A Fibonacci Analogue of Pascal's Triangle

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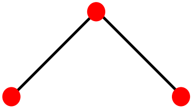
Note. P_{ib} is **upper homogeneous**, i.e., for all $t \in P_{ib}$, we have $\{s \in P_{ib} : s \geq t\} \cong P_{ib}$.

Construction of $\mathfrak{F} := P_{23}$



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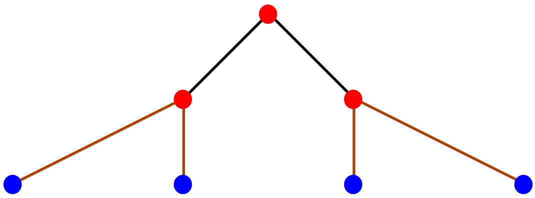

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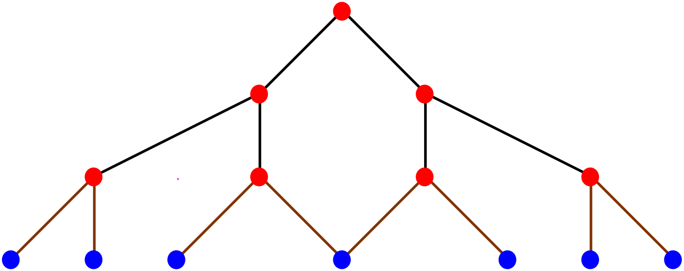

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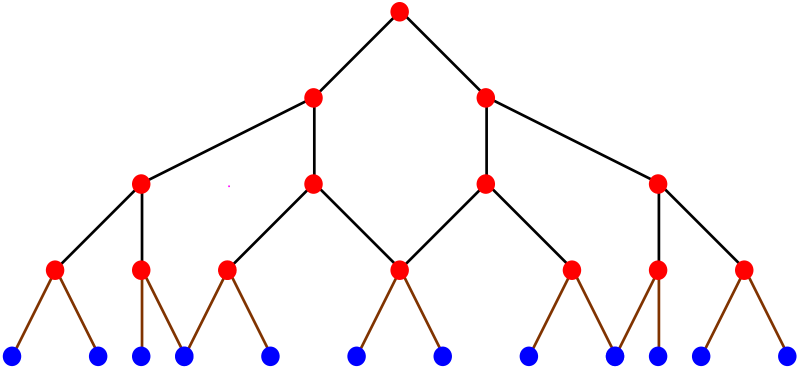

Fibonacci poset

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Initial conditions: $p_{ib}(n) = i^n$, $0 \le n \le b-1$

$$\Rightarrow \sum_{n\ge 0} p_{ib}(n)x^n = \frac{1}{1-ix+(i-1)x^b}.$$

The special case $i = 2, \; b = 3$

$$\begin{aligned} \sum_{n \geq 0} p_{23}(n) x^n &= \frac{1}{1 - 2x + x^3} \\ &= \frac{1}{(1-x)(1-x-x^2)} \\ \Rightarrow p_{23}(n) &= F_{n+2} - 1, \end{aligned}$$

where $F_1 = F_2 = 1, \; F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

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First connection with **Fibonacci numbers**.

The numbers $e(t)$

For $t \in P_{ib}$, let $\boldsymbol{e(t)}$ be the number of saturated chains from $\hat{0}$ to t.

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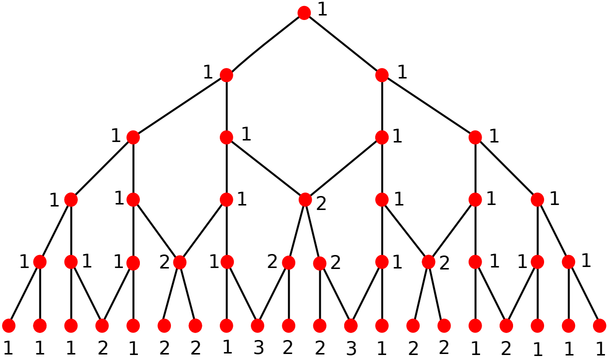
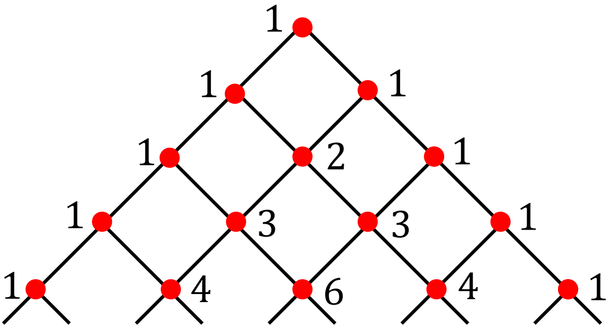
Example. $\mathfrak{F} = P_{23}$



