CHAPTER 10

K-theory

This is a brief treatment of K-theory, enough to discuss, and maybe even prove, the Atiyah-Singer index theorem. I am starting from the smoothing algebra discussed earlier in Chapter 4 in order to give a 'smooth' treatment of K-theory (this approach is in fact closely related to the currently-in-vogue subject of 'smooth K-theory').

10.1. What do I need for the index theorem?

Here is a summary of the parts of this chapter which are used in the proof of the index theorem to be found in Chapter 12

(1) Odd K-theory ($\mathrm{K}^1_{\mathrm{c}}(X)$) defined as stable homotopy classes of maps in $\mathrm{GL}(N,\mathbb{C})$.
(2) Even K-theory ($\mathrm{K}_{\mathrm{c}}(X)$) defined as stable isomorphism classes of $\mathbb{Z}_2$-graded bundles
(3) The gluing identification of $\mathrm{K}^1_{\mathrm{c}}(X)$ and $\mathrm{K}_{\mathrm{c}}(\mathbb{R}\times X)$.
(4) The isotropic index map $\mathrm{K}^1_{\mathrm{c}}(\mathbb{R}\times X)\longrightarrow \mathrm{K}_{\mathrm{c}}(X)$ using the eigenprojections of the harmonic oscillator to stabilize the index.
(5) Bott periodicity – proof that this map is an isomorphism and hence that $\mathrm{K}_{\mathrm{c}}(X)\equiv \mathrm{K}_{\mathrm{c}}(\mathbb{R}^2\times X)$.
(6) Thom isomorphism $\mathrm{K}_{\mathrm{c}}(V)\longrightarrow \mathrm{K}_{\mathrm{c}}(X)$ for a complex (or symplectic) vector bundle over X. In particular the identification of the 'Bott element' $b\in \mathrm{K}_{\mathrm{c}}(V)$ which generates $\mathrm{K}_{\mathrm{c}}(V)$ as a module over $\mathrm{K}_{\mathrm{c}}(X)$.

With this in hand you should be able to proceed to the proof of the index theorem in K-theory in Chapter 12. If you want the 'index formula' which is a special case of the index theorem in cohomology you need a bit more, namely the discussion of the Chern character and Todd class below.

10.2. Odd K-theory

First recall the group

$$G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)=\left\{A\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n);\ \exists\ B\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n),\ \mathrm{Id}+B=(\mathrm{Id}+A)^{-1}\right\}. \quad (10.1)$$

Note that the notation is potentially confusing here. Namely I am thinking of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ as the subset consisting of those $A\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ such that $\mathrm{Id}+A$ is invertible. The group product is then not the usual product on $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ since

$$(\mathrm{Id}+A_1)(\mathrm{Id}+A_2)=\mathrm{Id}+A_1+A_2+A_1A_2.$$

Just think of the operator as 'really' being $\mathrm{Id}+A$ but the identity is always there so it is dropped from the notation.

One consequence of the fact that $\mathrm{Id}+A$ is invertible if and only if $\det(\mathrm{Id}+A) \neq 0$ is that[1]

$$G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \subset \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^{2n}) = \dot{\mathcal{C}}^\infty(\overline{\mathbb{R}^{2n}}) \text{ is open.} \tag{10.2}$$

In view of this there is no problem in understanding what a smooth (of if you prefer just continuous) map into $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is. Namely, it is a map into $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ which has range in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and the following statment can be taken as a definition of smoothness, but it is just equivalent to the standard notion of a smooth map with values in a topological vector space. Namely if X is a manifold then

$$\begin{gathered}\mathcal{C}^\infty(X;G^{-\infty}) = \\ \{a \in \mathcal{C}^\infty(X\times\overline{\mathbb{R}^{2n}}); a \equiv 0 \text{ at } X\times\mathbb{S}^{2n-1},\ a(x)\in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)\ \forall\ x\in X\},\\ \mathcal{C}^\infty_c(X;G^{-\infty}) = \{a \in \mathcal{C}^\infty(X\times\overline{\mathbb{R}^{2n}}); a \equiv 0 \text{ at } X\times\mathbb{S}^{2n-1},\\ a(x)\in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)\ \forall\ x\in X, \exists\ K \Subset X \text{ s.t. } a(x)=0\ \forall\ x\in X\setminus K\}.\end{gathered} \tag{10.3}$$

The two spaces in (10.3) (they are the same if X is compact) are groups. They are in fact examples of gauge groups (with an infinite-dimensional target group), where the composite of a and b is the map $a(x)b(x)$ given by composition in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Two elements $a_0,\ a_1 \in \mathcal{C}^\infty_c(X;G^{-\infty}_{\mathrm{iso}})$ are said to be homotopic (in fact smoothly homotopic, but that is all we will use) if there exists $a \in \mathcal{C}^\infty_c(X\times[0,1]_t;G^{-\infty}_{\mathrm{iso}})$ such that $a_0 = a\big|_{t=0}$ and $a_1 = a\big|_{t=1}$. Clearly if b_0 and b_1 are also homotopic in this sense then a_0b_0 is homotopic to a_1b_1, with the homotopy just being the product of homotopies. This gives the group property in the following definition:-

Definition 10.1. *For any manifold*

$$\mathrm{K}^1_c(X) = \mathcal{C}^\infty_c(X;G^{-\infty})/\sim \tag{10.4}$$

is the group of equivalence classes of elements under homotopy.

Now, we need to check that this is a reasonable definition, and in particular see how is it related to K-theory in the usual sense. To misquote Atiyah, K-theory is the topology of linear algebra. So, the basic idea is that $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is just a version of $\mathrm{GL}(N,\mathbb{C})$ where $N=\infty$. To make this concrete, recall that finite rank elements are actually dense in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Using the discussion of the harmonic oscillator in Chapter 4 we can make this even more concrete. Let $\pi_{(N)}$ be the projection onto the span of the first N eigenvalues of the harmonic oscillator (so if $n>1$ it is projecting onto space of dimension a good deal larger than N, but no matter). Thus $\pi_{(N)} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is an operator of finite rank, exactly the sum of the dimensions of these eigenspaces. Then, from the discussion in Chapter 4

$$\begin{gathered} f\in\mathcal{S}(\mathbb{R}^n) \Longrightarrow \pi_{(N)}f \longrightarrow f \text{ in } \mathcal{S}(\mathbb{R}^n) \text{ as } N\to\infty,\\ A\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \Longrightarrow \pi_{(N)}A\pi_{(N)} \to A \text{ in } \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \text{ as } N\to\infty.\end{gathered} \tag{10.5}$$

The range of $\pi_{(N)}$ is just a finite dimensional vector space, so isomorphic to $\mathbb{C}^M$ (where $M=M(N,n)$ and $M=N$ if $n=1$ to keep things simple), we are choosing a fixed linear isomorphism to $\mathbb{C}^M$ by choosing a particular basis of eigenfunctions of the harmonic oscillator. If $a\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then $\pi_{(N)}a\pi_{(N)}$ becomes a linear operator on $\mathbb{C}^M$, so an element of the matrix algebra.

[1]See Problem 10.6 if you want a proof not using the Fredholm determinant.

Proposition 10.1. *The 'finite rank elements' in $\mathcal{C}^\infty_c(X;G^{-\infty}_{\text{iso}}(\mathbb{R}^n))$, those for which $\pi_{(N)}A = A\pi_{(N)} = A$ for some N, are dense in $\mathcal{C}^\infty_c(X;G^{-\infty}(\mathbb{R}^n))$.*

These elements are not finite rank of course, they are of the form $\operatorname{Id}+F$ with F of finite rank. The sloppy statement is in keeping with the principal that the 'identity is always there'.

