Chains and Antichains

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Write $t \geq s$ for $s \leq t$, $s < t$ for $s \leq t, s \neq t$, etc.

$s \| t$: s and t are **incomparable** (neither $s \leq t$ nor $t \leq s$)

chain of length n: $t_0 < t_1 < \cdots < t_n$

t **covers** s, s is **covered by** t: $s < t$, $\nexists u : s < u < t$. Denoted $s \lessdot t$ or $t \gtrdot s$.

More terminology

saturated chain: $t_0 \lessdot t_1 \lessdot \cdots \lessdot t_n$

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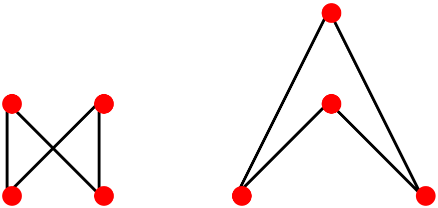
maximal chain: a chain C such that $t \notin C$ implies $C \cup \{t\}$ is not a chain

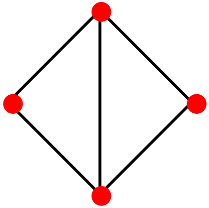
Maximal chains in a finite poset are saturated, but not conversely.

Hasse diagram

Hasse diagram of P: elements of P are drawn in the plane. If $s < t$ then t is above (larger y-coordinate than) s. An edge is drawn between all pairs $s \lessdot t$.



Hasse diagrams of isomorphic posets



Not a Hasse diagram

Unions of chains

Suppose $P = C_1 \cup \cdots \cup C_k$, where C_i is a chain. Let A be any antichain. Since $\#(C_i \cap A) \leq 1$, we have $k \geq \#A$. Thus:

Proposition. *Let k be the least integer such that P is a union of k chains. Let m be the size of the largest antichain of P. Then $k \geq m$.*

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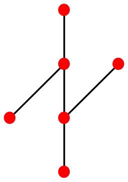

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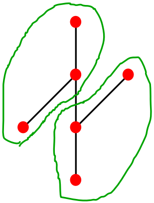

Proof of Dilworth's theorem (Galvin, 1994)

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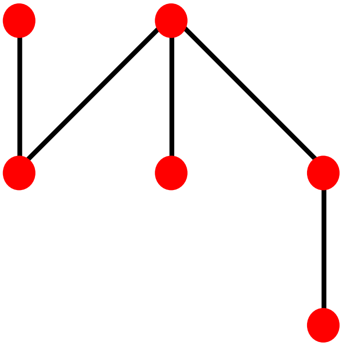
Let $P' = P - \{t\}$. By induction, let P' have an antichain A_0 of size k and a covering by chains $C_1, \dots, C_k$. Can assume $C_i \cap C_j = \varnothing$ for $i \neq j$. Now $A_0 \cap C_i \neq \varnothing$ for $1 \leq i \leq k$. For $1 \leq i \leq k$, let s_i be the maximal element of C_i that belongs to an antichain of size k in P', and set $A = \{s_1, \dots, s_k\}$.

Claim. A is an antichain.

Proof. Let A_i be an antichain of size k that contains s_i. Fix $j \neq i$. Then $A_i \cap C_j \neq \varnothing$. Let $u \in A_i \cap C_j$. Then $u \leq s_j$ by definition of s_j. Now $s_i \neq u$ since $s_i \in C_i$ and $u \in C_j$. Also $s_i \not> u$ since A_i is an antichain. Hence $s_i \not\geq u$. Since $u \leq s_j$, we have $s_i \not\geq s_j$. By symmetry, also $s_j \not\geq s_i$. Thus $s_i \parallel s_j$, so A is an antichain.

