CHAPTER 3

Residual, or Schwartz, algebra

The standard algebra of operators discussed in the previous chapter is not really representative, in its global behaviour, of the algebra of pseudodifferential operators on a compact manifold. Of course this can be attributed to the non-compactness of $\mathbb{R}^n$. However, as we shall see below in the discussion of the isotropic algebra, and then again in the later discussion of the scattering algebra, there are closely related global algebras of pseudodifferential operators on $\mathbb{R}^n$ which behave much more as in the compact case.

The 'non-compactness' of the algebra $\Psi^{\infty}_{\infty}(\mathbb{R}^n)$ is evidenced by the fact the the elements of the 'residual' algebra $\Psi^{-\infty}_{\infty}(\mathbb{R}^n)$ are not all compact as operators on $L^2(\mathbb{R}^n)$, or any other interesting space on which they act. In this chapter we consider a smaller algebra of operators in place of $\Psi^{-\infty}_{\infty}(\mathbb{R}^n)$. Namely

$$A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \Longleftrightarrow A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n),\quad A\phi(x) = \int_{\mathbb{R}^n} A(x,y)\phi(y)dy,\ A \in \mathcal{S}(\mathbb{R}^{2n}). \tag{3.1}$$

The notation here, as the residual part of the isotropic algebra – which has not yet been defined – is rather arbitrary. However it seems better than introducing a notation which will be retired later.

By definition then, $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is the algebra which corresponds to the non-commutative product on $\mathcal{S}(\mathbb{R}^{2n})$ given by

$$A \circ B(x,y) = \int_{\mathbb{R}^n} A(x,z)B(z,y)dz. \tag{3.2}$$

The properties we discuss here have little direct relation to the 'microlocal' concepts which are the central point of these notes. Rather they are more elementary, or at least familiar, results which are needed (and in particular are generalized) later in the discussion of global properties. In this sense this chapter could be considered more as an appendix.

3.1. The residual algebra

The residual algebra in both the isotropic and scattering calculi, discussed below, has two important properties not shared by the residual algebra $\Psi^{-\infty}_{\infty}(\mathbb{R}^n)$, of which it is a subalgebra (and in fact in which it is an ideal). The first is that as operators on $L^2(\mathbb{R}^n)$ the residual isotropic operators are compact.

Proposition 3.1. *Elements of $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ are characterized amongst continuous operators on $\mathcal{S}(\mathbb{R}^n)$ by the fact that they extend by continuity to define continuous linear maps*

$$A : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{3.3}$$

In particular the image of a bounded subset of $L^2(\mathbb{R}^n)$ *under an element of* $\Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ *is contained in a compact subset.*

PROOF. The kernels of elements of $\Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ are in $\mathcal{S}(\mathbb{R}^{2n})$ so the mapping property (3.3) follows.

The norm $\sup_{|\alpha|\le 1} |\langle x\rangle^{n+1} D^\alpha u(x)|$ is continuous on $\mathcal{S}(\mathbb{R}^n)$. Thus if $S \subset L^2(\mathbb{R}^n)$ is bounded and $A \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ the continuity of $A : L^2(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ implies that $A(S)$ is bounded with respect to this norm. The theorem of Arzela-Ascoli shows that any sequence in $A(S)$ has a strongly convergent subsequence in $\langle x\rangle^n \mathcal{C}^0_\infty(\mathbb{R}^n)$ and such a sequence converges in $L^2(\mathbb{R}^n)$. Thus $A(S)$ has compact closure in $L^2(\mathbb{R}^n)$ which means that A is compact. □

The second important property of the residual algebra is that it is 'bi-ideal' or a 'corner' in the bounded operators on $L^2(\mathbb{R}^n)$. Note that it is not an ideal.

LEMMA 3.1. *If* A_1, $A_2 \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ *and* B *is a bounded operator on* $L^2(\mathbb{R}^n)$ *then* $A_1BA_2 \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$.

PROOF. The kernel of the composite $C = A_1BA_2$ can be written as a distributional pairing

(3.4)

$$C(x,y) = \int_{\mathbb{R}^{2n}} B(x',y')A_1(x,x')A_2(y',y)dx'dy' = (B, A_1(x,\cdot)A_2(\cdot,y)) \in \mathcal{S}(\mathbb{R}^{2n}).$$

Thus the result follows from the continuity of the exterior product, $\mathcal{S}(\mathbb{R}^{2n}) \times \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}(\mathbb{R}^{4n})$. □

In fact the same conclusion, with essentially the same proof, holds for any continuous linear operator B from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

3.2. The augmented residual algebra

Recall that a bounded operator is said to have finite rank if its range is finite dimensional. If we consider a bounded operator B on $L^2(\mathbb{R}^n)$ which is of finite rank then we may choose an orthonormal basis f_j, $j = 1,\dots,N$ of the range $BL^2(\mathbb{R}^n)$. The functionals $u \longmapsto \langle Bu, f_j\rangle$ are continuous and so define non-vanishing elements $g_j \in L^2(\mathbb{R}^n)$. It follows that the Schwartz kernel of B is

$$B = \sum_{j=1}^{N} f_j(x)\overline{g_j(y)}. \tag{3.5}$$

If $B \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ then the range must lie in $\mathcal{S}(\mathbb{R}^n)$ and similarly for the range of the adjoint, so the functions f_j are linearly dependent on some finite collection of functions $f'_j \in \mathcal{S}(\mathbb{R}^n)$ and similarly for the g_j. Thus it can be arranged that the f_j and g_j are in $\mathcal{S}(\mathbb{R}^n)$.