Proof. This just requires a uniform version of the argument above, which in fact follows from the pointwise version, to show that

$$A \in \mathcal{C}^\infty_c(X;\Psi^{-\infty}_{\text{iso}}) \Longrightarrow \pi_{(N)}A\pi_{(N)} \to A \text{ in } \mathcal{C}^\infty_c(X;\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)). \tag{10.6}$$

From this it follows that if $A \in \mathcal{C}^\infty_c(X;G^{-\infty}_{\text{iso}}(\mathbb{R}^n))$ (meaning if you look back, that $\operatorname{Id}+A$ is invertible) then $\operatorname{Id}+\pi_{(N)}A$ is invertible for N large enough (since it vanishes outside a compact set). □

Corollary 10.1. *The groups $\mathrm{K}^1_c(X)$ are independent of n, the dimension of the space on which the group acts (as is already indicated by the notation).*

In fact this shows that $\pi_{(N)}a\pi_{(N)}$ and a are homotopic in $\mathcal{C}^\infty_c(X;G^{-\infty}_{\text{iso}}(\mathbb{R}^n))$ provided N is large enough. Thus each element of $\mathrm{K}^1_c(X)$ is represented by a finite rank family in this sense (where the order N may depend on the element). Any two elements can then be represented by finite approximations for the same N. Thus there is a natural isomophism between the groups corresponding to different n's by finite order approximation. In fact this approximation argument has another very important consequence.

Proposition 10.2. *For any manifold $\mathrm{K}^1_c(X)$ is an Abelian group, i.e. the group product is commutative.*

Proof. I can now write the proof for $n = 1$ so assuming that N and the rank of $\pi_{(N)}$ are the same. As shown above, given two elements $[a]$, $[b] \in \mathrm{K}^1_c(X)$ we can choose representatives a, $b \in \mathcal{C}^\infty_c(X;G^{-\infty}_{\text{iso}}(\mathbb{R}^n))$ such that $\pi_{(N)}a = a\pi_{(N)} = a$ and $\pi_{(N)}b = b\pi_{(N)} = b$. Thus they are represented by elements of $\mathcal{C}^\infty_c(X;\mathrm{GL}(N,\mathbb{C}))$ for some large N. Now, the range of $\pi_{(2N)}$ contains two N dimensional spaces, the ranges of $\pi_{(N)}$ and $\pi_{(2N)} - \pi_{(N)}$. Since we are picking bases in each, we can identify these two N dimensional spaces and then represent an element of the $2N$-dimensional space as a 2-vector of N-vectors. This decomposes $2N \times 2N$ matrices as 2×2 matrices with $N \times N$ matrix elements. In fact this tensor product of the 2×2 and $N \times N$ matrix algebras gives the same product as $2N \times 2N$ matrices (as follows easily from the definitions). Now, consider a rotation in 2 dimensions, represented by the rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{10.7}$$

This rotates the standard basis e_1, e_2 to e_2, $-e_1$ as θ varies from 0 to $\pi/2$. If we interpret it as having entries which are multiples of the identity as an $N\times N$ matrix, and then conjugate by it, we get a curve

$$\begin{aligned} a(x,\theta) &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \operatorname{Id}_N \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} a\cos^2\theta + \sin^2\theta & (\operatorname{Id}-a)\sin\theta\cos\theta \\ (\operatorname{Id}-a)\sin\theta\cos\theta & \cos^2\theta + a\sin^2\theta \end{pmatrix}. \end{aligned} \tag{10.8}$$

This is therefore an homotopy between a represented as an $N \times N$ matrix and the same element acting on the *second* N dimensional subspace, i.e. it becomes

$$\begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & a \end{pmatrix}. \tag{10.9}$$

This commutes with the second element which acts only in the first N dimensional space, so the product in $\mathrm{K}^1_{\mathrm{c}}(X)$ is commutative. □

So now we see that $\mathrm{K}^1_{\mathrm{c}}(X)$ is an Abelian group associated quite naturally to the space X. I should say that the notation is not quite standard. Namely the standard notation would be $\mathrm{K}^1(X)$, without any indication of the 'compact supports' that are involved in the definition. I prefer to put this in explicitly. Of course if X is compact it is not necessary.

Now, what about the other 'even' group. We can define this using the following computation.

Proposition 10.3. *For any manifold the natural inclusions*

$$\begin{gathered} \mathcal{C}^\infty_c(X; G^{-\infty}_{\mathrm{iso}}) \longrightarrow \mathcal{C}^\infty_c(X \times \mathbb{S}; G^{-\infty}_{\mathrm{iso}}), \\ \left\{ a \in \mathcal{C}^\infty_c(X \times \mathbb{S}; G^{-\infty}_{\mathrm{iso}}); a(x,1) = \mathrm{Id} \ \forall \ x \in X \right\} \longrightarrow \mathcal{C}^\infty_c(X \times \mathbb{S}; G^{-\infty}_{\mathrm{iso}}) \end{gathered} \tag{10.10}$$

where the first inclusion is as pullback under the projection $X \times \mathbb{S} \longrightarrow X$, define complementary subgroups of $\mathrm{K}^1_c(X)$ which therefore splits as

$$\mathrm{K}^1_c(X \times \mathbb{S}) = \mathrm{K}^1_c(X) \oplus \mathrm{K}^0_c(X), \tag{10.11}$$

defining the second group which can also be naturally identified as

$$K^0_c(X) = K^1_c(\mathbb{R} \times X). \tag{10.12}$$

Proof. The first inclusion is a left inverse to the restriction map

$$\mathcal{C}^\infty_c(X \times \mathbb{S}; G^{-\infty}_{\mathrm{iso}}) \ni (x, \theta) \longmapsto a(x,1) \in \mathcal{C}^\infty_c(X; G^{-\infty}_{\mathrm{iso}}). \tag{10.13}$$

This restriction clearly gives a short exact sequence of groups

$$\begin{gathered} \left\{ a \in \mathcal{C}^\infty_c(X \times \mathbb{S}; G^{-\infty}_{\mathrm{iso}}); a(x,1) = \mathrm{Id} \ \forall \ x \in X \right\} \longrightarrow \\ \mathcal{C}^\infty_c(X \times \mathbb{S}; G^{-\infty}_{\mathrm{iso}}) \longrightarrow \mathcal{C}^\infty_c(X; G^{-\infty}_{\mathrm{iso}}) \end{gathered} \tag{10.14}$$

which therefore splits. Under homotopy this becomes the direct sum decomposition (10.11).

So the definition of $K^0_{\mathrm{c}}(X)$ reduces to equivalence classes of elements $a \in \mathcal{C}^\infty_c(\mathbb{S} \times X; G^{-\infty}_{\mathrm{iso}})$ such that $a(1,x) = \mathrm{Id}$ and the same is required for homotopies. Since all supports are compact it is easy to see that any such element is homotopic to one which satisfies $a(\theta, x) = \mathrm{Id}$ for $|\theta - 1| < \epsilon$, i.e. is equal to the identity in a neighbourhood of $\{1\} \times X$ – and hence the same can be arranged for homotopies. If we identify $\mathbb{S} \setminus \{1\}$ with $\mathbb{R}$ this reduces precisely to (10.12). □

So, now we have the two Abelian groups $\mathrm{K}^1_{\mathrm{c}}(X)$ and $\mathrm{K}^0_{\mathrm{c}}(X)$ associated to the manifold X. The direct sum, here just $\mathrm{K}^1_{\mathrm{c}}(\mathbb{S} \times X)$ is often just denoted $\mathrm{K}^*_{\mathrm{c}}(X)$ (in fact usually without the c suffix) so

$$\mathrm{K}^*_{\mathrm{c}}(X) = \mathrm{K}^1_{\mathrm{c}}(\mathbb{S} \times X). \tag{10.15}$$

If you know a little topology, you will see that the discussion here starts from the premise that $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is a *classifying space* for odd K-theory. So this is true by fiat. The corresponding classifying space for even K-theory is then the pointed loop group, the set of maps

$$G^{-\infty}_{\mathrm{iso},s} = \{a \in \mathcal{C}^\infty(\mathbb{S}; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n); a(1) = \mathrm{Id}\}. \tag{10.16}$$

If you look at Proposition 10.3 you will see that it amounts to defining $\mathrm{K}^0_{\mathrm{c}}(X)$ as the homotopy classes of maps in $\mathcal{C}^\infty_c(X;_{\mathrm{iso},s})$. This is somewhat backwards compared to the usual definition and in fact this group is really more naturally denoted $\mathrm{K}^2_{\mathrm{c}}(X)$. Fortunately it is naturally isomorphic to the 'true' $\mathrm{K}^0_{\mathrm{c}}(X)$.