An example of the antichain A



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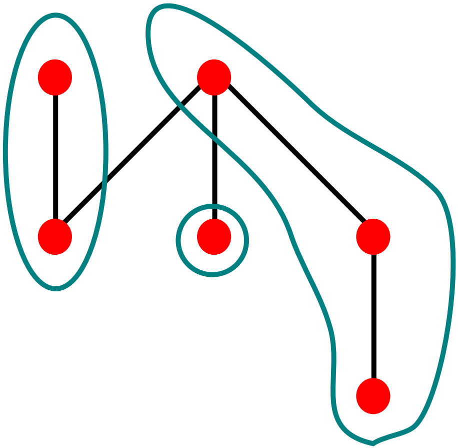

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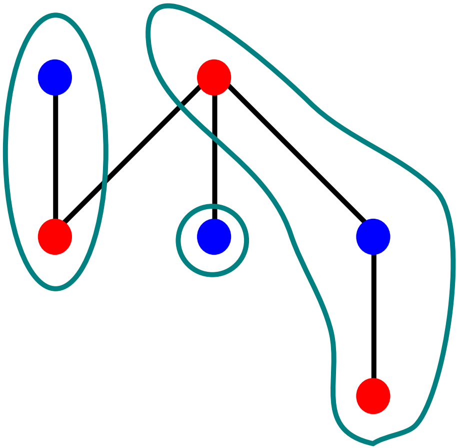

Conclusion of proof

Return to P.

Case 1. $t \geq s_i$ for some $1 \leq i \leq k$.

K: the chain $\{t\} \cup \{u \in C_i : u \leq s_i\}$

By definition of s_i, $P - K$ does not have an antichain of size k. Since $A - \{s_i\}$ is an antichain of size $k - 1$ in $P - K$, $P - K$ is a union of $k - 1$ chains (by the induction hypothesis). Thus P is a union of k chains.

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Case 2. Now $t \ngeq s_i$ for all $1 \leq i \leq k$. Thus $A \cup \{t\}$ is an antichain of size $k + 1$ in P (since t is maximal in P, so $t \nleq s_i$). Then P is a union of the $k + 1$ chains $\{t\}, C_1, \ldots, C_k$. $\square$

“Dual” of Dilworth’s theorem

Suppose $P = A_1 \cup \cdots \cup A_k$, where A_i is an antichain. Let C be any chain. Since $\#(A_i \cap C) \leq 1$, we have $k \geq \#C$. Thus:

Proposition. *Let k be the least integer such that P is a union of k antichains. Let m be the size of the largest chain of P. Then $k \geq m$.*

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Proof. Let A_1 be the set of minimal elements of P, then A_2 the set of minimal elements of $P - A_1$, etc. This gives a decomposition of P into a union of m antichains. $\square$

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Note how trivial the proof is compared to Dilworth’s theorem!

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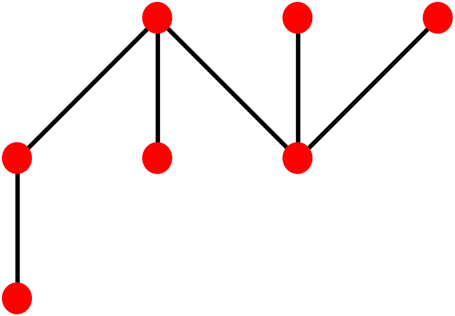

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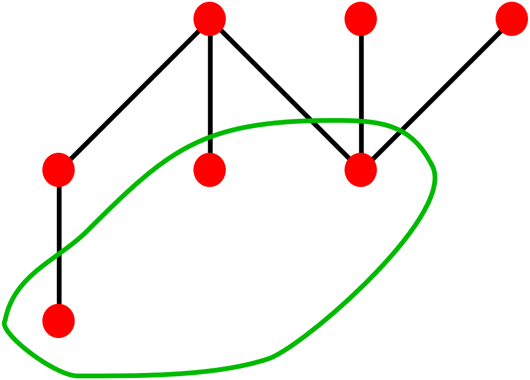

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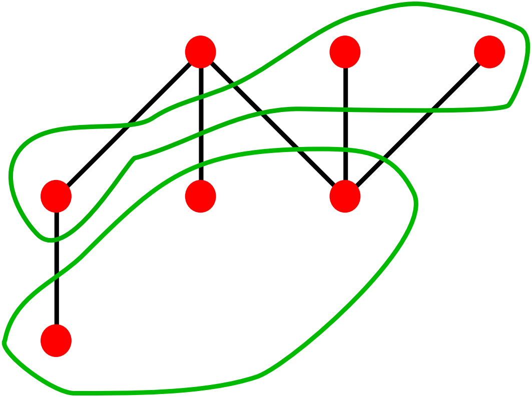