A familiar example: P_{22}



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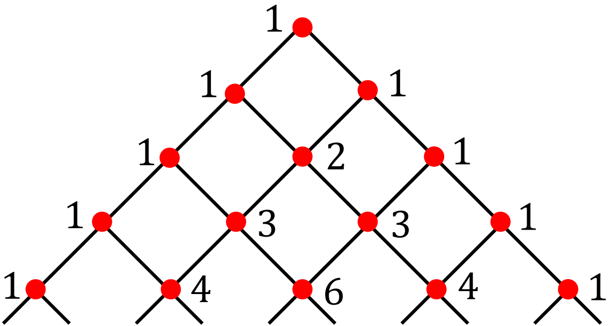

Pascal's triangle

A generating function for the $e(t)$'s

Fix i and b.

t_{nk}: kth element from left in the nth row of P_{ib}, beginning with $k = 0$.

$$\begin{bmatrix} n \\ k \end{bmatrix} = e(t_{nk})$$

q_n: number of elements of P_{ib} of rank n

$$r_n = \frac{q_n - q_{n-1}}{i-1} \in \mathbb{P} = \{1, 2, \dots\}$$

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Theorem. $\displaystyle\sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k = \prod_{j=1}^{n} \left(1 + x^{r_j} + x^{2r_j} + \cdots + x^{(i-1)r_j}\right)$

(analogue of binomial theorem, the case $i = b = 2$)

A Fibonacci product

Recall: $F_1 = F_2 = 1, \ F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$

$$I_n(x) = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$$

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$$\begin{aligned} I_4(x) &= (1+x)(1+x^2)(1+x^3)(1+x^5) \\ &= 1 + x + x^2 + 2x^3 + x^4 + 2x^5 + 2x^6 + x^7 + 2x^8 + x^9 + x^{10} + x^{11} \end{aligned}$$

When $i = 2, b = 3$ (so $P_{23} = \mathfrak{F}$), the previous theorem gives:

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k = I_n(x).$$

Sum of rth powers

$\boldsymbol{v_r(n)}$: sum of rth powers of coefficients of $I_n(x)$

$$\boldsymbol{V_r(x)} = \sum_{n\geq 0} v_r(n)x^n$$

Recursive structure of $\mathfrak{F}$ leads to a system of linear recurrences from which there follows:

Theorem. *For all $r \geq 0$, $V_r(x)$ is a rational function.*

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Compare Pascal's triangle ($i = b = 2$): $V_2(x)$ is algebraic but not rational, and $V_r(x)$ for $r \geq 3$ is D-finite but not algebraic.

Some small values of $V_r(x)$

Theorem. $V_1(x) = \dfrac{1}{1-2x}$

$$
\begin{aligned}
V_2(x) &= \frac{1-2x^2}{1-2x-2x^2+2x^3}\\
V_3(x) &= \frac{1-4x^2}{1-2x-4x^2+2x^3}\\
V_4(x) &= \frac{1-7x^2-2x^4}{1-2x-7x^2-2x^4+2x^5}\\
V_5(x) &= \frac{1-11x^2-20x^4}{1-2x-11x^2-8x^3-20x^4+10x^5}\\
V_6(x) &= \frac{1-17x^2-88x^4-4x^6}{1-2x-17x^2-28x^3-88x^4+26x^5-4x^6+4x^7}
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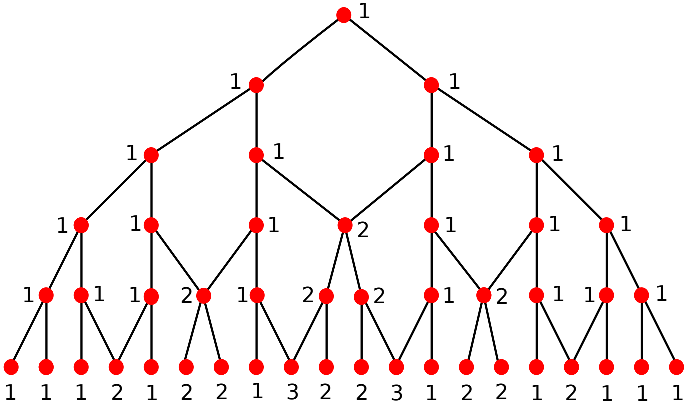
Note. Numerator is "even part" of denominator. Why?