10.3. Computations

Let us pause for a moment to compute some simple cases. Namely

Lemma 10.1.

$$\mathrm{K}^1(\{pt\}) = \{0\},\ \mathrm{K}^1_c(\mathbb{R}) = \mathbb{Z},\ \mathrm{K}^1(\mathbb{S}) = \mathbb{Z}. \tag{10.17}$$

Proof. These two statements follow directly from the next two results. □

Lemma 10.2. *The group $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is connected.*

Proof. If $a \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, the curve $[0,1] \ni t \longmapsto (1-t)aa + t\pi_{(N)}a\pi_{(N)}$ lies in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ for N sufficiently large. Thus it suffices to show that $\mathrm{GL}(n,\mathbb{C})$ is connected for large N; of course[2]

$$\mathrm{GL}(N,\mathbb{C}) \text{ is connected for all } N \geq 1. \tag{10.18}$$

□

Proposition 10.4. *A closed loop in $\gamma : S \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is contractible (homotopic through loops to a constant loop) if and only if the composite map*

$$\tilde{\gamma} = \det \circ \gamma : \mathbb{S} \longrightarrow \mathbb{C}^* \tag{10.19}$$

is contractible, so

$$\pi_1(G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)) = \mathbb{Z} \tag{10.20}$$

with the identification given by the winding number of the Fredholm determinant.

Proof. Again, as in the previous proof but now a loop can be deformed into $\mathrm{GL}(N,\mathbb{C})$ so it is certainly enough to observe that[3]

$$\pi_1(\mathrm{GL}(N,\mathbb{C})) = \mathbb{Z} \text{ for all } N \geq 1. \tag{10.21}$$

□

[2]See Problem 10.8

[3]Proof in Problem 10.9

10.4. Vector bundles

The notion of a complex vector bundle was briefly discussed earlier in Section 6.2. Recall from there the notion of a bundle isomorphism and that a bundle is said to be trivial (over some set) if there is a bundle isomorphism to $X \times \mathbb{C}^k$ (over this set). The direct sum of vector bundles and the tensor product are also briefly discussed there (I hope).

To see that there is some relationship between K-theory as discussed above and vector bundles consider $K^1(X)$ for a compact manifold, X. First note that if V is a complex vector bundle over X and $e : V \longrightarrow V$ is a bundle isomorphism, then e defines an element of $\mathrm{K}^1(X)$. To see this we first observe we can always find a complement to V.

Proposition 10.5. *Any vector bundle V which is trivial outside a compact subset of X can be complemented to a trivial bundle, i.e. there exists a vector bundle E and a bundle isomorphism*

$$V \oplus E \longrightarrow X \times \mathbb{C}^N. \tag{10.22}$$

Proof. This follows from the local triviality of V. Choose a finite open cover U_i of X with M elements in which one set is $U_0 = X \setminus K$ for K compact and such that V is trivial over each U_i. Then choose a partition of unity subordinate to U_i – so only the $\phi_0 \in \mathcal{C}^\infty(X)$ with support in U_0 does not have compact support. If $f_i : V\big|_{U_i} \longrightarrow \mathbb{C}^N \times U_i$ is a trivialization over U_i (so the one over U_0 is given by the assumed triviality outside a compact set) consider

$$F : V \longrightarrow X \times \mathbb{C}^{NM},\ u(x) \longmapsto \bigoplus_{i=1}^{M} f_i(\phi_i(u(x))). \tag{10.23}$$

This embeds V as a subbundle of a trivial bundle of dimension NM since the map F is smooth, linear on the fibres and injective. Then we can take E to be the orthocomplement of the range of F which is identified with V. □

Thus, a bundle isomorphism e of V can be extended to a bundle isomorphism $e \oplus \mathrm{Id}_E$ of the trivial bundle. This amounts to a map $X \longrightarrow \mathrm{GL}(MN, \mathbb{C})$ which can then be extended to an element of $\mathcal{C}^\infty(X; G^{-\infty}_{\text{iso}}(\mathbb{R}^n))$ and hence gives an element of $\mathrm{K}^1_{\mathrm{c}}(X)$ as anticipated. It is straightforward to check that the element defined in $\mathrm{K}^1(X)$ does not depend on choices made in its construction, only on e (and through it of course on V.)

This is one connection between bundles and $\mathrm{K}^1_{\mathrm{c}}$. There is another, similar, connection which is more important. Namely from a class in $\mathrm{K}^1_{\mathrm{c}}(X)$ we can construct a bundle over $\mathbb{S} \times X$. One way to do this is to observe that Proposition 10.5 associates to a bundle V a smooth family of projections $\pi_V \in \mathcal{C}^\infty_c(X; M(N, \mathbb{C}))$ which is trivial outside a compact set, in the sense that it reduces to a fixed projection there. Namely, π_V is just (orthogonal) projection onto the range of V. We will need to think about equivalence relations later, but certainly such a projection defines a bundle as well, namely its range.

For the following construction choose a smooth function $\Theta : \mathbb{R} \longrightarrow (0, 2\pi)$ which is non-decreasing, constant with the value 0 on some $(-\infty, -T]$ and with the value 2π on $[T, \infty)$ and strictly increasing otherwise. We also assume that Θ is 'odd' in the sense that

$$\Theta(-t) = 2\pi - \Theta(t). \tag{10.24}$$

This is just a function which we can used to progressively 'rotate' through angle 2π but staying constant initially and near the end. In fact if $a \in \mathcal{C}^\infty(X;\mathbb{C}^N)$ then

$$\begin{aligned} \mathbb{R}\times X \ni (t,x) &\longmapsto R_a(t,x) \\ &= \begin{pmatrix} \cos(\Theta(t))\,\mathrm{Id}_N & -\sin(\Theta(t))a(x) \\ \sin(\Theta(t))a(x)^{-1} & \cos(\Theta(t))\,\mathrm{Id}_N \end{pmatrix} \in \mathrm{GL}(2N,\mathbb{C}) \end{aligned} \tag{10.25}$$

has inverse $R_a(-t,x)$ and is equal to the identity in $|t|>T$. The idea is that it 'rotates once' between the identity and a. Now consider the family of projections

$$\begin{aligned} \Pi_a(t,x) &= R_{\mathrm{Id}}(t)\Pi'_a(t,x)R_{\mathrm{Id}}(-t) \\ \Pi'_a(t,x) &= R_a(-t,x)\begin{pmatrix}\mathrm{Id} & 0\\ 0 & 0\end{pmatrix}R_a(t,x) \\ &= \begin{pmatrix} \cos^2(\Theta(t))\,\mathrm{Id}_N & -\cos(\Theta(t))\sin(\Theta(t))a(x) \\ -\sin(\Theta(t))\cos(\Theta(t))a^{-1}(x) & \sin^2(\Theta(t)) \end{pmatrix}. \end{aligned} \tag{10.26}$$

Note, for later reference that

$$\Pi_a(t,x) \text{ has entries linear in } a \text{ and } a^{-1}. \tag{10.27}$$

Lemma 10.3. *An element $a\in\mathcal{C}^\infty_c(X;\mathrm{GL}(N,\mathbb{C}))$ defines a smooth family of matrices with values in the projections, $\Pi_a\in\mathcal{C}^\infty(\mathbb{R}\times X;M(2n,\mathbb{C}))$, which is constant outside a compact subset and so defines a vector bundle over $\mathbb{R}\times X$ which is trivial outside a compact set.*

Proof. That Π_a is a projection (sometimes people say idempotent when it may not be self-adjoint as in this case) follows from the definition in (10.26), since $\Pi_a^2=\Pi_a$. Moreover, where $a=\mathrm{Id}$, which is the case outside a compact subset of X, $R_a(t,x)R_{\mathrm{Id}}(-t,x)=\mathrm{Id}$ so Π_a is the constant projection corresponding to projection on the first N coefficients. The same is true in $|t|>T$ so indeed Π_a is constant outside a compact subset of $\mathbb{R}\times X$. □

So, by now it should not be so surprising that the K-groups introduced above are closely related to the 'Grothendieck group' constructed from vector bundles. The main issue is the equivalence relation.

