ON THE RING OF COOPERATIONS FOR 2-PRIMARY CONNECTIVE TOPOLOGICAL MODULAR FORMS

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1. Introduction

Note: This is a preliminary draft. In particular, parts of the last section are in the process of being written!

The Adams-Novikov spectral sequence based on a connective spectrum E (E-ANSS) is perhaps the best available tool for computing stable homotopy groups. For example, $H\mathbb{F}_p$ and BP give the classical Adams spectral sequence and the Adams Novikov spectral sequence respectively.

To begin to compute with the E-ANSS, one needs to know the structure of the smash powers $E^{\wedge k}$. When E is one of $H\mathbb{F}_p$, MU, or BP, the situation is simpler than in general, since in this case $E \wedge E$ is an infinite wedge of suspensions of E itself, which allows for an algebraic description of the E_2-term. This is not the case

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for bu, bo, or tmf, in which case the E_2 page is harder to describe, and in fact, has not yet been described in the the case of tmf.

Mahowald and his collaborators have studied the 2-primary bo-ANSS to a great effect: it gives the most efficient calculation of the v_1-periodic homotopy in the sphere spectrum [LM87, Mah81]. The starting input in that calculation is a complete description of $bo \wedge bo$ as an infinite wedge product of spectra that are a smash product of certain finite complexes with bo (as in [Mil75] and others). The finite complexes involved are the so-called integral Brown-Gitler spectra.

Mahowald has worked on a similar description for $tmf \wedge tmf$, but concluded that no analogous result could hold. In this paper we use his insights to explore four different perspectives on 2-primary tmf-cooperations. While we do not arrive at a complete and closed-form description of $tmf \wedge tmf$, we believe our results have the potential to be very useful as a computational tool.

(1) The E_2 term of the 2-primary Adams spectral sequence for $\mathrm{tmf} \wedge \mathrm{tmf}$ admits a splitting in terms of bo-Brown-Gitler modules:
$$\mathrm{Ext}(\mathrm{tmf} \wedge \mathrm{tmf}) \cong \bigoplus_i \mathrm{Ext}(\Sigma^{8i}\mathrm{tmf} \wedge \mathrm{bo}_i).$$
(2) Modulo torsion, $\mathrm{TMF}_*\mathrm{TMF}$ is isomorphic to a subring of the ring of integral two variable modular forms.
(3) $K(2)$-locally, the ring spectrum $(\mathrm{TMF} \wedge \mathrm{TMF})_{K(2)}$ is given by an equivariant function spectrum:
$$(\mathrm{TMF} \wedge \mathrm{TMF})_{K(2)} \simeq \mathrm{Map}(\mathbb{G}_2/G_{48}, E_2)^{hG_{48}}.$$
(4) $\mathrm{TMF}_*\mathrm{TMF}$ injects into a certain product of homotopy groups of topological modular forms with level structures.
$$\mathrm{TMF} \wedge \mathrm{TMF} \hookrightarrow \prod_{\substack{i \in \mathbb{Z}, \\ j \geq 0}} \mathrm{TMF}_0(3^j) \times \mathrm{TMF}_0(5^j).$$

The purpose of this paper is to describe and investigate the relationship between these different perspectives.

1.1. **Conventions.** In this paper we shall always be implicitly working 2-locally. Homology will be taken with mod 2 coefficients, unless specified otherwise. We will use $\mathrm{Ext}(X)$ to abbreviate $\mathrm{Ext}_{A_*}(\mathbb{F}_2, H_*X)$, the E_2-term of the Adams spectral sequence (ASS) for π_*X, and will let $C^*_{A_*}(H^*X)$ denote the corresponding cobar complex. Given an element $x \in \pi_*X$, we shall let $[x]$ denote the coset of the ASS E_2-term which detects x.

2. Motivation: analysis of $\mathrm{bo}_*\mathrm{bo}$

In analogy with the four perspectives described in the introduction, there are four primary perspectives on the ring of cooperations for real K-theory.