PROPOSITION 3.2. *If* $A \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ *then* $\operatorname{Id} + A$ *has, as an operator on* $L^2(\mathbb{R}^n)$, *finite dimensional null space and closed range which is the orthocomplement of the null space of* $\operatorname{Id} + A^*$. *There is an element* $B \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ *such that*

$$(\operatorname{Id} + A)(\operatorname{Id} + B) = \operatorname{Id} - \Pi_1,\ (\operatorname{Id} + B)(\operatorname{Id} + A) = \operatorname{Id} - \Pi_0 \tag{3.6}$$

where Π_0, $\Pi_1 \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ *are the orthogonal projections onto the null spaces of* $\operatorname{Id} + A$ *and* $\operatorname{Id} + A^*$ *and furthermore, there is an element* $A' \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ *of rank* *equal to the dimension of the null space such that* $\mathrm{Id} + A + sA'$ *is an invertible operator on* $L^2(\mathbb{R}^n)$ *for all* $s \neq 0$.

PROOF. Most of these properties are a direct consequence of the fact that A is compact as an operator on $L^2(\mathbb{R}^n)$.

We have shown, in Proposition 3.1 that each $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is compact. It follows that

$$N_0 = \mathrm{Nul}(\mathrm{Id} + A) \subset L^2(\mathbb{R}^n) \tag{3.7}$$

has compact unit ball. Indeed the unit ball, $B = \{u \in \mathrm{Nul}(\mathrm{Id} + A)\}$ satisfies $B = A(B)$, since $u = -Au$ on B. Thus B is closed (as the null space of a continuous operator) and precompact, hence compact. Any Hilbert space with a compact unit ball is finite dimensional, so $\mathrm{Nul}(\mathrm{Id} + A)$ is finite dimensional.

Now, let $R_1 = \mathrm{Ran}(\mathrm{Id} + A)$ be the range of $\mathrm{Id} + A$; we wish to show that this is a closed subspace of $L^2(\mathbb{R}^n)$. Let $f_k \to f$ be a sequence in R_1, converging in $L^2(\mathbb{R}^n)$. For each k there exists a unique $u_k \in L^2(\mathbb{R}^n)$ with $u_k \perp N_0$ and $(\mathrm{Id} + A)u_k = f_k$. We wish to show that $u_k \to u$. First we show that $\|u_k\|$ is bounded. If not, then along a subsequent $v_j = u_{k(j)}$, $\|v_j\| \to \infty$. Set $w_j = v_j/\|v_j\|$. Using the compactness of A, $w_j = -Aw_j + f_{k(j)}/\|v_j\|$ must have a convergent subsequence, $w_j \to w$. Then $(\mathrm{Id} + A)w = 0$ but $w \perp N_0$ and $\|w\| = 1$ which are contradictory. Thus the sequence u_k is bounded in $L^2(\mathbb{R}^n)$. Then again $u_k = -Au_k + f_k$ has a convergent subsequence with limit u which is a solution of $(\mathrm{Id} + A)u = f$; hence R_1 is closed. The orthocomplement of the range of a bounded operator is always the null space of its adjoint, so R_1 has a finite-dimensional complement $N_1 = \mathrm{Nul}(\mathrm{Id} + A^*)$. The same argument applies to $\mathrm{Id} + A^*$ so gives the orthogonal decompositions

$$\begin{aligned} L^2(\mathbb{R}^n) &= N_0 \oplus R_0,\ N_0 = \mathrm{Nul}(\mathrm{Id} + A),\ R_0 = \mathrm{Ran}(\mathrm{Id} + A^*) \\ L^2(\mathbb{R}^n) &= N_1 \oplus R_1,\ N_1 = \mathrm{Nul}(\mathrm{Id} + A^*),\ R_1 = \mathrm{Ran}(\mathrm{Id} + A). \end{aligned} \tag{3.8}$$

Thus we have shown that $\mathrm{Id} + A$ induces a continuous bijection $\tilde{A} : R_0 \longrightarrow R_1$. From the closed graph theorem the inverse is a bounded operator $\tilde{B} : R_1 \longrightarrow R_0$. In this case continuity also follows from the argument above.[1] Thus $\tilde{B}$ is the generalized inverse of $\mathrm{Id} + A$ in the sense that $B = \tilde{B} - \mathrm{Id}$ satisfies (3.6). It only remains to show that $B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. This follows from (3.6), the identities in which show that

$$\begin{aligned} B = -A - AB - \Pi_1,\ -B = A + BA + \Pi_0 \\ \Longrightarrow B = -A + A^2 + ABA - \Pi_1 + A\Pi_0. \end{aligned} \tag{3.9}$$

All terms here are in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$; for ABA this follows from Proposition 3.1.