Definition 10.2. *For a manifold X, $\mathrm{K}_c(X)$ is defined as the set of equivalence classes of pairs of complex vector bundles (V,W), both trivial outside a compact set and with given trivializations a, b there, under the relation $(V_1,W_1;a_1,b_1)\sim(V_2,W_2;a_2,b_2)$ if and only if there is a bundle S and a bundle isomorphism*

$$T: V_1\oplus W_2\oplus S \longrightarrow V_2\oplus W_1\oplus S \tag{10.28}$$

which is equal to $(a_2\oplus b_2)^{-1}(a_1\oplus b_2)\oplus\mathrm{Id}_S$ outside some compact set.

Note that if X is compact then the part about the trivializations is completely void, then we just have pairs of vector bundles (V,W) and the equivalence relation is the existence of a stabilizing bundle S and a bundle isomorphism (10.28).

This is again an Abelian group with the group structure given at the level of pairs of bundles (V_i,W_i), $i=1$, 2 by[4]

$$[(V_1,W_1)]+[V_2,W_2)]=[(V_1\oplus V_2,W_1\oplus W_2)] \tag{10.29}$$

[4]See Problem 10.7 for the details.

with the trivializations $(a_1 \oplus a_2), (b_1 \oplus b_2)$. In particular $[(V, V)]$ is the zero element for any bundle V (trivial outside a compact set).

The equivalence relation being (stable) bundle isomorphism rather than some sort of homotopy may seen strange, but it actually more general.

LEMMA 10.4. *If V is a vector bundle over $[0,1]_t \times X$ which is trivial outside a compact set then $V_0 = V\big|_{t=0}$ and $V_1 = V\big|_{t=1}$ are bundle isomorphic over X with an isomorphism which is trivial outside a compact set.*

PROOF. The proof is 'use a connection and integrate'. We can do this explicitly as follows. First we can complement V to a trivial bundle so that it is identified with a constant projection outside a compact set, using Proposition 10.5. Let the family of projections be $\pi_V(t,x)$ in $M \times M$ matrices. We want to differentiate sections of the bundle with respect to t. Since they are M-vectors we can do this, but we may well not get sections this way. However defining the (partial) connection by

$$\nabla_t v(t) = v'(t) - \pi'_V v(t) \Longrightarrow (\mathrm{Id} - \pi_V)\nabla_t v(t) = ((\mathrm{Id} - \pi_V)v(t))' = 0 \tag{10.30}$$

if $\pi_V v = v$, i.e. if v is a section. Now this is just a system of ordinary differential equations, so $\nabla_t v(t) = 0$ has a unique solution with $v(0) = v_0 \in V_0$ fixed. Then define $F : V_0 \longrightarrow V_1$ by $Fv_0 = v(1)$. This is a bundle isomorphism. □

PROPOSITION 10.6. *For any manifold X the construction in Lemma 10.3 gives an isomorphism*

$$\mathrm{K}^1_c(X) \ni [a] \longrightarrow [(\Pi_a, \Pi_a^\infty)] = \mathrm{K}_c(\mathbb{R} \times X) \tag{10.31}$$

where Π_a^∞ is the constant projection to which Π_a restricts outside a compact set.

PROOF. The vector bundle fixed by Π_a in Lemma 10.3 fixes an element of $\mathrm{K_c}(\mathbb{R} \times X)$ but we need to see that it is independent of the choice of a representing $[a] \in \mathrm{K}^1_{\mathrm{c}}(X)$. A homotopy of a gives a bundle over $[0,1] \times X$ and then Lemma 10.4 shows that the resulting bundles are isomorphic. Stabilizing a, i.e. enlarging it by an identity matrix adds a trivial bundle to Π_a and the same trivial projection to Π_a^∞. Thus the map in (10.31) is well defined. So we need to show that it is an isomorphism. First we should show that it is additive – see Problem 10.1.

If V is a bundle over $\mathbb{R} \times X$ which is trivial outside a compact set, we can embed it as in Proposition 10.5 so it is given by a family of projections π_V (this of course involves a bundle isomorphism). Now, using the connection as in (10.30) we can define an isomorphism of the trivial bundle π_V^∞. Namely, integrating from $t = -T$ to $t = T$ defines an isomorphism a. The claim is that $(\Pi_a, \Pi_a^\infty) = (V, V^\infty)$. I leave the details to you, there is some help in Problem 10.2. Conversely, this construction recovers a from Π_a so shows that (10.31) is injective and surjective. □

PROBLEM 10.1. Additivity of the map (10.31).

PROBLEM 10.2. Details that (10.31) is an isomorphism.

10.5. Isotropic index map

Now, (10.31) is part of Bott periodicity. The remaining part is that, for any manifold X there is a natural isomorphism

$$\mathrm{K}^1_{\mathrm{c}}(\mathbb{R} \times X) \longrightarrow \mathrm{K_c}(X). \tag{10.32}$$

If we regard this as an identification (and one has to be careful about orientations here) then it means that we have identified

$$\mathrm{K}^0_{\mathrm{c}}(X) = \mathrm{K}^1_{\mathrm{c}}(\mathbb{R} \times X) = \mathrm{K}_{\mathrm{c}}(X) = \mathrm{K}_{\mathrm{c}}(\mathbb{R}^2 \times X) \tag{10.33}$$

as is discussed more below. For the moment what we will work on is the definition of the map in (10.32). This is the 'isotropic' (or 'Toeplitz'[5]) index map.

Problem 10.3. Toeplitz instead of isotropic.

Finally we get to the start of the connection of this stuff with index theory. An element of $\mathrm{K}^1_{\mathrm{c}}(\mathbb{R} \times X)$ is represented by a map from $\mathbb{R} \times X$ to $\mathrm{GL}(N, \mathbb{C})$, for some N, and with triviality outside a compact set. In particular this map reduces to the identity near $\pm\infty$ in $\mathbb{R}$ so we can join the ends and get a map

$$\tilde{a} \in \mathcal{C}^\infty(\mathbb{S} \times X; \mathrm{GL}(N, \mathbb{C})),\ \tilde{a} = \mathrm{Id} \ \text{near } \{1\} \times X \text{ and outside a compact set.}$$

This indeed is close to the original definition of $\mathrm{K}^0_{\mathrm{c}}(X)$ above. Now, we can interpret $\tilde{a}$ as the principal symbol of an elliptic family in $\Psi^0_{\mathrm{iso}}(\mathbb{R}; \mathbb{C}^N)$ depending smoothly on $x \in X$ (and reducing to the identity outside a compact set). Let's start with the case $X = \{\mathrm{pt}\}$ so there are no parameters.

Proposition 10.7. *If $A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}; \mathbb{C}^N)$ is elliptic with principal symbol $a = \sigma_0(A) \in \mathcal{C}^\infty(\mathbb{S}; \mathrm{GL}(N, \mathbb{C}))$ then the index of A is given by the winding number of the determinant of the symbol*

$$\mathrm{Ind}_a(A) = -\,\mathrm{wn}(\det(a)) = -\frac{1}{2\pi i}\int_{\mathbb{S}} \mathrm{tr}(a^{-1}\frac{da}{d\theta})d\theta \tag{10.34}$$

and if $a = \mathrm{Id}$ near $\{1\} \in \mathbb{S}$ then $\mathrm{Ind}_a(A) = 0$ if and only if $[a] = 0 \in \mathrm{K}^1_c(\mathbb{S})$.