(1) The spectrum $\mathrm{bo} \wedge \mathrm{bo}$ admits a decomposition (at the prime 2)

$$\mathrm{bo} \wedge \mathrm{bo} \simeq \bigvee_{i \geq 0} \mathrm{bo} \wedge \mathrm{H}\mathbb{Z}_i,$$

where $\mathrm{H}\mathbb{Z}_i$ is the ith integral Brown-Gitler spectrum.

(2) There is an isomorphism $\mathrm{KO}_*\mathrm{KO} \cong \mathrm{KO}_* \otimes_{\mathrm{KO}_0} \mathrm{KO}_0\mathrm{KO}$, and $\mathrm{KO}_0\mathrm{KO}$ is isomorphic to a subring of the ring of numerical functions.

(3) $K(1)$-locally, the ring spectrum $(\mathrm{KO} \wedge \mathrm{KO})_{K(1)}$ is given by the function spectrum:

$$(\mathrm{KO} \wedge \mathrm{KO})_{K(1)} \simeq \mathrm{Map}(\mathbb{Z}_2^\times/\{\pm 1\}, \mathrm{KO}_2^\wedge).$$

(4) $\mathrm{KO}_*\mathrm{KO}$ injects into a product of copies of KO:

$$\mathrm{KO} \wedge \mathrm{KO} \hookrightarrow \prod_{i \in \mathbb{Z}} \mathrm{KO}.$$

2.1. **Integral Brown-Gitler spectra.** The decomposition of $\mathrm{bo} \wedge \mathrm{bo}$ above is a topological realization of a homology decomposition (see [Mah81], [Mil75]). Endow the monomials of the A_*-comodule

$$H_*\mathrm{H}\mathbb{Z} = \mathbb{F}_2[\bar{\xi}_1^2, \bar{\xi}_2, \bar{\xi}_3, \ldots]$$

with a multiplicative weight by defining $wt(\bar{\xi}_i) = 2^{i-1}$. The comodule $H_*\mathrm{H}\mathbb{Z}$ admits an increasing filtration by integral Brown-Gitler comodules $\underline{\mathrm{H}\mathbb{Z}}_i$, where $\underline{\mathrm{H}\mathbb{Z}}_i$ is spanned by elements of weight less than $2i$. These A_*-comodules are realized by integral Brown-Gitler spectra $\mathrm{H}\mathbb{Z}_i$, so that

$$H_*\mathrm{H}\mathbb{Z}_i \cong \underline{\mathrm{H}\mathbb{Z}}_i.$$

There is a decomposition of $A(1)_*$-comodules:

$$H_*\mathrm{bo} = (A//A(1))_* \cong_{A(1)_*} \bigoplus_{i \geq 0} \Sigma^{4i} \underline{\mathrm{H}\mathbb{Z}}_i$$

. This results in a decomposition on the level of Adams E_2-terms

$$\begin{aligned} \mathrm{Ext}(\mathrm{bo} \wedge \mathrm{bo}) &\cong \bigoplus_{i \geq 0} \mathrm{Ext}(\Sigma^{4i}\mathrm{bo} \wedge \mathrm{H}\mathbb{Z}_i) \\ &\cong \bigoplus_{i \geq 0} \mathrm{Ext}_{A(1)_*}(\Sigma^{4i}\mathrm{H}\mathbb{Z}_i). \end{aligned}$$

This algebraic splitting is topologically realized by a splitting

$$\mathrm{bo} \wedge \mathrm{bo} \simeq \bigvee_{i \geq 0} \mathrm{bo} \wedge \mathrm{H}\mathbb{Z}_i.$$

The goal of this section is to calculate the images of the maps

$$\mathrm{bo} \wedge \mathrm{H}\mathbb{Z}_i \longrightarrow \mathrm{bo} \wedge \mathrm{bo}$$

in the decomposition above in order to illustrate the method used in our analysis of $\mathrm{tmf} \wedge \mathrm{tmf}$. Even in this case our perspective has some novel elements which provide a conceptual explanation for formulas obtained by Lellmann and Mahowald in [LM87].