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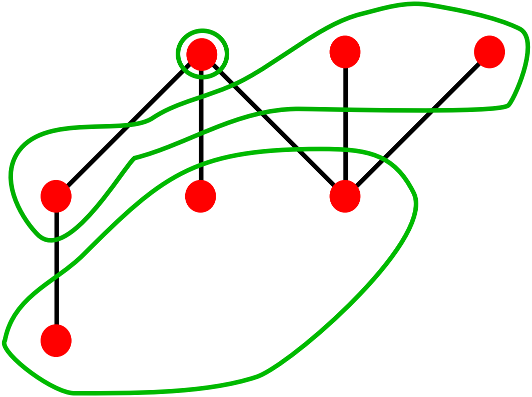

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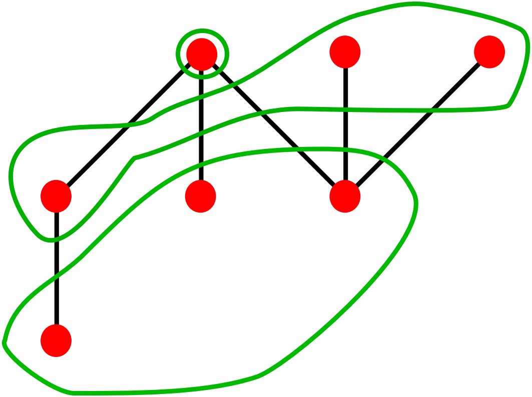

Note. Largest antichain of P has four elements.

Largest union of j chains

Define $\boldsymbol{\lambda_1, \lambda_2, \ldots}$ by:

The size of the largest union of j chains in P is $\lambda_1 + \lambda_2 + \cdots + \lambda_j$.

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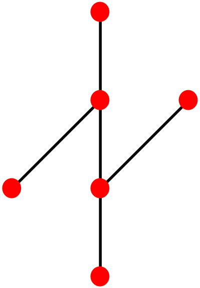
Clear (by Dilworth's theorem). Let $\#P = p$. Then $\lambda_i \geq 0$, and if the largest antichain of P has m elements, then

$$\begin{aligned} \lambda_1 + \cdots + \lambda_m &= p \\ \lambda_{m+1} = \lambda_{m+2} = \cdots &= 0. \end{aligned}$$

An example



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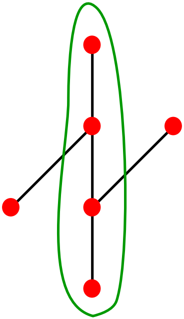

$$\lambda_1 = 4$$

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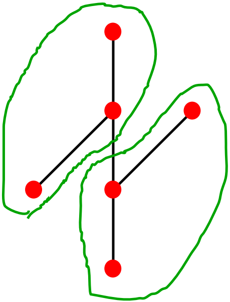


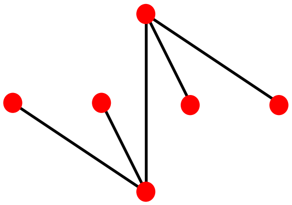
$$\begin{aligned} \lambda_1 &= 4 \\ \lambda_1 + \lambda_2 &= 6 \Rightarrow \lambda_2 = 2 \end{aligned}$$

Largest union of j antichains

Completely analogous definition for antichains:
Define $\boldsymbol{\mu_1, \mu_2, \ldots}$ by:

The size of the largest union of j antichains in P is $\mu_1 + \mu_2 + \cdots + \mu_j$.

An example



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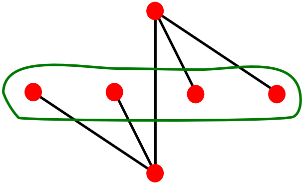

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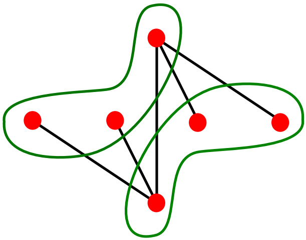

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Greene's theorem (1976)

P: p-element poset

Theorem. (a) $\lambda_1 \geq \lambda_2 \geq \cdots$ *and* $\mu_1 \geq \mu_2 \geq \cdots$. *In other words,* $\boldsymbol{\lambda(P)} = (\lambda_1, \lambda_2, \dots)$ *and* $\boldsymbol{\mu(P)} = (\mu_1, \mu_2, \dots)$ *are* **partitions** *of* p.

(b) $\lambda(P)' = \mu$, *where* $\boldsymbol{\lambda(P)'}$ *is the* **conjugate** *partition to* $\lambda(P)$.