Proof. This follows from Proposition 10.4. First, recall what the winding number is. Then check that it defines the identification (10.20). Observe that the index is stable under homotopy and stabilization and that the index of a product is the sum of the indices. Then check one example with index 1, namely for the annihilation operator will suffices. For general A with winding number m, compose with m factors of the creation operator – the adjoint of the annihilation operator. This gives an operator with symbol for which the winding number is trivial. By Proposition 10.4 it can be deformed to the identity after stabilization, so its index vanishes and (10.34) follows. □

Now for the analytic step that allows us to define the full (isotropic) index map.

Proposition 10.8. *If $a \in \mathcal{C}^\infty_c(\mathbb{R} \times X; \mathrm{GL}(N, \mathbb{C}))$ (so it reduces to the identity outside a compact set) then there exists $A \in \mathcal{C}^\infty(X; \Psi^0_{\mathrm{iso}}(\mathbb{R}))$ with $\sigma_0(A) = a$, A constant in $X \setminus K$ for some compact K and such that $\mathrm{null}(A)$ is a (constant) vector bundle over X.*

Proof. We can choose a $B \in \mathcal{C}^\infty(X; \Psi^0_{\mathrm{iso}}(\mathbb{R}))$ with $\sigma(B) = a$ by the surjectivity of the symbol map. Moreover, taking a function $\psi \in \mathcal{C}^\infty(X)$ which is equal to 1 outside a compact set in X but which vanishes where $a \neq \mathrm{Id}$, $(1 - \psi)B + \psi\,\mathrm{Id}$ has the same principal symbol and reduces to Id outside a compact set.

The problem with this initial choice is that the dimension of the null space may change from point to point. However, we certainly have a parametrix $G_B \in$

[5] See Problem 10.3 for this alternative approach.

$\mathcal{C}^\infty(X;\Psi^0_{\rm iso}(\mathbb{R}))$ which we can take to be equal to the identity outside a compact set, by the same method, and which then satisfies

$$G_BB = \operatorname{Id} + R_1,\ BG_B = \operatorname{Id} + R_2,\ R_i \in \mathcal{C}^\infty_c(X;\Psi^{-\infty}_{\rm iso}(\mathbb{R}). \tag{10.35}$$

So, recall the finite rank projection $\pi_{(N)}$ onto the span of the first N eigenspaces. We know that $R_1\pi_{(N)} \to R_1$ in $\Psi^{-\infty}_{\rm iso}(\mathbb{R})$ and this is true uniformly on X since the support in X is compact. So, if N is large enough $\sup_{x\in X}\|R_1(x)(\operatorname{Id} - \pi_{(N)})\| < \frac{1}{2}$. Composing the first equation in (10.35) on the right with $\operatorname{Id} - \pi_{(N)}$ we find that

$$G_BB(\operatorname{Id} - \pi_{(N)}) = (\operatorname{Id} + R_1(\operatorname{Id} - \pi_{(N)}))(\operatorname{Id} - \pi_{(N)}) \tag{10.36}$$

where the fact that $\operatorname{Id} - \pi_{(N)}$ is a projection is also used. Now $(\operatorname{Id} + R_1(\operatorname{Id} - \pi_{(N)}))^{-1} = \operatorname{Id} + S_1$ where $S_1 \in \mathcal{C}^\infty_c(X;\Psi^{-\infty}_{\rm iso}(\mathbb{R}))$ by the openness of $G^{-\infty}_{\rm iso}(\mathbb{R})$. So if we set $A = B(\operatorname{Id} - \pi_{(N)})$ and $G = (\operatorname{Id} + S_1)G_B$ we see that

$$GA = \operatorname{Id} - \pi_{(N)}. \tag{10.37}$$

In particular the null space of $A(x)$ for each x is exactly the span of $\pi_{(N)}$ – it certainly annihilates this set but can annihilate no more in view of (10.37). Moreover A has the same principal symbol as B and is constant outside a compact set in X. $\square$

Now, once we have chosen A as in Proposition 10.8 it follows from the constancy of the index that family $A(x)^*$ also has null spaces of constant finite dimension, and indeed these define a smooth bundle over X which, if X is not compact, reduces to $\pi_{(N)}$ near infinity – since $A = \operatorname{Id} - \pi_{(N)}$ there. Thus we arrive at the index map.

Proposition 10.9. *If A is as in Proposition 10.8 the the null spaces of $A^*(x)$ form a smooth vector bundle R over X defining a class $[(\pi_{(N)}, R)] \in \mathrm{K}_c(X)$ which depends only on $[a] \in \mathrm{K}^1_c(\mathbb{R}\times X)$ and so defines an additive map*

$$\operatorname{Ind}_a : \mathrm{K}^1_c(\mathbb{R}\times X) \longrightarrow \mathrm{K}_c(X). \tag{10.38}$$

Proof. In the earlier discussion of isotropic operators it was shown that an elliptic operator has a generalized inverse. So near any particular point $\bar{x} \in X$ we can add an element $E(\bar{x}) \in \Psi^{-\infty}_{\rm iso}(\mathbb{R};\mathbb{C}^N)$ to $G(\bar{x})$ so that $H(\bar{x}) = G(\bar{x}) + E(\bar{x})$ is a generalized inverse, $H(\bar{x})A(\bar{x}) = \operatorname{Id} - \pi_{(N)}$, $A(\bar{x}_H(\bar{x}) = \operatorname{Id} - \pi'(\bar{x})$ where $\pi'(\bar{x})$ is a finite rank projection onto a subspace of $\mathcal{S}(\mathbb{R})$. Then $H(x) = G(x) + E(\bar{x})$ is still a parametrix nearby and

$$H(x)A(x) = \operatorname{Id} - \pi_{(N)},\ A(x)H(x) = \operatorname{Id} - p(x) \text{ near } \bar{x} \tag{10.39}$$

where $p(x)$ must have constant rank. Indeed, it follows that $p(x)\pi'(\bar{x})$ is a smooth bundle isomorphism, near $\bar{x}$, from the range of $\pi'(\bar{x})$ to the null space of A^*. This shows that the null spaces of the $A^*(x)$ form a bundle, which certainly reduces to $\pi_{(N)}$ outside a compact set. Thus

$$[(\pi_{(N)}, \operatorname{null}(A^*))] \in \mathrm{K}_c(X). \tag{10.40}$$

Next note the independence of this element of the choice of N. It suffices to show that increasing N does not change the class. In fact increasing N to $N+1$ replaces A by $A(\operatorname{Id} - \pi_{(N+1)})$ which has null bundle increased by the trivial line bundle $(\operatorname{Id}_{(N+1)} - \pi_{(N)})$. The range of A then decreases by the removal of the trivial bundle $A(x)(\operatorname{Id}_{(N+1)} - \pi_{(N)})$ and $\operatorname{null}(A^*)$ increases correspondingly. So the class in (10.40) does not change.

To see that the class is independent of the choice of A, for fixed a, consider two such choices. Initially the choice was of an operator with a as principal symbol, two choices are smoothly homotopic, since $tA + (1-t)A'$ is a smooth family with constant symbol. The same construction as above now gives a pair of bundles over $[0,1] \times X$, trivialized outside a compact set, and it follows from Lemma 10.4 that the class is constant. A similar discussion shows that homotopy of a is just a family over $[0,1] \times X$ so the discussion above applies to it and shows that the bundles can be chosen smoothly, again from Lemma 10.4 the class is constant. □

It is important to understand what the index tell us.