Structure of two consecutive ranks



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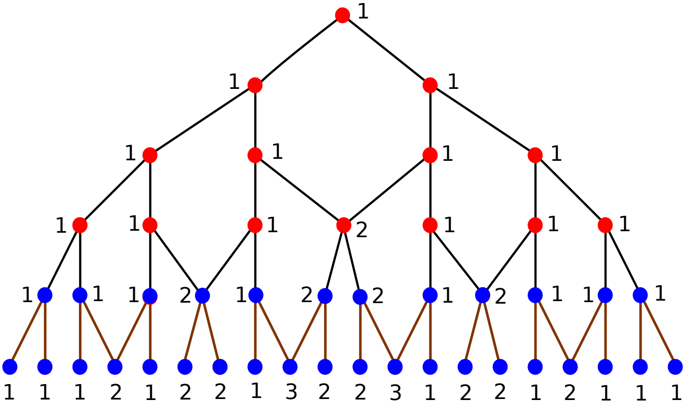

string sizes on last rank: $2, 3, 2, 3, 3, 2, 3, 2$

The limiting string size sequence

As $n \to \infty$, we get a "limiting sequence"

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Theorem. *The limiting sequence* $(c_1, c_2, \ldots)$ *is given by*

$$c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor .$$

Properties of $c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor$

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \ldots.$$

- $\gamma = (c_2, c_3, \ldots)$ characterized by invariance under $2 \to 3$, $3 \to 32$ (**Fibonacci word** in the letters 2,3).

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- $\gamma = z_1 z_2 \cdots$ (concatenation), where $z_1 = 3$, $z_2 = 23$, $z_k = z_{k-2} z_{k-1}$

$$3 \cdot 23 \cdot 323 \cdot 23323 \cdot 32323323 \cdots$$

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- Sequence of number of 3's between consecutive 2's is the original sequence with 1 subtracted from each term.

$$2 \underbrace{3}_{1} 2 \underbrace{33}_{2} 2 \underbrace{3}_{1} 2 \underbrace{33}_{2} 2 \underbrace{33}_{2} 2 \underbrace{3}_{1} 2 \underbrace{33}_{2} 2 \ldots.$$

Coefficients of $I_n(x)$

$$I_n(x) = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$$

Coefficient of x^m: number of ways to write m as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$.

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Example. Coefficient of x^8 in $(1+x)(1+x^2)(1+x^3)(1+x^5)(1+x^8)$ is 3:

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Can we see these sums from $\mathfrak{F}$? Each path from the top to a point $t \in \mathfrak{F}$ should correspond to a sum.

An edge labeling of $\mathfrak{F}$

The edges between ranks $2k$ and $2k+1$ are labelled alternately $0, F_{2k+2}, 0, F_{2k+2}, \ldots$ from left to right.