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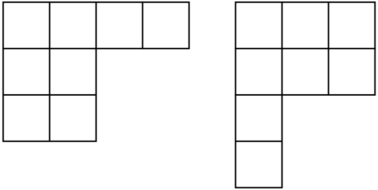

$\lambda = (4,2,2)$ $\lambda' = (3,3,1,1)$

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Dual Dilworth's theorem. $\lambda_1 = \mu'_1$

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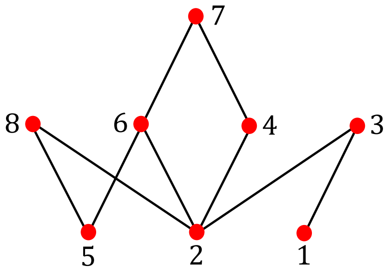

$w = 52864713$

Increasing and decreasing subsequences

If $i_1 < i_2 < \cdots < i_k$ is a chain in I_w, then $i_1, i_2, \ldots, i_k$ is an **increasing subsequence** of w.

If $i_1 <_{\mathbb{Z}} i_2 <_{\mathbb{Z}} \cdots <_{\mathbb{Z}} i_k$ is an antichain in I_w, then $i_k, \ldots, i_2, i_1$ is a **decreasing subsequence** of w.

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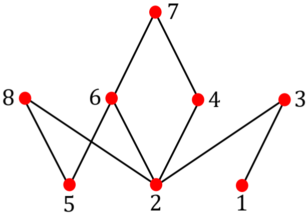

$w = 52864713$

$5, 6, 7$ is an increasing subsequence

$8, 6, 4, 3$ is a decreasing subsequence

Corollary to Greene's theorem

Given $w \in \mathfrak{S}_n$, let $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ be largest size of the union of k increasing subsequences of w, and let $\mu_1 + \mu_2 + \cdots + \mu_k$ be the largest size of the union of k decreasing subsequences of w.

Corollary (**Greene**, 1974). *Both* $\lambda = (\lambda_1, \lambda_2, \dots)$ *and* $\mu = (\mu_1, \mu_2, \dots)$ *are partitions of* n*, and* $\mu = \lambda'$*.*

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Note (**Greene**). If $w \overset{\mathrm{rsk}}{\rightarrow} (P, Q)$, then $\mathrm{shape}(P) = \mathrm{shape}(Q) = \lambda$.

Natural question

Natural question. Given a (finite) poset P, determine λ and μ.

Note. Even determining μ_1 (the size of the largest antichain) is interesting and subtle. For instance, if Π_n is the lattice of partitions of an n-set, then $\mu_1(\Pi_n)$ is not known.

Some definitions

P is **graded of rank n** if $P = P_0 \dot\cup P_1 \dot\cup \cdots \dot\cup P_n$ (disjoint union) and every maximal chain has the form $t_0 < t_1 < \cdots < t_n$, where $t_i \in P_i$. The set P_i is the ith **level** or ith **rank** of P.

Let $\boldsymbol{p_i} = \#P_i$. If P is graded of rank n, then P is **rank-symmetric** if $p_i = p_{n-i}$ for all i, and **rank-unimodal** if

$$p_0 \le p_1 \le \cdots \le p_j \ge p_{j+1} \ge \cdots \ge p_n$$

for some j.

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Note. Rank-symmetric and rank-unimodal implies $j = \lfloor n/2 \rfloor$.

The strong Sperner property

P graded of rank n, $p_i = \#P_i$

Definition. P is **strongly Sperner** (or has the **strong Sperner property**) if $\mu(P) = \mathrm{sort}_{\geq}(p_0, p_1, \ldots, p_n)$

Symmetric chain decompositions

P: finite, graded of rank n, rank-symmetric, rank-unimodal

symmetric chain decomposition: $P = C_1 \dot\cup C_2 \dot\cup \cdots \dot\cup C_k$, where each C_i is a **saturated** chain symmetric about the middle level (n even) or middle two levels (n odd)

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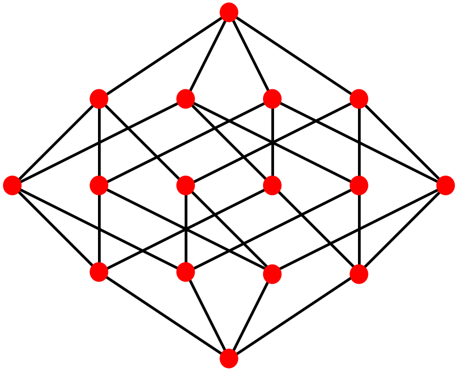
Exercise. If P has a symmetric chain decomposition, then P is strongly Sperner.