PROPOSITION 10.10. *If* $a \in \mathcal{C}^\infty_c(\mathbb{R} \times X; \mathrm{GL}(N,\mathbb{C}))$ *then* $\mathrm{Ind}_a(a) = 0$ *if and only if there is a family* $A \in \mathcal{C}^\infty(X; \Psi^0_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^N))$ *with* $\sigma_0(A) = a$ *which is constant outside a compact set in* X *and everywhere invertible.*

PROOF. The definition of the index class above shows that a may be quantized to an operator with smooth null bundle and range bundle such with $\mathrm{Ind_a}(a)$ represented by $(\pi_{(N)}, p')$ where p' is the null bundle of the adjoint. If A can be chosen invertible this class is certainly zero. Conversely, if the class vanishes then after stabilizing with a trivial bundle $\pi_{(N)}$ and p' become bundle isomorphic. This just means that they are isomorphic for sufficiently large N with the isomorphism being the trivial one near infinity. However this isomorphism is itself an element of $\mathcal{C}^\infty(X; \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^N))$ which is trivial near infinity. Adding it to A gives an invertible realization of the symbol, proving the Proposition. □

10.6. Bott periodicity

Now to the proof of Bott periodicity. Choose a 'Bott' element, which in this case is a smooth function

$$\beta(t) = e^{i\Theta(t)} \Longrightarrow \begin{cases} \beta : \mathbb{R} \longrightarrow \mathbb{C}^*,\ \beta(t) = 1 \text{ for } |t| > T, \\ \arg \beta(t) \text{ increasing over } (0, 2\pi) \text{ for } t \in (-T, T) \end{cases} \tag{10.41}$$

where Θ satisfies (10.24) and the preceeding conditions. Thus β has winding number one but is constant near infinity.

We first show

PROPOSITION 10.11. *The map* (10.38) *is surjective with explicit left inverse generated by mapping a smooth projection (constant near infinity)*

$$(\pi_V, \pi_V^\infty) \longmapsto \beta(t)^{-1}\pi_V + (\mathrm{Id} - \pi_V) \in \mathcal{C}^\infty_c(\mathbb{R} \times X; \mathrm{GL}(N,\mathbb{C})). \tag{10.42}$$

PROOF. The surjectivity follows from the existence of a left inverse, so we need to investigate (10.42). Observe that $\beta(t)^{-1}$, when moved to the circle, is a symbol with winding number 1. By Proposition 10.7 we may choose an elliptic operator $b \in \Psi^0_{\mathrm{iso}}(\mathbb{R})$ which has a one-dimensional null space and has symbol in the same class in $\mathrm{K}^1_{\mathrm{c}}(\mathbb{R})$ as β^{-1}. In fact we could take the annihilation operator, normalized to have order 0. Then we construct an elliptic family $B_V \in \Psi^0_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^N)$ by setting

$$B_V = \pi_V(x) b + (\mathrm{Id} - \pi_V(x)),\ x \in X. \tag{10.43}$$

The null space of this family is clearly $\pi_V \times N$, where N is the fixed one-dimensional vector space null(b). Thus indeed

$$\mathrm{Ind_a}(B_V) = [(\pi_V, \pi_V^\infty)] \in \mathrm{K_c}(X). \tag{10.44}$$

This proves the surjectivity of $\mathrm{Ind_a}$ in this isotropic setting. □

With some danger of repeating myself, if X is compact the 'normalizing term' at infinity π_V^∞ is dropped. You shall now see why we have been dragging this non-compact case along, it is rather handy even if interest is in the compact case.

This followign proof that $\mathrm{Ind_a}$ is injective is a variant of the 'clever' argument of Atiyah (maybe it is very clever – look at the original proof by Bott or the much more computational, but actually rather enlightening, argument in [**1**]).

Proposition 10.12. *For any manifold X, the isotropic index map in* (10.32), (10.38) *is an isomorphism*

$$\mathrm{K}_c^1(\mathbb{R}\times X)\simeq \mathrm{K}_c(X). \tag{10.45}$$

Proof. Following Proposition 10.11 only the injectivity of the map remains to be shown. Rather than try to do this directly we use another carefully chosen homotopy.

So, we need to show that if $a\in\mathcal{C}_c^\infty(\mathbb{R}\times X;\mathrm{GL}(N,\mathbb{C}))$ has $\mathrm{Ind_a}(a)=0$ then $0=[a]\in\mathrm{K}_\mathrm{c}^1(\mathbb{R}_s\times X)$. As a first step we use the construction of Proposition 10.6 and Lemma 10.3 to construct the image of $[a]$ in $\mathrm{K_c}(\mathbb{R}^2\times X)$. It is represented by the projection-valued matrix

$$\Pi_a(t,s,x)\in\mathcal{C}_c^\infty(\mathbb{R}^2\times X;M(2N,\mathbb{C})) \tag{10.46}$$

which is constant near infinity. Then we use the surjectivity of the index map in the case

$$\mathrm{Ind_a}:\mathrm{K_c}(\mathbb{R}\times(\mathbb{R}^2\times X))\longrightarrow\mathrm{K_c}(\mathbb{R}^2\times X) \tag{10.47}$$

and the explicit lift (10.43) to construct

$$\begin{gathered}e\in\mathcal{C}_c^\infty(\mathbb{R}^2\times X;\mathrm{GL}(2N,\mathbb{C})),\ e(r,t,s,x)=\beta(r)\Pi_a(t,s,x)+(\mathrm{Id}-\Pi_a(t,s,x)),\\ \mathrm{Ind_a}(e)=[\Pi_a,\Pi_a^\infty]\in\mathrm{K_c}(\mathbb{R}^2\times X).\end{gathered} \tag{10.48}$$

Here the 'r' variable is the one which is interpreted as the variable in the circle at infinity on $\mathbb{R}^2$ to turn e into a symbol and hence a family of elliptic operators with the given index. However we can rotate between the variables r and s, which is an homotopy replacing $e(r,t,s,x)$ by $e(-s,t,r,x)$. Since the index map is homotopy invariant, this symbol must give the same index class. Now, the third variable here is the argument of a, the original symbol. So the quantization map just turns a and a^{-1} which appears in the formula for Π_a – see (10.27) – into any operator with these symbols. By Proposition 10.10 a (mabye after a little homotopy) is the symbol of an invertible family. Inserting this in place of a and its inverse for a^{-1} gives an invertible family of operators with symbol $e(-s,t,r,x)$[6]. Thus $\mathrm{Ind_a}(e)=0$, but this means that

$$0=[(\Pi_a,\Pi_a^\infty)]\in\mathrm{K_c}(\mathbb{R}^2\times X)\Longrightarrow 0=[a]\in\mathrm{K}_\mathrm{c}^1(\mathbb{R}\times X). \tag{10.49}$$

This shows the injectivity of the isotropic index map and so proves Bott periodicity. □

Problem 10.4.

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[6] See Problem 10.4 for some more details

What does this tell us? Well, as it turns out, lots of things! For one thing the normalization conditions extend to all Euclidean space:-

$$\mathrm{K}^1_{\mathrm{c}}(\mathbb{R}^k) = \begin{cases} \{0\} & k \text{ even} \\ \mathbb{Z} & k \text{ odd,} \end{cases} \mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^k) = \begin{cases} \mathbb{Z} & k \text{ even} \\ \{0\} & k \text{ odd.} \end{cases} \tag{10.50}$$

This in turn means that we understand a good deal more about $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

Theorem 10.1 (Bott periodicity). *The homotopy groups $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ are*

$$\pi_j(G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)) = \begin{cases} \{0\} & k \text{ even} \\ \mathbb{Z} & k \text{ odd.} \end{cases} \tag{10.51}$$

Indeed Bott proved this rather directly using Morse theory.

10.7. Semiclassical quantization

Odd to odd.

10.8. Symplectic bundles

10.9. Thom isomorphism

10.10. Chern-Weil theory and the Chern character

I would not take this section seriously yet, I am going to change it.

Let's just think about the finite-dimensional groups $\mathrm{GL}(N,\mathbb{C})$ for a little while. Really these can be replaced by $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, as I will do below, but it may be a strain to do differential analysis and differential topology on such an infinite dimensional manifold, so I will hold off for a while.