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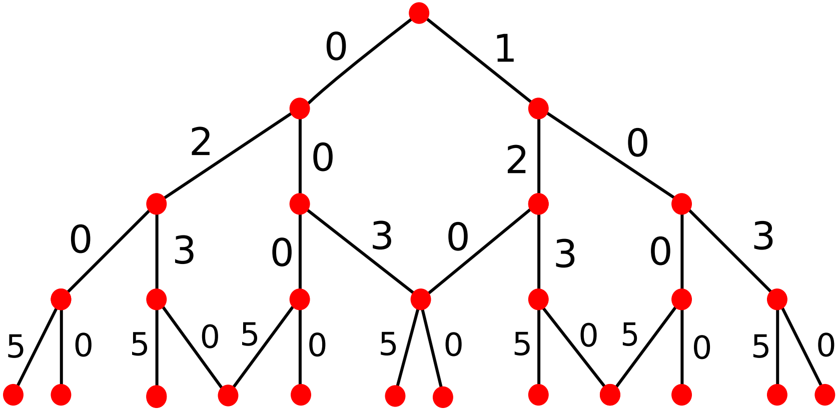
The edges between ranks $2k-1$ and $2k$ are labelled alternately $F_{2k+1}, 0, F_{2k+1}, 0, \ldots$ from left to right.

Diagram of the edge labeling



Connection with sums of Fibonacci numbers

Let $t \in \mathfrak{F}$. All paths (saturated chains) from the top to t have the same sum of their elements $\sigma(t)$.

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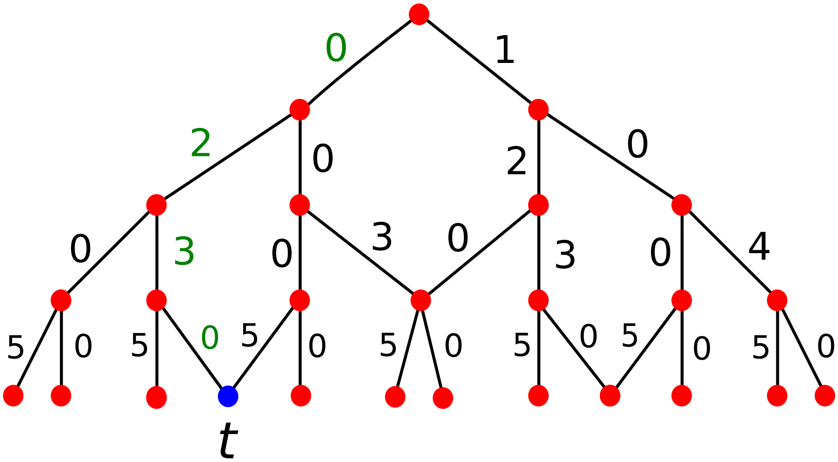
If $\mathrm{rank}(t) = n$, this gives all ways to write $\sigma(t)$ as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$.

An example



$$2 + 3 = F_3 + F_4$$

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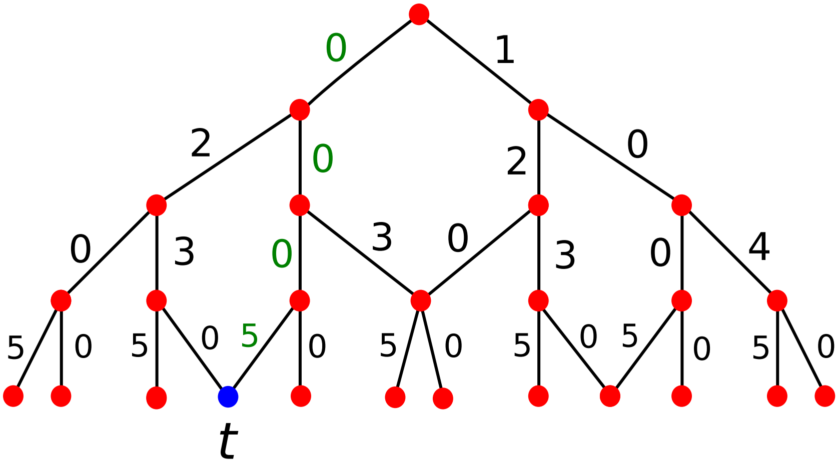
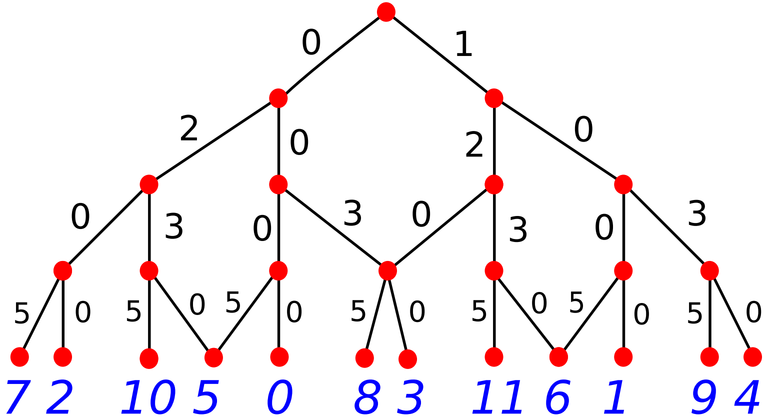