It remains to show the existence of the finite rank perturbation A'. This is equivalent to the vanishing of the index, that is

$$\mathrm{Ind}(\mathrm{Id} + A) = \dim \mathrm{Nul}(\mathrm{Id} + A) - \dim \mathrm{Nul}(\mathrm{Id} + A^*) = 0. \tag{3.10}$$

Indeed, let f_j and g_j, $j = 1, \dots, N$, be respective bases of the two finite dimensional spaces $\mathrm{Nul}(\mathrm{Id} + A)$ and $\mathrm{Nul}(\mathrm{Id} + A^*)$. Then

$$A' = \sum_{j=1}^{N} g_j(x) \overline{f_j(y)} \tag{3.11}$$

[1] We need to show that $\|\tilde{B}f\|$ is bounded when $f \in R_1$ and $\|f\| = 1$. This is just the boundedness of $u \in R_0$ when $f = (\mathrm{Id} + A)u$ is bounded in R_1.

is an isomorphism of N_0 onto N_1 which vanishes on R_0. Thus $\mathrm{Id} + A + sA'$ is the direct sum of $\mathrm{Id} + A$ as an operator from R_0 to R_1 and sA' as an operator from N_0 to N_1, invertible when $s \neq 0$.

There is a very simple proof[2] of the equality (3.10) if we use the trace functional discussed in Section 3.5 below; this however is logically suspect as we use (although not crucially) approximation by finite rank operators in the discussion of the trace and this in turn might appear to use the present result via the discussion of ellipticity and the harmonic oscillator. Even though this is not really the case we give a clearly independent, but less elegant proof.

Consider the one-parameter family of operators $\mathrm{Id} + tA$, $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. We shall see that the index, the difference in dimension between $\mathrm{Nul}(\mathrm{Id} + tA)$ and $\mathrm{Nul}(\mathrm{Id} + tA^*)$ is locally constant. To see this it is enough to consider a general A near the point $t = 1$. Consider the pieces of A with respect to the decompositions $L^2(\mathbb{R}^n) = N_i \oplus R_i$, $i = 0, 1$, of domain and range. Thus A is the sum of four terms which we write as a 2×2 matrix

$$A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}.$$

Since $\mathrm{Id} + A$ has only one term in such a decomposition, $\tilde{A}$ in the lower right, the solution of the equation $(\mathrm{Id} + tA)u = f$ can be written

$$(3.12)\quad (t-1)A_{00}u_0 + (t-1)A_{01}u_\perp = f_1,\ (t-1)A_{10}u_0 + (A' + (t-1)A_{11})u_\perp = f_\perp$$

Since $\tilde{A}$ is invertible, for $t - 1$ small enough the second equation can be solved uniquely for $u_\perp$. Inserted into the first equation this gives

$$(3.13)\quad \begin{aligned} &G(t)u_0 = f_1 + H(t)f_\perp, \\ &G(t) = (t-1)A_{00} - (t-1)^2 A_{01}(A' + (t-1)A_{11})^{-1}A_{10}, \\ &H(t) = -(t-1)A_{01}(A' + (t-1)A_{11})^{-1}. \end{aligned}$$

The null space is therefore isomorphic to the null space of $G(t)$ and a complement to the range is isomorphic to a complement to the range of $G(t)$. Since $G(t)$ is a finite rank operator acting from N_0 to N_1 the difference of these dimensions is constant in t, namely equal to $\dim N_0 - \dim N_1$, near $t = 1$ where it is defined.

This argument can be applied to tA so the index is actually constant in $t \in [0, 1]$ and since it certainly vanishes at $t = 0$ it vanishes for all t. In fact, as we shall note below, $\mathrm{Id} + tA$ is invertible outside a discrete set of $t \in \mathbb{C}$. $\square$

Corollary 3.1. *If $\mathrm{Id} + A$, $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is injective or surjective on $L^2(\mathbb{R}^n)$, in particular if it is invertible as a bounded operator, then it has an inverse of the form $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.*

Corollary 3.2. *If $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then as an operator on $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$, $\mathrm{Id} + A$ is Fredholm in the sense that its null space is finite dimensional and its range is closed with a finite dimensional complement.*

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[2]Namely the trace of a finite rank projection, such as either Π_0 or Π_1, is its rank, hence the dimension of the space onto which it projects. From the identity satisfied by the generalized inverse we see that

$$\mathrm{Ind}(\mathrm{Id} + A) = \mathrm{Tr}(\Pi_0) - \mathrm{Tr}(\Pi_1) = \mathrm{Tr}\left((\mathrm{Id} + B)(\mathrm{Id} + A) - (\mathrm{Id} + A)(\mathrm{Id} + B)\right) = \mathrm{Tr}([B, A]) = 0$$

from the basic property of the trace.