Recall that for a Lie group G the tangent space at the identity (thought of as given by an equivalence to second order on curves through Id), $\mathfrak{g}$, has the structure of a Lie algebra. In the case of most interest here, $\mathrm{GL}(n,\mathbb{C}) \subset M(N,\mathbb{C})$ is an open subset of the algebra of $N\times N$ matrices, namely the complement of the hypersurface where $\det = 0$. Thus the tangent space at Id is just $M(N,\mathbb{C})$ and the Lie algebra structure is given by the commutator

$$[a,b] = ab - ba,\ \ a,b \in \mathfrak{gl}(N,\mathbb{C}) = M(N,\mathbb{C}). \tag{10.52}$$

At any other point, g, of the group the tangent space may be naturally identified with $\mathfrak{g}$ by observing that if $c(t)$ is a curve through g then $g^{-1}c(t)$ is a curve through Id with the equivalence relation carrying over. This linear map from T_gG to $\mathfrak{g}$ is herlpfully denoted

$$g^{-a}dg : T_gG \longrightarrow \mathfrak{g}. \tag{10.53}$$

In this notation 'dg' is the differential of the identity map of G at g. This 'Maurier-Cartan' form as a well-defined 1-form on G with values in } – which is a fixed vector space.

The fundamental property of this form is that

$$d(g^{-1}dg) = -\frac{1}{2}[g^{-1}dg, g^{-1}dg]. \tag{10.54}$$

In the case of $\mathrm{GL}(N,\mathbb{C})$ this can be checked directly, and written slightly differently. Namely in this case as a 'function' 'g' is the identity on G but thought of as the canonical embedding $\mathrm{GL}(N,\mathbb{C}) \subset M(N,\mathbb{C})$. Thus it takes values in $M(N,\mathbb{C})$, a vector space, and we may differentiate directly to find that

$$d(g^{-1}dg) = -dgg^{-1}dg \wedge dg \tag{10.55}$$

where the product is that in the matrix algebra. Here we are just using the fact that $dg^{-1} = -g^{-1}dgg^{-1}$ which comes from differentiating the defining identity $g^{-1}g = \mathrm{Id}$. Of course the right side of (10.55) is antisymmetric as a function on the tangent space $T_gG \times T_gG$ and so does reduce to (10.54) when the product is repalced by the Lie product, i.e. the commutator.

Since we are dealing with matrix, or infinite matrix, groups throughout, I will use the 'non-intrinsic' form (10.55) in which the product is the matrix product, rather than the truly intrinsic (and general) form (10.54).

Proposition 10.13 (Chern forms). *If* tr *is the trace functional on* $N \times N$ *matrices then on* $\mathrm{GL}(N,\mathbb{C})$,

$$\begin{gathered}\mathrm{tr}((g^{-1}dg)^{2k}) = 0 \ \forall\ k \in \mathbb{N},\\ \beta_{2k-1} = \mathrm{tr}((g^{-1}dg)^{2k-1}) \text{ is closed } \forall\ k \in \mathbb{N}.\end{gathered} \tag{10.56}$$

Proof. This is the effect of the antisymmetry. The trace idenitity, $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ means precisely that tr vanishes on commutators. In the case of an even number of factors, for clarity evaluation on $2k$ copies of $T_g\,\mathrm{GL}(N,\mathbb{C})$, given for $a_i \in M(N,\mathbb{C})$, $i = 1,\dots,2k$, by the sum over

$$\begin{gathered}\mathrm{tr}((g^{-1}dg)^{2k})(a_1,a_2,\dots,a_{2k}) = \sum_e \mathrm{sgn}(e)\,\mathrm{tr}(g^{-1}a_{e(1)}g^{-1}a_{e(2)}\dots g^{-1}a_{e(2k)}) = \\ -\sum_e \mathrm{sgn}(e)\,\mathrm{tr}(g^{-1}a_{e(2k)}g^{-1}a_{e(1)}\dots g^{-1}a_{e(2k-1)}) = -\,\mathrm{tr}((g^{-1}dg)^{2k})(a_1,a_2,\dots,a_{2k}).\end{gathered} \tag{10.57}$$

In the case of an odd number of factors the same manipulation products a trivial identity. However, notice that

$$g^{-1}dgg^{-1} = -d(g^{-1}) \tag{10.58}$$

is closed, as is dg. So in differentiating the odd number of wedge products each pair $g^{-1}dgg^{-1}dg$ is closed, so (tr being a fixed functional)

$$d\beta_{2k-1} = \mathrm{tr}(dg^{-1})(g^{-1}dgg^{-1}dg)^{2k-2}) = -\,\mathrm{tr}((g^{-1}dg)^{2k}) = 0 \tag{10.59}$$

by the previous discussion. □

Now, time to do this in the infinite dimensional case. First we have to make sure we know that we are talking about.

Definition 10.3 (Fréchet differentiability). *A function on an open set of a Fréchet space,* $O \subset F$, $f : O \longrightarrow V$, *where* V *is a locally convex topological space (here it will also be Fréchet, and might be Banach) differentiable at a point* $u \in O$ *if there exists a continuous linear map* $D : F \longrightarrow V$ *such that for each continuous seminorm* $\|\cdot\|_\alpha$ *on* V *there is a continuous norm* $\|\cdot\|_i$ *on* F *such that for each* $\epsilon > 0$ *there exists* $\delta > 0$ *for which*

$$\|v\|_i < \delta \Longrightarrow \|f(u+v) - f(u) - Tv\|_\alpha \le \epsilon\|v\|_i. \tag{10.60}$$

This is a rather strong definition of differentiability, stronger than the Gâteaux definition – which would actually be enough for most of what we want, but why not use the stronger condition when it holds?

Proposition 10.14. *The composition of smoothing operators defines a bilinear smooth map*

$$\Psi^{-\infty}_{\rm iso}(\mathbb{R}^n) \times \Psi^{-\infty}_{\rm iso}(\mathbb{R}^n) \longrightarrow \Psi^{-\infty}_{\rm iso}(\mathbb{R}^n),\ \|ab\|_k \le C_k\|a\|_{k+N}\|b\|_{k+N} \tag{10.61}$$

(where the kth norm on u is for instance the $\mathcal{C}^k$ norm on $\langle z\rangle^k u$ and inversion is a smooth map

$$G^{-\infty}_{\rm iso}(\mathbb{R}^n) \longrightarrow G^{-\infty}_{\rm iso}(\mathbb{R}^n). \tag{10.62}$$

Proof. I did not define smoothness above, but it is iterated differentiability, as usual. In fact linear maps are always differentiable, as follows immediately from the definition. The same is true of jointly continuous bilinear maps, so the norm estimates in (10.61) actually prove the regularity statement. The point is that the derivative of a bilinear map P at $(\bar a, \bar b)$ is the linear map

$$Q_{\bar a,\bar b}(a,b) = P(a,\bar b)+P(\bar a,b), P(\bar a+a,\bar b+b)-P(\bar a,\bar b)-Q_{\bar a,\bar b}(a,b) = P(a,b). \tag{10.63}$$

The bilinear estimates themselves follow directly by differentiating and estimating the integral composition formula

$$a\circ b(z,z') = \int a(z,z'')b(z'',z')dz''. \tag{10.64}$$

The shift in norm on the right compared to the left is to get a negative factor of $\langle z''\rangle$ to ensure integrability.

Smoothness of the inverse map is a little more delicate. Of course we do know what the derivative at the point g, evaluated on the tangent vector a is, namely $g^{-1}ag^{-1}$. So to get differentiability we need to estimate

$$(g+a)^{-1} - g^{-1} + g^{-1}ag^{-1} = g^{-1}a\left(\sum_{k\ge 0}(-1)^{k+1}g^{-1}(ag^{-1})^k\right)ag^{-1}. \tag{10.65}$$

This is the Neumann series for the inverse. If a is close to 0 in $\Psi^{-\infty}_{\rm iso}(\mathbb{R}^n)$ then we know that $\|a\|_{L^2}$ is small, i.e. it is bounded by some norm on $\Psi^{-\infty}_{\rm iso}(\mathbb{R}^n)$. Thus the series on the right converges in bounded operators on $L^2(\mathbb{R}^n)$. However the smoothing terms on both sides render the whole of the right side smoothing and with all norms small in $\Psi^{-\infty}_{\rm iso}(\mathbb{R}^n)$ when a is small.

