CHAPTER 12

The index theorem and formula

Using the earlier results on K-theory and cohomology the families index theorem of Atiyah and Singer is proved using a variant of their 'embedding' proof. The index formula in cohomology (including of course the formula for the numerical index) is then derived from this.

12.1. Outline

The index theorem of Atiyah and Singer is proved here in K-theory, using the results from Chapter 10 and then the cohomological version is derived from this. Here are the main steps carried out below:-

(1) Fibrations of manifolds, $M \longrightarrow B$, are discussed and shown to be embedable in a trivial fibration following Whitney's embedding theorem.
(2) The notion of a family of pseudodifferential operators on the fibres of a fibration is introduced and ellipticity is described. The symbol is shown to define an element of $\mathrm{K_c}(T^*(M/B))$.
(3) (Optional) Spectral sections for self-adjoint elliptic families are defined, and shown to exist for perturbatively-invertible families.
(4) (Optional) The group of homotopy classes of sections of the bundle $G^{-\infty}(M/B;E)$ is shown to reduce to $\mathrm{K_c}(B)$.
(5) The stabilization of the index bundle is discussed and the analytic index map $\mathrm{Ind_a}: \mathrm{K_c}(T^*(M/B)) \longleftrightarrow \mathrm{K_c}(B)$ is defined.
(6) The topological index map is defined using embeddings and the Thom isomorphism.
(7) The original proof of Atiyah and Singer is outlined
(8) The 'odd analytic index' is defined using the semiclassical calculus and shown to be equal to the analytic index.
(9) A variant of the embedding argument is used to show the equality of odd analytic and topological index and hence the index theorem

$$\mathrm{K_c}(T^*(M/B)) \underset{\mathrm{Ind_t}}{\overset{\mathrm{Ind_a}}{\underset{\longrightarrow}{\overset{\longrightarrow}{=}}}} \mathrm{K_c}(B) \tag{12.1}$$

is proved.

Subsequently the special case of Dirac operators is treated and the formula for the Chern character of the index bundle is deduced.

Maybe other things will go in here, η forms, determinant bundle etc.

12.2. Fibrations

Instead of just considering families of pseudodifferential operators on a manifold but depending smoothly on parameters in some other manifold we allow 'twisting by the diffeomorphism group' and consider the more general setting of a family of pseudodifferential operators on the fibres of a fibration, so the parameters are the variables in the base of the fibration and the operators act on the fibres, which are diffeomorphic to a fixed manifold. This indeed is the setting for the 'families index theorem' of Atiyah and Singer.

So, first we need a preliminary discussion of fibrations. A map between two manifolds

$$\phi : M \longrightarrow B \tag{12.2}$$

is a fibration, with typical fibre a manifold Z, if it is smooth, surjective and has the 'local product' property:-

(12.3)

Each $b \in B$ has an open neighbourhood $U \subset B$

for which there exists a diffeomorphism F_U giving a commutative diagramme

$$\begin{tikzcd} \phi^{-1}(U) \arrow[rr, "F_U"] \arrow[dr, "\phi"'] & & Z \times U \arrow[dl, "\pi_U"] \\ & U. & \end{tikzcd}$$

Here of course, π_U is projection onto the second factor. In particular this means that each fibre $\phi^{-1}(b) = Z_b$ is diffeomorphic to Z, and in such a way that the diffeomorphism can be chosen locally to be smooth in $b \in B$. However there is no *chosen* diffeomorphism and of course in general the diffeomorphism cannot be chosen globally smoothly in b – other wise the fibration is trivial in the sense that there exists a diffeomorphism giving a commutative diagramme

$$\begin{tikzcd} M \arrow[rr, "F"] \arrow[dr, "\phi"'] & & Z \times B \arrow[dl, "\pi_B"] \\ & B. & \end{tikzcd} \tag{12.4}$$

I use the notation

$$\begin{tikzcd} Z \arrow[r, dash] & M \arrow[d, "\phi"] \\ & B \end{tikzcd} \tag{12.5}$$

to denote a fibration, the headless arrow meaning that there is no chosen diffeomorphism onto the fibres; often people put an arrow there.

One standard source of fibrations is the implict function theorem.