PROOF. This follows from the existence of the generalized inverse of the form $\mathrm{Id} + B$, $B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. $\square$

3.3. Exponential and logarithm

PROPOSITION 3.3. *The exponential*

$$\exp(A) = \sum_j \frac{1}{j!} A^j : \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \tag{3.14}$$

is a globally defined, entire, function with range containing a neighbourhood of the identity and with inverse on such a neighbourhood given by the analytic function

$$\log(\mathrm{Id} + A) = \sum_j \frac{(-1)^j}{j} A^j,\ A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n),\ \|A\|_{L^2} < 1 \tag{3.15}$$

3.4. The residual group

By definition, $\mathcal{G}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is the set (if you want to be concrete you can think of them as operators on $L^2(\mathbb{R}^n)$) of invertible operators in $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. If we identify this topologically with $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then, as follows from Corollary 3.1, $\mathcal{G}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is open. We will think of it as an infinite-dimensional manifold modeled, of course, on the linear space $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \simeq \mathcal{S}(\mathbb{R}^{2n})$. Since I have no desire to get too deeply into the general theory of such Fréchet manifolds I will keep the discussion as elementary as possible.

The dual space of $\mathcal{S}(\mathbb{R}^p)$ is $\mathcal{S}'(\mathbb{R}^p)$. If we want to think of $\mathcal{S}(\mathbb{R}^p)$ as a manifold we need to consider smooth functions and forms on it. In the finite-dimensional case, the exterior bundles are the antisymmetric parts of the tensor powers of the dual. Since we are in infinite dimensions the tensor power needs to be completed and the usual choice is the 'projective' tensor product. In our case this is something quite simple, namely the k-fold completed tensor power of $\mathcal{S}'(\mathbb{R}^p)$ is just $\mathcal{S}'(\mathbb{R}^{kp})$. Thus we set

$$\begin{aligned}\Lambda^k \mathcal{S}(\mathbb{R}^p) = \{u \in \mathcal{S}'(\mathbb{R}^{kp});\ &\text{for any permutation} \\ &e,\ u(x_{e(1)}, \dots x_{e(h)}) = \operatorname{sgn}(e) u(x_1, \dots x_k)\}\,.\end{aligned} \tag{3.16}$$

In view of this it is enough for us to consider smooth functions on open sets $F \subset \mathcal{S}(\mathbb{R}^p)$ with values in $\mathcal{S}'(\mathbb{R}^p)$ for general p. Thus

$$v : F \longrightarrow \mathcal{S}'(\mathbb{R}^p),\ F \subset \mathcal{S}(\mathbb{R}^n) \text{ open} \tag{3.17}$$

is continuously differentiable on F if there exists a continuous map

$$\begin{aligned}v' : F \longrightarrow \mathcal{S}'(\mathbb{R}^{n+p}) &\text{ and each } u \in F \text{ has a neighbourhood } U \\ &\text{such that for each } N \ \exists\ M \ \text{ with} \\ &\|v(u+u') - v(u) - v'(u; u')\|_N \le C\|u'\|^2_M,\ \forall\ u, u+u' \in U.\end{aligned}$$

Then, as usual we define smoothness as infinite differentiability by iterating this definition. The smoothness of v in this sense certainly implies that if $f : X \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is a smooth from a finite dimensional manifold then $v \circ F$ is smooth.

Thus we define the notion of a smooth *form* on $F \subset \mathcal{S}(\mathbb{R}^n)$, an open set, as a smooth map

$$\alpha : F \to \Lambda^k \mathcal{S}(\mathbb{R}^p) \subset \mathcal{S}'(\mathbb{R}^{kp})\,. \tag{3.18}$$

In particular we know what smooth forms are on $\mathcal{G}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

The de Rham differential acts on forms as usual. If $v : F \to \mathbb{C}$ is a function then its differential at $f \in F$ is $dv : F \longrightarrow \mathcal{S}'(\mathbb{R}^n) = \Lambda^1\mathcal{S}(\mathbb{R}^n)$, just the derivative. As in the finite-dimensional case d extends to forms by enforcing the condition that $dv = 0$ for constant forms and the distribution identity over exterior products

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta\,. \tag{3.19}$$

3.5. Traces on the residual algebra

The algebras we are studying are topological algebras, so it makes sense to consider continuous linear functionals on them. The most important of these is the *trace*. To remind you what it is we consider first its properties for matrix algebras.

Let $M(N;\mathbb{C})$ denote the algebra of $N \times N$ complex matrices. We can simply define

$$\mathrm{Tr} : M(N;\mathbb{C}) \to \mathbb{C}, \quad \mathrm{Tr}(A) = \sum_{i=1}^{N} A_{ii} \tag{3.20}$$

as the sum of the diagonal entries. The fundamental property of this functional is that

$$\mathrm{Tr}([A, B]) = 0 \ \forall \ A, B \in M(N;\mathbb{C})\,. \tag{3.21}$$

To check this it is only necessary to write down the definition of the composition in the algebra. Thus

$$(AB)_{ij} = \sum_{k=1}^{N} A_{ik}B_{kj}\,.$$

It follows that

$$\begin{aligned}\mathrm{Tr}(AB) &= \sum_{i=1}^{N} (AB)_{ii} = \sum_{i,k=1}^{N} A_{ik}B_{ki} \\ &= \sum_{k=1}^{N}\sum_{i=1}^{N} B_{ki}A_{ik} = \sum_{k=1}^{N} (BA)_{kk} = \mathrm{Tr}(BA)\end{aligned}$$

which is just (3.21).

Of course any multiple of Tr has the same property (3.21) but the normalization condition

$$\mathrm{Tr}(\mathrm{Id}) = N \tag{3.22}$$

distinguishes it from its multiples. In fact (3.21) and (3.22) together distinguish $\mathrm{Tr} \in M(N;\mathbb{C})'$ as a point in the N^2 dimensional linear space which is the dual of $M(N;\mathbb{C})$.