$$5 = F_5$$

An ordering of $\mathbb{N}$



In the limit as rank $\to \infty$, get an interesting (dense) linear ordering $\prec$ of $\mathbb{N}$.

When is $m \succ 0$?

Zeckendorf's theorem. *Every nonnegative integer has a unique representation as a sum of nonconsecutive Fibonacci numbers, where a summand equal to 1 is always taken to be* F_2.

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Theorem. *Let $m > 0$. Then $m \succ 0$ if and only the smallest Fibonacci number in the Zeckendorf representation of m has even index.*

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Theorem. *Let* $m > 0$. *Then* $m \succ 0$ *if and only the smallest Fibonacci number in the Zeckendorf representation of* m *has even index.*

Example. $45 \succ 0$ since F_4 has even index 4.

Second proof concerning $\sum \begin{bmatrix} n \\ k \end{bmatrix}^2$

Recall: for $P_{23} = \mathfrak{F}$, we define

$$
\begin{aligned}
v_2(n) &= \sum_{\substack{t \in \mathfrak{F} \\ \mathrm{rk}(t) = n}} e(t)^2 \\
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where $\prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right) = \sum_k c_k x^k$.

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Theorem. $V_2(x) := \sum_{n \geq 0} v_2(n) x^n = \dfrac{1 - 2x^2}{1 - 2x - 2x^2 + 2x^3}$

Tautological interpretation of $v_2(n)$

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where each a_i and b_i is 0 or 1.

A concatenation product

$$\mathcal{M}_n := \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} : \sum a_i F_{i+1} = \sum b_i F_{i+1} \right\}$$

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Let

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix} \in \mathcal{M}_n, \quad \beta = \begin{pmatrix} c_1 & \cdots & c_m \\ d_1 & \cdots & d_m \end{pmatrix} \in \mathcal{M}_m.$$

Define

$$\alpha\beta = \begin{pmatrix} a_1 & \cdots & a_n & c_1 & \cdots & c_m \\ b_1 & \cdots & b_n & d_1 & \cdots & d_m \end{pmatrix},$$

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Easy to check: $\alpha\beta \in \mathcal{M}_{n+m}$

The monoid $\mathcal{M}$

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Suppose $\mathcal{G}$ freely generates $\mathcal{M}$, and let $\boldsymbol{G(x)} = \sum_{n\geq 1} \#(\mathcal{M}_n \cap \mathcal{G})x^n$. Then

$$\begin{aligned} \sum_n v_2(n)x^n &= \sum_n \#\mathcal{M}_n \cdot x^n \\ &= 1 + G(x) + G(x)^2 + \cdots \\ &= \frac{1}{1-G(x)}. \end{aligned}$$

Free generators of $\mathcal{M}$

Theorem. *$\mathcal{M}$ is freely generated by the following elements:*

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & * & 1 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \\ 00 & * & 0 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \end{pmatrix}$$

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Example. $\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$: $1+2+3+5=3+8$

$G(x)$

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Let $\boldsymbol{k}$ be the number of columns of $*$'s. Length is $2k+3$. Thus

$$\begin{aligned} G(x) &= 2x + 2\sum_{k\geq 0} 2^k x^{2k+3} \\ &= 2x + \frac{2x^3}{1-2x^2}. \end{aligned}$$

Completion of proof

$$\begin{aligned}\sum_n v_2(n)x^n &= \frac{1}{1-G(x)} \\ &= \frac{1}{1-\left(2x+\frac{2x^3}{1-2x^2}\right)} \\ &= \frac{1-2x^2}{1-2x-2x^2+2x^3} \quad \square\end{aligned}$$

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Reference: **arXiv:2101.02131**

The End