of $M_{\mathcal{A}}$* exactly *divisible by $(q-1)^i$. Then*

$$\chi(\mathcal{A}, x) = \sum_i (-1)^i a_i x^{n-i},$$

the characteristic polynomial of $\mathcal{A}$.

What about the highest power of $q+1$ dividing the SNF entries?

Transparencies available at:

http://www-math.mit.edu/~rstan/trans.html

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