Lemma 3.2. *If $F : M(N;\mathbb{C}) \to \mathbb{C}$ is a linear functional satisfying* (3.21) *and $B \in M(N;\mathbb{C})$ is any matrix such that $F(B) \neq 0$ then $F(A) = \frac{F(B)}{\mathrm{Tr}(B)}\,\mathrm{Tr}(A)$.*

Proof. Consider the basis of $M(N;\mathbb{C})$ given by the elementary matrices E_{jk}, where E_{jk} has jk-th entry 1 and all others zero. Thus

$$E_{jk}E_{pq} = \delta_{kp}E_{jq}.$$

If $j \neq k$ it follows that

$$E_{jj}E_{jk} = E_{jk},\ E_{jk}E_{jj} = 0.$$

Thus

$$F([E_{jj}, E_{jk}]) = F(E_{jk}) = 0 \text{ if } j \neq k.$$

On the other hand, for any i and j

$$E_{ji}E_{ij} = E_{jj},\ E_{ij}E_{ji} = E_{ii}$$

so

$$F(E_{jj}) = F(E_{11}) \ \forall\ j.$$

Since the E_{jk} are a basis,

$$\begin{aligned} F(A) &= F(\sum_{j,k=1}^{N} A_{ij}E_{ij}) \\ &= \sum_{j,l=1}^{N} A_{jj}F(E_{ij}) \\ &= F(E_{11})\sum_{j=1}^{N} A_{jj} = F(E_{11})\,\mathrm{Tr}(A). \end{aligned}$$

This proves the lemma. □

For the isotropic smoothing algebra we have a similar result.

PROPOSITION 3.4. *If* $F : \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \simeq \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathbb{C}$ *is a continuous linear functional satisfying*

$$F([A, B]) = 0 \ \forall\ A, B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \tag{3.23}$$

then F *is a constant multiple of the functional*

$$\mathrm{Tr}(A) = \int_{\mathbb{R}^n} A(x, x)\, dx. \tag{3.24}$$

PROOF. Recall that $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \subset \Psi^{\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is an ideal so $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and $B \in \Psi^{\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ implies that AB, $BA \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and it follows that the equality $F(AB) = F(BA)$, or $F([A, B]) = 0$, is meaningful. To see that it holds we just use the continuity of F. We know that if $B \in \Psi^{\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then there is a sequence $B_n \to B$ in the topology of $\Psi^{m}_{\mathrm{iso}}(\mathbb{R}^n)$ for some m. Since this implies $AB_n \to AB$, $B_nA \to BA$ in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ we see that

$$F([A, B]) = \lim_{n\to\infty} F([A, B_n]) = 0\,.$$

We use this identity to prove (3.24). Take $B = x_j$ or D_j, $j = 1, \dots, n$. Thus for any $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$

$$F([A, x_j]) = F([A, D_j]) = 0\,.$$

Now consider F as a distribution acting on the kernel $A \in \mathcal{S}(\mathbb{R}^{2n})$. Since the kernel of $[A, x_j]$ is $A(x, y)(y_j - x_j)$ and the kernel of (A, D_j) is $-(D_{y_j} + D_{x_j})A(x, y)$ we conclude that, as an element of $\mathcal{S}'(\mathbb{R}^{2n})$, F satisfies

$$(x_j - y_j)F(x, y) = 0,\ (D_{x_j} + D_{y_j})F(x, y) = 0.$$

If we make the linear change of variables to $p_i = \frac{x_i+y_i}{2}$, $q_i = x_i - y_i$ and set $\tilde{F}(p,q) = F(x,y)$ these conditions become

$$D_{q_i}\tilde{F} = 0,\ p_i\tilde{F} = 0,\ i = 1,\dots,N.$$

As we know from Lemmas 1.2 and 1.3, this implies that $\tilde{F} = c\delta(p)$ so

$$F(x,y) = c\delta(x-y)$$

as a distribution. Clearly $\delta(x-y)$ gives the functional Tr defined by (3.24), so the proposition is proved. $\square$

We still need to justify the use of the same notation, Tr, for these two functionals. However, if $L \subset \mathcal{S}(\mathbb{R}^n)$ is any finite dimensional subspace we may choose an orthonal basis $\varphi_i \in L$, $i = 1,\dots,l$,

$$\int_{\mathbb{R}^n} |\varphi_i(x)|^2\,dx = 0,\ \int_{\mathbb{R}^n} \varphi_i(x)\overline{\varphi_j}(x)\,dx = 0,\ i \neq j\,.$$

Then if a_{ij} is an $l \times l$ matrix,

$$A = \sum_{i,j=1}^{\ell} a_{ij}\varphi_i(x)\overline{\varphi_j(y)} \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)\,.$$

From (3.24) we see that

$$\begin{aligned}\operatorname{Tr}(A) &= \sum_{ij} a_{ij}\operatorname{Tr}(\varphi_i\bar{\varphi}_j)\\ &= \sum_{ij} a_{ij}\int_{\mathbb{R}^n}\varphi_i(x)\overline{\varphi_j}(x)\,dx\\ &= \sum_{i=1}^{n} a_{ii} = \operatorname{Tr}(a)\,.\end{aligned}$$

Thus the two notions of trace coincide. In any case this already follows, up to a constant, from the uniqueness in Lemma 3.2.