Proposition 12.1. [1] *If $\phi : M \longrightarrow B$ is a smooth map between connected smooth compact manifolds which is a submersion, i.e. the differential $\phi_* : T_m M \longrightarrow T_{\phi(m)} B$ is surjective for every $m \in M$, then ϕ is a fibration.*

It is easy to see that this implication can fail if M is not compact.

We will discuss operators on the fibres of a fibration below. First however we consider one of the important steps in the proof of the Atiyah-Singer theorem, namely the embedding of a fibration.

[1] See Problem 12.1

PROPOSITION 12.2. *Any fibration of compact manifolds can be embedded in a trivial fibration to give a commutative diagramme*

$$\begin{tikzcd} M \arrow[rr, "\iota"] \arrow[dr, "\phi"'] & & \mathbb{R}^M \times B \arrow[dl, "\pi_B"] \\ & B. & \end{tikzcd} \tag{12.6}$$

PROOF. Following Whitney, simply embed M in $\mathbb{R}^M$ for some M. This is easy to do, much the same way as vector bundle can be complemented to a trivial bundle.[2] Then let ι be the product of this embedding and ϕ, giving a map into $\mathbb{R}^M \times B$. □

Vector bundles give particular examples of fibrations. There are various standard constructions on fibrations, in particular the fibre product.

LEMMA 12.1. *If $\phi_i : M_i \longrightarrow B$, $i = 1, 2$ are two fibrations with the same base and typical fibres Z_i, then*

$$M_1 \times_B M_2 = \{(m_1, m_2) \in M_1 \times M_2; \phi_1(m_1) = \phi_2(m_2)\} \subset M_1 \times M_2$$

is an embedded submanifold and the restriction of $\phi_1 \times \phi_2$ to it gives a fibration

$$\begin{tikzcd} Z_1 \times Z_2 \arrow[r, dash] & M_1 \times_B M_2 \arrow[d, "\phi_1 \times \phi_2"] \\ & B. \end{tikzcd} \tag{12.7}$$

PROOF. Just look at local trivializations. □

It has become standard to denote 'relative objects' for a fibration, meaning objects on the fibres, using the formal notation M/B for the fibres. Thus $T(M/B)$ is the *fiber tangent bundle*. It is a bundle over the total space M with fibre at $m \in M$ the tangent space to the fibre through m, $\phi^{-1}(\phi(m))$, at m. To see that it is a bundle, just look at local trivializations of the fibration. Its dual bundle is $T^*(M/B)$, with fibre at m the cotangent space for the fibre. This will play a significant role in what we do below.

12.3. Smoothing families

Philosophically it is often a good idea to think of a space like $\mathcal{C}^\infty(M)$, the smooth functions (or more generally sections of some vector bundle) on the total space of a fibration as an infinite-dimensional bundle over the base. The fibre at b is just $\mathcal{C}^\infty(Z_b)$, the smooth functions on the fibre, and a local trivialization of the fibration gives a local trivialization of this bundle. To be consistent with the notation above I suppose this bundle should be denoted $\mathcal{C}^\infty(M/B) = \mathcal{C}^\infty(M)$ (or $\mathcal{C}^\infty_c(M/B) = \mathcal{C}^\infty_c(M)$ if M is not compact but B is) thought of as a bundle over B.

Next let us consider smoothing operators on the fibres of a fibration from this point of view. Recall that the densities on a manifold form a trivial, but not canonically trivial, real line bundle over the manifold. If this bundle is trivialized

[2]See Problem 12.2 for more details.

then the smoothing operators on Z are identified with the smooth functions (their Schwartz kernels) on $Z \times Z$. Really this is more invariantly written

$$\Psi^{-\infty}(Z) = \mathcal{C}^\infty(Z \times Z; \pi_R^*\Omega(Z)) \tag{12.8}$$

where $\pi_R^*\Omega(Z)$ is the density bundle over Z, pulled back to the product under the projection onto to the right-hand factor.