An example: the boolean algebra B_4



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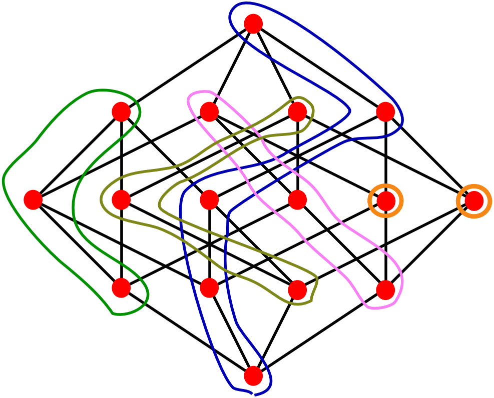

Boolean algebra B_4

$$\lambda(P) = (5,3,3,3,1,1), \quad \mu(P) = (6,4,4,1,1)$$

Which posets have symmetric chain decompositions?

Examples of posets with SCD.

- p-element chains $\boldsymbol{p}$

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- p-element chains $\boldsymbol{p}$
- P, Q SCD $\Rightarrow P \times Q$ SCD

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- the Bruhat order of a finite Coxeter group

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B_n: subsets of $\{1, \ldots, n\}$, ordered by $\subseteq$

Example of one of the chains for $n = 10$. Put n spaces in a line:

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Choose a set of well-formed left and right parentheses

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Start with positions of left parentheses (2,3,9). Adjoin blank positions one at a time from right-to-left:

$$239 \subset 2389 \subset 23789 \subset 236789 \subset 1236789$$

The poset B_n/G

The symmetric group $\mathfrak{S}_n$ acts on B_n by

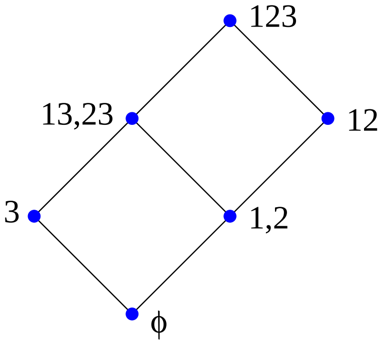
$$w \cdot \{a_1, \ldots, a_k\} = \{w \cdot a_1, \ldots, w \cdot a_k\}.$$

If G is a subgroup of $\mathfrak{S}_n$, define the **quotient poset** B_n/G to be the poset on the orbits of G (acting on B_n), with

$$\mathfrak{o} \le \mathfrak{o}' \quad \Leftrightarrow \quad \exists S \in \mathfrak{o}, T \in \mathfrak{o}', \quad S \subseteq T.$$

An example

$$n = 3, \quad G = \{(1)(2)(3), (1,2)(3)\}$$



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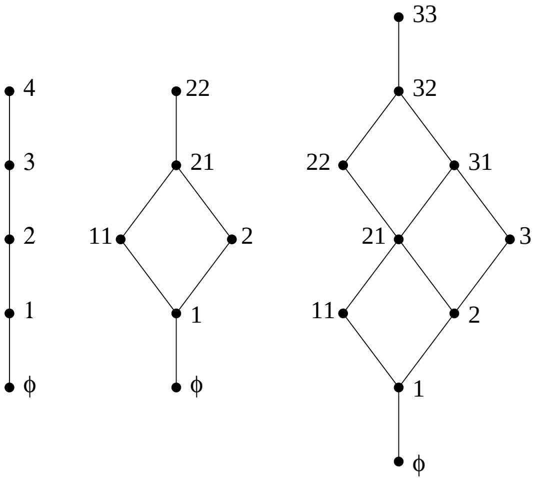
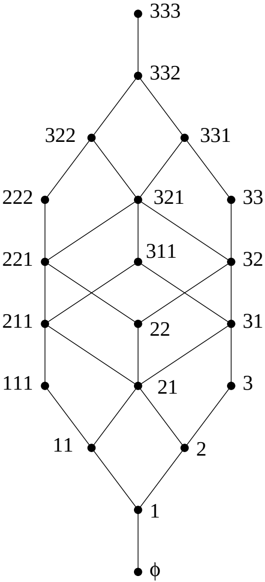
Note. $L(m, n)$ is graded of rank mn.