3.6. Fredholm determinant

For $N \times N$ matrices, the determinant is a multiplicative polynomial map

$$\det : M(N;\mathbb{C}) \longrightarrow \mathbb{C},\ \det(AB) = \det(A)\det(B),\ \det(\mathrm{Id}) = 1. \tag{3.25}$$

It is not quite determined by these conditions, since $\det(A)^k$ also satisfies then. The fundamental property of the determinant is that it defines the group of invertible elements

$$\mathrm{GL}(N,\mathbb{C}) = \{A \in M(N;\mathbb{C}); \det(A) \neq 0\}. \tag{3.26}$$

A reminder of a direct definition is given in Problem 4.7.

The Fredholm determinant is an extension of this definition to a function on the ring $\mathrm{Id} + \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. This can be done in several ways using the density of finite rank operators, as shown in Corollary 4.2. We proceed by generalizing the formula relating the determinant to the trace. Thus, for any smooth curve with values in $\mathrm{GL}(N;\mathbb{C})$ for any N,

$$\frac{d}{ds}\det(A_s) = \det(A_s)\operatorname{tr}(A_s^{-1}\frac{A_s}{ds}). \tag{3.27}$$

In particular if (3.25) is augmented by the normalization condition

$$\frac{d}{ds}\det(\mathrm{Id}+sA)\big|_{s=0} = \mathrm{tr}(A)\ \forall\ A \in M(N;\mathbb{C}) \tag{3.28}$$

then it is determined.

A branch of the logarithm can be introduced along any curve, smoothly in the parameter, and then (3.27) can be rewritten

$$d\log\det(A) = \mathrm{tr}(A^{-1}dA). \tag{3.29}$$

Here $\mathrm{GL}(N;\mathbb{C})$ is regarded as a subset of the linear space $M(N;\mathbb{C})$ and dA is the canonical identification, at the point A, of the tangent space to $M(N,\mathbb{C})$ with $M(N,\mathbb{C})$ itself. This just arises from the fact that $M(N,\mathbb{C})$ is a linear space. Thus $dA(\frac{d}{ds}(A+sB)\big|_{s=0} = B$. This allows the expression on the right in (3.29) to be interpreted as a smooth 1-form on the manifold $\mathrm{GL}(N;\mathbb{C})$. Note that it is independent of the local choice of logarithm.

To define the Fredholm determinant we shall extend the 1-form

$$\alpha = \mathrm{Tr}(A^{-1}dA) \tag{3.30}$$

to the group $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \hookrightarrow \mathrm{Id}+\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Here dA has essentially the same meaning as before, given that Id is fixed. Thus at any point $A = \mathrm{Id}+B \in \mathrm{Id}+\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ it is the identification of the tangent space with $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ using the linear structure:

$$dA(\frac{d}{ds}(\mathrm{Id}+B+sE)\big|_{s=0}) = E,\ E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n).$$

Since dA takes values in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, the trace functional in (3.30) is well defined.

The 1-form α is closed. In the finite-dimensional case this follows from (3.29). For (3.30) we can compute directly. Since $d(dA) = 0$, essentially by definition, and

$$dA^{-1} = -A^{-1}dAA^{-1} \tag{3.31}$$

we see that

$$d\alpha = -\mathrm{Tr}(A^{-1}(dA)A^{-1}(dA)) = 0. \tag{3.32}$$

Here we have used the trace identity, and the antisymmetry of the implicit wedge product in (3.32), to conlcude that $d\alpha = 0$. For a more detailed discussion of this point see Problem 4.8.

From the fact that $d\alpha = 0$ we can be confident that there is, locally near any point of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, a function f such that $df = \alpha$; then we will define the Fredholm determinant by $\det_{\mathrm{Fr}}(A) = \exp(f)$. To define $\det_{\mathrm{Fr}}$ globally we need to see that this is well defined.

Lemma 3.3. *For any smooth closed curve $\gamma : \mathbb{S}^1 \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ the integral*

$$\int_\gamma \alpha = \int_{\mathbb{S}^1} \gamma^*\alpha \in 2\pi i\mathbb{Z}. \tag{3.33}$$

That is, α defines an integral cohomology class, $[\frac{\alpha}{2\pi i}] \in H^1(G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n);\mathbb{Z})$.