2.2. **Exact sequences relating $H\mathbb{Z}_i$.** Just as with $\underline{\mathrm{H}\mathbb{Z}}_i$ we define $\underline{\mathrm{bo}}_i$ to be the the submodule of

$$(A//A(1))_* \cong \mathbb{F}_2[\bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_3, \dots]$$

generated by elements of weight less than $4i$. These submodules are discussed more thoroughly at the beginning of Section 4. With these in hand we have the following exact sequences:

Lemma 2.1. There are short exact sequences of $A(1)_*$-comodules

$$0 \to \Sigma^{4j} \underline{\mathrm{H}\mathbb{Z}}_j \to \underline{\mathrm{H}\mathbb{Z}}_{2j} \to \underline{\mathrm{bo}}_{j-1} \otimes (A(1)//A(0))_* \to 0, \tag{2.2}$$

$$0 \to \Sigma^{4j} \underline{\mathrm{H}\mathbb{Z}}_j \otimes \underline{\mathrm{H}\mathbb{Z}}_1 \to \underline{\mathrm{H}\mathbb{Z}}_{2j+1} \to \underline{\mathrm{bo}}_{j-1} \otimes (A(1)//A(0))_* \to 0. \tag{2.3}$$

(Here $\underline{\mathrm{bo}}_i$ is the subspace of $H_*\mathrm{bo}$ spanned by monomials of weight $\leq 4i$.)

Proof. These short exact sequences are the analogs for integral Brown-Gitler modules of a pair of short exact sequences for bo-Brown-Gitler modules (see Propositions 7.1 and 7.2 of [BHHM08]). The proof is almost identical to that given in [BHHM08]. On the level of basis elements, the maps

$$\Sigma^{4j} \underline{\mathrm{H}\mathbb{Z}}_j \to \underline{\mathrm{H}\mathbb{Z}}_{2j}$$
$$\Sigma^{4j} \underline{\mathrm{H}\mathbb{Z}}_j \otimes \underline{\mathrm{H}\mathbb{Z}}_1 \to \underline{\mathrm{H}\mathbb{Z}}_{2j+1}$$

are given respectively by

$$\bar{\xi}_1^{2i_1} \bar{\xi}_2^{i_2} \cdots \mapsto \bar{\xi}_1^{a} \bar{\xi}_2^{2i_1} \bar{\xi}_3^{i_2} \cdots,$$
$$\bar{\xi}_1^{2i_1} \bar{\xi}_2^{i_2} \cdots \otimes \{1, \bar{\xi}_1^2, \bar{\xi}_2\} \mapsto (\bar{\xi}_1^{a} \bar{\xi}_2^{2i_1} \bar{\xi}_3^{i_2} \cdots) \cdot \{1, \bar{\xi}_1^2, \bar{\xi}_2\}$$

where a is taken to be $4j - wt(\bar{\xi}_2^{2i_1} \bar{\xi}_3^{i_2} \cdots)$. The maps

$$\underline{\mathrm{H}\mathbb{Z}}_{2j} \to \underline{\mathrm{bo}}_{j-1} \otimes (A(1)//A(0))_*$$
$$\underline{\mathrm{H}\mathbb{Z}}_{2j+1} \to \underline{\mathrm{bo}}_{j-1} \otimes (A(1)//A(0))_*$$

are given by

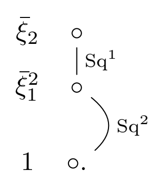
$$\bar{\xi}_1^{4i_1+2\epsilon_1} \bar{\xi}_2^{2i_2+\epsilon_2} \bar{\xi}_3^{i_3} \cdots \mapsto \begin{cases} \bar{\xi}_1^{4i_1} \bar{\xi}_2^{2i_2} \bar{\xi}_3^{i_3} \cdots \otimes \bar{\xi}_1^{2\epsilon_1} \bar{\xi}_2^{\epsilon_2}, & wt(\bar{\xi}_1^{4i_1} \bar{\xi}_2^{2i_2} \bar{\xi}_3^{i_3} \cdots) \leq 4j-4, \\ 0, & \text{otherwise,} \end{cases}$$

where $\epsilon_s \in \{0, 1\}$. $\square$

Define

$$\frac{\mathrm{Ext}_{A(1)_*}(X)}{v_1\text{-tor}} := \mathrm{Image}\left(\mathrm{Ext}_{A(1)_*}(X) \to v_1^{-1}\,\mathrm{Ext}_{A(1)_*}(X)\right).$$