$L(1,4)$, $L(2,2)$, $L(2,3)$



$L(3,3)$



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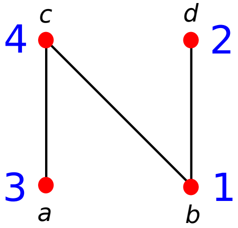
Reference. RS, *Algebraic Combinatorics*, Chapters 5–6.

Linear extensions

A **linear extension** of a p-element poset P is a bijection $\varphi\colon P \to \{1,\ldots,p\}$ such that $s <_P t \Rightarrow \varphi(s) <_{\mathbb{Z}} \varphi(t)$.

Can identify φ with the permutation $t_1,\ldots,t_p$ of the elements of P by $t_i = \varphi^{-1}(i)$.

An example



$a\ b\ c\ d$
$b\ a\ c\ d$
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$b\ a\ d\ c$
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Removing an element from P

Proposition. *Let $t \in P$. Then $\lambda(P)$ is obtained from $\lambda(P - t)$ by adding 1 to some part of $\lambda(P - t)$ or adding a new part equal to 1. Thus the diagram of $\lambda(P)$ is obtained from that of $\lambda(P - t)$ by adding a single box.*

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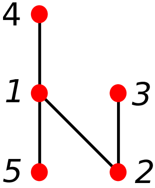
Corollary. *Let $t_1, \ldots, t_p$ be* **any** *ordering of the elements of P. Let $P_i = \{t_1, \ldots, t_i\}$ (a subposet of P). Then the sequence*

$$\varnothing, \lambda(P_1), \lambda(P_2), \ldots, \lambda(P_t)$$

defines a standard Young tableau (SYT) of shape $\lambda(P)$.

An example



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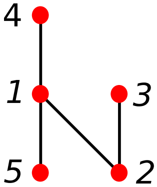

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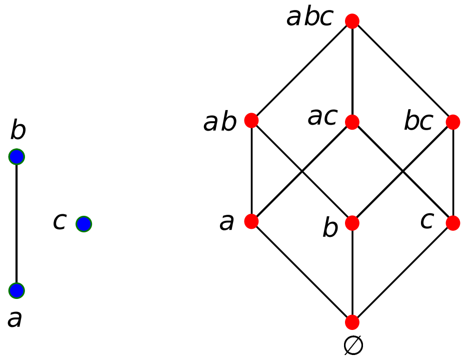
1	2	4
3	5	

A map from B_p to partitions



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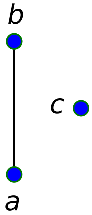

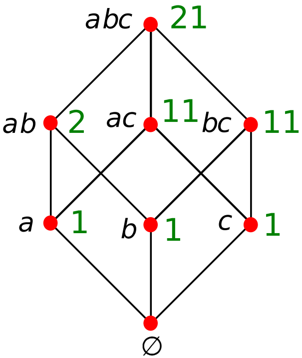

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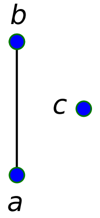

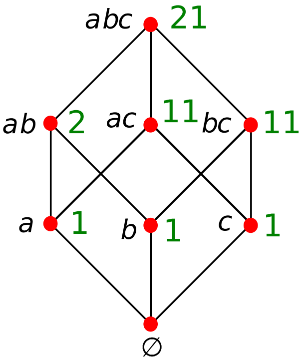

map from B_p to **Young's lattice** Y (partitions of all $n \geq 0$ ordered component-wise), order preserving, rank preserving

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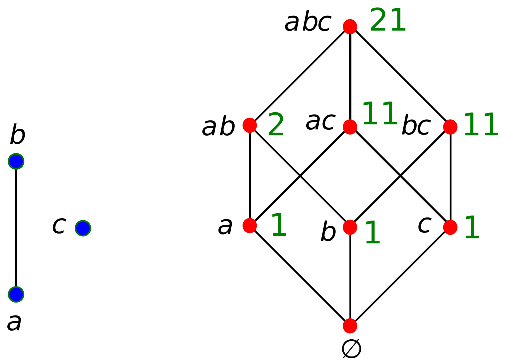

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Can anything be done with this?