Proof. This is where we use the approximability by finite rank operators. If π_N is the orthogonal projection onto the span of the eigenspaces of the smallest N eigenvalues of the harmonic oscillator then we know from Section 4.3 that $\pi_N E\pi_N \to E$ in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ for any element. In fact it follows that for the smooth curve that $\gamma(s) = \mathrm{Id}+E(s)$ and $E_N(s) = \pi_N E(s)\pi_N$ converges uniformly with all

s derivatives. Thus, for some N_0 and all $N > N_0$, $\mathrm{Id} + E_N(s)$ is a smooth curve in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and hence $\gamma_N(s) = \mathrm{Id}_N + E_N(s)$ is a smooth curve in $\mathrm{GL}(N;\mathbb{C})$. Clearly

$$\int_{\gamma_N} \alpha \longrightarrow \int_{\gamma} \alpha \text{ as } N \to \infty, \tag{3.34}$$

and for finite N it follows from the identity of the trace with the matrix trace (see Section 3.5) that $\int_N \gamma_N^* \alpha$ is the variation of $\arg \log \det(\gamma_N)$ around the curve. This gives (3.33). □

Now, once we have (3.33) and the connectedness of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ we may define

$$\det{}_{\mathrm{Fr}}(A) = \exp(\int_\gamma \alpha),\ \gamma : [0,1] \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n),\ \gamma(0) = \mathrm{Id},\ \gamma(1) = A. \tag{3.35}$$

Indeed, Lemma 3.3 shows that this is independent of the path chosen from the identity to A. Notice that the connectedness of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ follows from the connectedness of the $\mathrm{GL}(N,\mathbb{C})$ and the density argument above.

The same arguments and results apply to $G^{-2n-\epsilon}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ using the fact that the trace functional extends continuously to $\Psi^{-2n-\epsilon}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ for any $\epsilon > 0$.

PROPOSITION 3.5. *The Fredholm determinant, defined by* (3.35) *on* $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *(or* $G^{-2n-\epsilon}_{\mathrm{iso}}(\mathbb{R}^n)$ *for* $\epsilon > 0$) *and to be zero on the complement in* $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *(or* $\mathrm{Id} + \Psi^{-2n-\epsilon}_{\mathrm{iso}}(\mathbb{R}^n)$) *is an entire function satisfying*

$$\begin{aligned}\det{}_{\mathrm{Fr}}(AB) = \det{}_{\mathrm{Fr}}(A)\det{}_{\mathrm{Fr}}(B),\ A, B \in \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)\\ (\textit{or}\ \mathrm{Id} + \Psi^{-2n-\epsilon}_{\mathrm{iso}}(\mathbb{R}^n)),\ \det{}_{\mathrm{Fr}}(\mathrm{Id}) = 1.\end{aligned} \tag{3.36}$$

PROOF. We start with the multiplicative property of $\det_{\mathrm{Fr}}$ on $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus is $\gamma_1(s)$ is a smooth curve from Id to A_1 and $\gamma_2(s)$ is a smooth curve from Id to A_2 then $\gamma(s) = \gamma_1(s)\gamma_2(s)$ is a smooth curve from Id to A_1A_2. Consider the differential on this curve. Since

$$\frac{d(A_1(s)A_2(s))}{ds} = \frac{dA_1(s)}{ds}A_2(s) + A_1(s)\frac{dA_2(s)}{ds}$$

the 1-form becomes

$$\gamma^*(s)\alpha(s) = \mathrm{Tr}(A_2(s)^{-1}\frac{dA_2(s)}{ds}) + \mathrm{Tr}(A_2(s)^{-1}A_1(s)^{-1}\frac{dA_2(s)}{ds}A_2(s)). \tag{3.37}$$

In the second term on the right we can use the trace identity, since $\mathrm{Tr}(GA) = \mathrm{Tr}(AG)$ if $G \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$ and $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus (3.37) becomes

$$\gamma^*(s)\alpha(s) = \gamma_1^*\alpha + \gamma_2^*\alpha.$$

Inserting this into the definition of $\det_{\mathrm{Fr}}$ gives (3.36) when both factors are in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Of course if either factor is not invertible, then so is the product and hence both $\det_{\mathrm{Fr}}(AB)$ and at least one of $\det_{\mathrm{Fr}}(A)$ and $\det_{\mathrm{Fr}}(B)$ vanishes. Thus (3.36) holds in general when $\det_{\mathrm{Fr}}$ is extended to be zero on the non-invertible elements.

Thus it remains to establish the smoothness. That $\det_{\mathrm{Fr}}(A)$ is smooth in any real parameters in which $A \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ depends, or indeed is holomorphic in holomorphic parameters, follows from the definition since α clearly depends smoothly, or holomorphically, on parameters. In fact the same follows if holomorphy is examined as a function of E, $A = \mathrm{Id} + E$, for $E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus it is only smoothness across the non-invertibles that is at issue. To prove this we use the multiplicativity just established.

If $A = \mathrm{Id} + E$ is not invertible, $E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then it has a generalized inverse $\mathrm{Id} + E'$ as in Proposition 4.3. Since A has index zero, we may actually replace E' by $E' + E''$, where E'' is an invertible linear map from the orthocomplement of the range of A to its null space. Then $\mathrm{Id} + E' + E'' \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and $(\mathrm{Id} + E' + E'')A = \mathrm{Id} - \Pi_0$. To prove the smoothness of $\det_{\mathrm{Fr}}$ on a neighbourhood of A it is enough to prove the smoothness on a neighbourhood of $\mathrm{Id} - \Pi_0$ since $\mathrm{Id} + E' + E''$ maps a neighbourhood of the first to a neighbourhood of the second and $\det_{\mathrm{Fr}}$ is multiplicative. Thus consider $\det_{\mathrm{Fr}}$ on a set $\mathrm{Id} - \Pi_0 + E$ where E is near 0 in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, in particular we may assume that $\mathrm{Id} + E \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus

$$\det_{\mathrm{Fr}}(\mathrm{Id} + E - \Pi_0) = \det(\mathrm{Id} + E)\det(\mathrm{Id} - \Pi_0 + (G_E - \mathrm{Id})\Pi_0)$$

were $G_E = (\mathrm{Id} + E)^{-1}$ depends holomorphically on E. Thus it suffices to prove the smoothness of $\det_{\mathrm{Fr}}(\mathrm{Id} - \Pi_0 + H\Pi_0)$ where $H \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$