The following lemma follows from a simple induction, using the fact that $\underline{\mathrm{H}\mathbb{Z}}_1$ is given by



Lemma 2.4. We have

$$\frac{\mathrm{Ext}_{A(1)_*}(\underline{\mathrm{H}\mathbb{Z}}_1^{\otimes i})}{v_1\text{-tor}} \cong \begin{cases} \mathrm{Ext}(\mathrm{bo}^{\langle i \rangle}), & i \text{ even}, \\ \mathrm{Ext}(\mathrm{bsp}^{\langle i-1 \rangle}), & i \text{ odd}. \end{cases}$$

Here, $X^{\langle i \rangle}$ denotes the ith Adams cover.

We deduce the following well known result (cf. [LM87, Thm. 2.1]).

Proposition 2.5.

$$\frac{\mathrm{Ext}_{A(1)_*}(\underline{\mathrm{H}\mathbb{Z}}_i)}{v_1\text{-tor}} \cong \begin{cases} \mathrm{Ext}(\mathrm{bo}^{\langle 2i-\alpha(i) \rangle}), & i \text{ even}, \\ \mathrm{Ext}(\mathrm{bsp}^{\langle 2i-\alpha(i)-1 \rangle}), & i \text{ odd}. \end{cases}$$

Here, $\alpha(i)$ denotes the number of 1's in the dyadic expansion of i.

Proof. This may be established by induction on i using the short exact sequences of Lemma 2.1, by augmenting Lemma 2.4 with the following facts.

(1) All v_0-towers in $\mathrm{Ext}_{A(1)_*}(\underline{\mathrm{H}\mathbb{Z}}_i)$ are v_1-periodic. This can be seen as $\mathrm{Ext}_{A(1)_*}(\underline{\mathrm{H}\mathbb{Z}}_i)$ is a summand of $\mathrm{Ext}(\mathrm{bo} \wedge \mathrm{bo})$, and after inverting v_0, the latter has no v_1-torsion. Explicitly we have

$$v_0^{-1}\,\mathrm{Ext}(\mathrm{bo} \wedge \mathrm{bo}) = \mathbb{F}_2[v_0^{\pm 1}, u^2, v^2].$$

(2) We have

$$\begin{aligned} \frac{\mathrm{Ext}_{A(1)_*}((A(1)//A(0))_* \otimes \underline{\mathrm{bo}}_j)}{v_0\text{-tors}} &\cong \frac{\mathrm{Ext}_{A(0)_*}(\underline{\mathrm{bo}}_j)}{v_0\text{-tors}} \\ &\cong \mathbb{F}_2[v_0]\{1, \xi_1^4, \ldots, \xi_1^{4j}\}. \end{aligned}$$

This follows from the fact that

$$\frac{\mathrm{Ext}_{A(0)_*}(\underline{\mathrm{H}\mathbb{Z}}_j)}{v_0\text{-tors}} \cong \mathbb{F}_2[v_0],$$

which, for instance, can be established by induction using the short exact sequences of Lemma 2.1.

∎

2.3. **The cooperations of** KU **and** bu**.** In order to put the ring of cooperations for bo in the proper setting, we briefly review the story for bu. We begin by recalling the Adams-Harris determination of $\mathrm{KU}_*\mathrm{KU}$ [Ada74, Sec. II.13]. We have an arithmetic square

$$\begin{tikzcd} \mathrm{KU} \wedge \mathrm{KU} \arrow[r] \arrow[d] & (\mathrm{KU} \wedge \mathrm{KU})^\wedge_2 \arrow[d] \\ (\mathrm{KU} \wedge \mathrm{KU})_{\mathbb{Q}} \arrow[r] & ((\mathrm{KU} \wedge \mathrm{KU})^\wedge_2)_{\mathbb{Q}}, \end{tikzcd}$$