This proves differentiability, but in fact infinite differentiability follows, since the differentiability of g^{-1} and the smoothness of composition, discussed above, shows that $g^{-1}ag^{-1}$ is differentiable, and allows one to proceed on inductively. □

We also know that the trace functional extends to $\Psi^{-\infty}_{\rm iso}(\mathbb{R}^n)$ as a trace functional, i.e. vanishing on commutators. This means that the construction above of Chern classes on $\mathrm{GL}(N,\mathbb{C})$ extends to $G^{-\infty}_{\rm iso}(\mathbb{R}^n)$.

Proposition 10.15. *(Universal Chern forms) The statements* (10.56) *extend to the infinite-dimensional group $G^{-\infty}_{\rm iso}(\mathbb{R}^n)$ to define deRham classes $[\beta_{2k-1}]$ in each odd dimension.*

In fact these classes generate (not span, you need to take cup products as well) the cohomology, over $\mathbb{R}$, of $G^{-\infty}_{\rm iso}(\mathbb{R}^n)$.

Proof. We have now done enough to justify the earlier computations in this setting. □

Proposition 10.16. *If X is a manifold and $a \in \mathcal{C}_c^\infty(X; G_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ then the forms $a^*\beta_{2k-1}$ define deRham classes in ${\mathrm{H}_c}^{2k+1}(X;\mathbb{R})$ which are independent of the homotopy class and so are determined by $[a] \in \mathrm{K}_c^1(X)$. Combining them gives the (odd) Chern character*

$$\mathrm{Ch}_o([a]) = \sum_k c_{2k-1} a^*\beta_{2k-1}. \tag{10.66}$$

the particular constants chosen in (10.66) corresponding to multiplicativity under tensor products, which will be discussed below.

Proof. The independence of the (smooth) homotopy class follows from the computation above. Namely if $a_t \in \mathcal{C}_c^\infty(X \times [0,1]; G_{\text{iso}}^{-\infty}(\mathbb{R}^b n)$ then $B_{2k-1} = a_t^*\beta_{2k-1}$ is a closed $(2k-1)$-form on $X \times [0,1]$. If we split it into the two terms

$$B_{2k-1} = b_{2k-1}(t) + \gamma_{2k-1}(t) \wedge dt \tag{10.67}$$

where $b_{2k-1}(t)$ and $\gamma_{2k-1}(t)$ are respectively a t-dependent $2k-1$ and $2k-2$ form, then

(10.68)

$$dB_{2k-1} = 0 \Longleftrightarrow \frac{\partial}{\partial t} b_{2k-1}(t) = d_X \gamma_{2k-2}(t) \text{ and hence } b(1)_{2k-1} - b(0)_{2k-1} = d\mu_{2k-2},\ \mu_{2k-2} = \int_0^1 dt \gamma_{2k-2}(t)$$

shows that $b(1)_{2k-1}$ and $b(0)_{2k-1}$, the Chern forms of a_1 and a_0 are cohomologous. □

The even case is very similar. Note above that we have defined even K-classes on X as equivalence classes under homotopy of elements $a \in \mathcal{C}_c^\infty(X; G_{iso,s}^{-\infty}(\mathbb{R}^n)$. The latter group consists of smooth loops in $G_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ starting and ending at Id. This means there is a natural (smooth) map

$$T : G_{\text{iso},s}^{-\infty}(\mathbb{R}^n) \times \mathbb{S} \longrightarrow G_{\text{iso}}^{-\infty}(\mathbb{R}^n),\ (a, \theta) \longmapsto a(\theta). \tag{10.69}$$

This map may be used to pull back the Chern forms discussed above to the product and integrate over $\mathbb{S}$ to get forms in even dimensions:-

$$\beta_{2k} = \int_0^{2\pi} \mathrm{tr}(g^{-1}dg)^{2k+1},\ k = 0, 1, \dots. \tag{10.70}$$

Proposition 10.17. *The group $G_{\text{iso},s}^{-\infty}(\mathbb{R}^n)$ has an infinite number of components, labelled by the 'index' β_0 in (10.70), the other Chern forms define cohomology classes such that for any map*

$$\mathrm{Ch}([a]) = \sum_{k=0}^{\infty} c_{2k} a^*\beta_{2k} \tag{10.71}$$

defines a map $\mathrm{K}_c^0(X) \longrightarrow \mathrm{H}^{\text{even}}(X)$.

The range of this map spans the even cohomology, this is a form of a theorem of Atiyah-Hurzebruch.

If $f : X \longrightarrow Y$ is a smooth map then it induces a pull-back operation on vector bundles (see Problem 10.5) and this in turn induces an operation

Problem 10.5.

$$f^*:\mathrm{K}(Y)\longrightarrow \mathrm{K}(X). \tag{10.72}$$

Now we can interpret Proposition **??** in a more K-theoretic form.

10.11. Todd class

10.12. Stabilization

In which operators with values in $\Psi^{-\infty}_{\mathrm{iso}}$ are discussed.

10.13. Delooping sequence

The standard connection between even and odd classifying groups.

10.14. Looping sequence

The quantized connection between classifying groups.

10.15. $\mathcal{C}^*$ algebras

10.16. K-theory of an algeba

10.17. The norm closure of $\Psi^0(X)$

10.18. The index map

10.19. Problems

Problem 10.6. Remind yourself of the proof that $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)\subset\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is open. Since $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is a group, it suffices to show that a neighbourhood of $0\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is a neighbourhood of the identity. Show that the set $\|A\|_{\mathcal{B}(L^2)}<\frac{1}{2}$, given by the operator norm, fixes an open neighbourhood of $0\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ (this is the L^2 continuity estimate). The inverse of $\mathrm{Id}+A$ for A in this set is given by the Neumann series and the identity (which follows from the Neumann series)

$$(\mathrm{Id}+A)^{-1}=\mathrm{Id}+B=\mathrm{Id}-A+A^2-ABA \tag{10.73}$$

in which *a prioiri* $B\in\mathcal{B}(L^2)$ shows that $B\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ by the 'corner' property of smoothing operators (meaning $ABA'\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ if $A,A'\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and $B\in\mathcal{B}(L^2)$.

Problem 10.7. Check that (10.29) is well-defined, meaning that if (V_1,W_2) is replaced by an equivalent pair then the result is the same. Similarly check that the operation is commutative and that it make $\mathrm{K}(X)$ into a group.

Problem 10.8. Check that you do know how to prove (10.18). One way is to use induction over N, since it is certainly true for $N=1$, $\mathrm{GL}(1,\mathbb{C})=\mathbb{C}^*$. Proceeding by induction, note that an element $a\in\mathrm{GL}(N,\mathbb{C})$ is fixed by its effect on the standard basis, e_i. Choose $N-1$ elements ae_j which form a basis together with e_1. The inductive hypothesis allows these elements to be deformed, keeping their e_1 components fixed, to e_k, $k>1$. Now it is easy to see how to deform the resulting basis back to the standard one.

PROBLEM 10.9. Prove (10.21). Hint:- The result is very standard for $N = 1$. So proceed by induction over N. Given a smooth curve in $\mathrm{GL}(N, \mathbb{C})$, by truncating its Fourier series at high frequencies one gets, by the openness of $\mathrm{GL}(N, \mathbb{C})$, a homotopic curve which is real-analytic, denote it $a(\theta)$. Now there can only be a finite number of points at which $e_1 \cdot a(\theta)e_1 = 0$. Moreover, by deforming into the complex near these points they can be avoided, since the zeros of an analytic function are isolated. Thus after homotopy we can assume that $g(\theta) = e_1 \cdot a(\theta)e)1 \neq 0$. Composing with a loop in which e_1 is roatated in the complex by $1/g(\theta)$, and e_2 in the opposite direction, one reduces to the case that $e_1 \cdot a(\theta)e)1 = 0$ and then easily to the case $a(\theta)e_1 = e_1$, then induction takes over (with the determinant condition still holding). Thus it is enough to do the two-dimensional case, which is pretty easy, namely e_1 rotated in one direction and e_2 by the inverse factor.