Consider the deformation $H_s = \Pi_0 H \Pi_0 + s(\mathrm{Id} - \Pi_0)H\Pi_0$, $s \in [0, 1]$. If $\mathrm{Id} - \Pi_0 + H_s$ is invertible for one value of s it is invertible for all, since its range is always the range of $\mathrm{Id} - \Pi_0$ plus the range of $\Pi_0 H \Pi_0$. It follows that $\det_{\mathrm{Fr}}(\mathrm{Id} - \Pi_0 + H_s)$ is smooth in s; in fact it is constant. If the family is not invertible this follows immediately and if it is invertible then

$$\frac{d\det_{\mathrm{Fr}}(\mathrm{Id} - \Pi_0 + H_s)}{ds} \\ = \det_{\mathrm{Fr}}(\mathrm{Id} - \Pi_0 + H_s)\,\mathrm{Tr}\left((\mathrm{Id} - \Pi_0 + H_s)^{-1}(\mathrm{Id} - Pi_0)H\Pi_0)\right) = 0$$

since the argument of the trace is finite rank and off-diagonal with respect to the decomposition by Π_0.

Thus finally it is enough to consider the smoothness of $\det_{\mathrm{Fr}}(\mathrm{Id} - \Pi_0 + \Pi_0 H \Pi_0)$ as a function of $H \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Since this is just $\det(\Pi_0 H \Pi_0)$, interpreted as a finite rank map on the range of Π_0 the result follows from the finite dimensional case. □

3.7. Fredholm alternative

Since we have shown that $\det_{\mathrm{Fr}} : \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow \mathbb{C}$ is an entire function, we see that $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is the complement of a (singular) holomorphic hypersurface, namely the surface $\{\mathrm{Id} + E; \det_{\mathrm{Fr}}(\mathrm{Id} + E) = 0\}$. This has the following consequence, which is sometimes call the 'Fredholm alternative' and also part of 'analytic Fredholm theory'.

Lemma 3.4. *If $\Omega \subset \mathbb{C}$ is an open, connected set and $A : \Omega \longrightarrow \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is a holomorphic function then either $\mathrm{Id} + A(z)$ is invertible on all but a discrete subset of Ω and $(\mathrm{Id} + A(z))$ is meromorphic on Ω with all residues of finite rank, or else it is invertible at no point of Ω.*

Proof. Of course the point here is that $\det_{\mathrm{Fr}}(\mathrm{Id} + A(z))$ is a holomorphic function on Ω. Thus, either $\det_{\mathrm{Fr}}(A(z)) = 0$ is a discrete set, $D \subset \Omega$ or else $\det_{\mathrm{Fr}}(\mathrm{Id} + A(z)) \equiv 0$ on Ω; this uses the connectedness of Ω. Since this corresponds exactly to the invertibility of $\mathrm{Id} + A(z)$ the main part of the lemma is proved. It remains only to show that, in the former case, $(\mathrm{Id} + A(z))^{-1}$ is meromorphic. Thus consider a point $p \in D$. Thus the claim is that near p

$$(\mathrm{Id} + A(z))^{-1} = \mathrm{Id} + E(z) + \sum_{j=1}^{N} z^{-j} E_j,\ \ E_j \in \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n) \text{ of finite rank} \tag{3.38}$$

and where $E(z)$ is locally holomorphic with values in $\Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$.

If N is sufficiently large and Π_N is the projection onto the first N eigenspaces of the harmonic oscillator then $B(z) = \mathrm{Id} + E(z) - \Pi_N E(z) \Pi_N$ is invertible near p with the inverse being of the form $\mathrm{Id} + F(z)$ with $F(z)$ locally holomorphic. Now

$$\begin{aligned}(\mathrm{Id} + F(z))(\mathrm{Id} + E(z)) &= \mathrm{Id} + (\mathrm{Id} + F(z))\Pi_N E(z) \Pi_N \\ &= (\mathrm{Id} - \Pi_N) + \Pi_N M(z) \Pi_N + (\mathrm{Id} - \Pi_N) M'(z) \Pi_N.\end{aligned}$$

It follows that this is invertible if and only if $M(z)$ is invertible as a matrix on the range of Π_N. Since it must be invertible near, but not at, p, its inverse is a meromorphic matrix $K(z)$. It follows that the inverse of the product above can be written

$$\mathrm{Id} - \Pi_N + \Pi_N K(z) \Pi_N - (\mathrm{Id} - \Pi_N) M'(z) \Pi_N K(z) \Pi_N. \tag{3.39}$$

This is meromorphic and has finite rank residues, so it follows that the same is true of $A(z)^{-1}$. $\square$