LEMMA 12.2. *For a fibration* (12.5) *the densities bundles on the fibres form a trivial bundle, denoted* $\Omega(M/B)$, *over the total space and the bundle of (compactly-supported) smoothing operators on the fibres may be identified as*

$$\Psi_c^{-\infty}(M/B) = \mathcal{C}_c^\infty(M \times_B M; \pi_R^*\Omega(M/B)) \tag{12.9}$$

where π_R *is the right projection from the total space of the fibre product to the total space of the fibration.*

PROOF. Perhaps this is more a definition than a Lemma. The fibre density bundle is just the density bundle for $T(M/B)$. It is then easy to see that an element on the right in (12.9) defines a smoothing operator on each fibre of the fibration and these operators vary smoothly when identified in a local trivialization of the fibration. This leads to the notation on the left. □

Again $\Psi_c^{-\infty}(M/B)$ can be thought of as a (big) bundle over B.

So, now to something a little less formal. As noted above, one case of a fibration is a vector bundle. If we consider a symplectic (or complex) we have discussed the Thom isomoprhism in K-theory above. In doing this we have used, rather extensively, the projections $\pi_{(N)}$ onto the first N eigenspaces of the harmonic oscillators. Since the index theorem is an geometric extension, to a general fibration, of the Thom isomorphism, we need some replacement for these 'exhausting projections' in the general case. Unfortunately there is nothing[3] to take the place of the harmonic oscillators on the fibres. Of course there are similar objects, such as the Laplacians for some family of fibre metrics, but the eigenvalues of such operators are not constant. As a result the eigenspaces are not even smooth and there is not simple replacement for $\pi_{(N)}$. But we really want these, so we have to construct them a little more crudely. I will do this using the embedding construction above; this is a similar argument to the core of the proof of the Atiyah-Singer theorem but in a much simpler setting.

First we note an extension result using these same $\pi_{(N)}$'s, or just $\pi_{(1)}$, the projection onto the ground state of the harmonic oscllator.

PROPOSITION 12.3. *Let* W *be a symplectic vector bundle over a compact manifold* Z *then there is a natural embedding as a subalgebra*

$$\Psi^{-\infty}(Z) \hookrightarrow \dot{\Psi}^{-\infty}(\bar{W}; \Lambda^*), \tag{12.10}$$

into the algebra of smoothing operator on the total space of the bundle of radial of the fibres of W *which vanish to infinite order at the boundary, acting on sections of the exterior algebra, in which an operator on* Z *is identified with an operator on the ground state of the bundle of harmonic oscillators.*

[3] As far as I know, please correct me if you know better.

Proof. The point here is simply that the bundle of ground states of the (bundle of) harmonic oscillators is canonically trivial. Indeed all these functions (and the projections onto them) are positive, so there is a unique choice of unit length basis. A smoothing operator on the manifold is then lifted to the same smoothing operator acting on this line bundle, so as a smoothing operator on the total space it is projection onto this bundle, followed by the action of the smoothing operator. Clearly this forms a subalgebra as claimed, since the Schwartz functions correspond to the functions vanishing at infinity on the radial compactification. □

Now, suppose the total space of W is mapped diffeomorphically to an open subset of a smooth manifold in such a way that $\mathcal{S}(W)$, the space of functions which are Schwartz on the fibres, is identified with the smooth functions with support in the closure of the image set. Then the algebra on the right in (12.10) is identified as the subalgebra of the smoothing operators on this manifold with supports in the closure of the image.

12.4. Elliptic families

12.5. Spectral sections

12.6. Analytic index

12.7. Topological index

12.8. Proof of index theorem

12.9. Chern character of the index bundle

Problems

Problem 12.1. Proof of Proposition 12.1.

Problem 12.2. Embedding manifolds.

APPENDIX A

Bounded operators on Hilbert space

Some of the main properties of bounded operators on a complex Hilbert space, H, are recalled here; they are assumed at various points in the text.

(1) Boundedness equals continuity, $\mathcal{B}(H)$.
(2) $\|AB\| \le \|A\|\,\|B\|$
(3) $(A-\lambda)^{-1} \in \mathcal{B}(H)$ if $|\lambda| \ge \|A\|$.
(4) $\|A^*A\| = \|AA^*\| = \|A\|^2$.
(5) Compact operators, defined by requiring the closure of the image of the unit ball to be compact, form the norm closure of the operators of finite rank.
(6) Fredholm operators have parametrices up to compact errors.
(7) Fredholm operators have generalized inverses.
(8) Fredholm operators for an open subalgebra.
(9) Hilbert-Schmidt operators?
(10) Operators of trace class?
(11) General Schatten class?

[1]

[1]Known as Gerard, my PhD advisor

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