which results in a pullback square after applying π_*

$$\begin{tikzcd}
\mathrm{KU}_*\mathrm{KU} \arrow[r] \arrow[d] & \mathrm{Map}^c(\mathbb{Z}_2^\times, \pi_*\mathrm{KU}_2^\wedge) \arrow[d] \\
\mathbb{Q}[u^{\pm 1}, v^{\pm 1}] \arrow[r] & \mathrm{Map}^c(\mathbb{Z}_2^\times, \mathbb{Q}_2[u^{\pm 1}]).
\end{tikzcd}$$

Setting $w = v/u$, the bottom map in the above square is given by

$$f(u, v) = u^n f(1, w) \mapsto (\lambda \mapsto u^n f(1, \lambda))\,.$$

We therefore deduce that $\mathrm{KU}_*\mathrm{KU} = \mathrm{KU}_* \otimes_{\mathrm{KU}_0} \mathrm{KU}_0\mathrm{KU}$, and continuity implies that

$$\mathrm{KU}_0\mathrm{KU} = \{f(w) \in \mathbb{Q}[w^{\pm 1}] \ : \ f(k) \in \mathbb{Z}_{(2)}, \text{for all } k \in \mathbb{Z}_{(2)}^\times\}.$$

Note that we can perform a similar analysis for $\mathrm{KU}_*\mathrm{bu}$: since bu and KU are $K(1)$-locally equivalent, applying π_* to the arithmetic square yields a pullback square with the same terms on the right hand edge.

$$\begin{tikzcd}
\mathrm{KU}_*\mathrm{bu} \arrow[r] \arrow[d] & \mathrm{Map}^c(\mathbb{Z}_2^\times, \pi_*\mathrm{KU}_2^\wedge) \arrow[d] \\
\mathbb{Q}[u^{\pm 1}, v] \arrow[r] & \mathrm{Map}^c(\mathbb{Z}_2^\times, \mathbb{Q}_2[u^{\pm 1}]).
\end{tikzcd}$$

We therefore deduce that $\mathrm{KU}_*\mathrm{bu} = \mathrm{KU}_* \otimes_{\mathrm{KU}_0} \mathrm{KU}_0\mathrm{bu}$, with

$$\mathrm{KU}_0\mathrm{bu} = \{g(w) \in \mathbb{Q}[w] \ : \ g(k) \in \mathbb{Z}_{(2)}, \text{for all } k \in \mathbb{Z}_{(2)}^\times\}.$$

Consider the related space of 2-*local numerical polynomials*:

$$\mathrm{NumPoly}_{(2)} := \{h(x) \in \mathbb{Q}[x] \ : \ h(k) \in \mathbb{Z}_{(2)}, \text{for all } k \in \mathbb{Z}_{(2)}\}.$$

The theory of numerical polynomials states that $\mathrm{NumPoly}_{(2)}$ is the free $\mathbb{Z}_{(2)}$-module generated by the basis elements

$$h_n(x) := \binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}.$$

We can relate $\mathrm{KU}_0\mathrm{bu}$ to $\mathrm{NumPoly}_{(2)}$ by a change of coordinates. A function on $\mathbb{Z}_{(2)}^\times$ can be regarded as a function on $\mathbb{Z}_{(2)}$ via the change of coordinates

$$\begin{gathered}
\mathbb{Z}_{(2)} \xrightarrow{\approx} \mathbb{Z}_{(2)}^\times \\
k \mapsto 2k+1
\end{gathered}$$

Observe that

$$\begin{aligned}
\frac{k(k-1)\cdots(k-n+1)}{n!} &= \frac{2k(2k-2)\cdots(2k-2n+2)}{2^n n!} \\
&= \frac{(2k+1)((2k+1)-3)\cdots((2k+1)-(2n-1))}{2^n n!}.
\end{aligned}$$

We deduce that a $\mathbb{Z}_{(2)}$ basis for $\mathrm{KU}_0\mathrm{bu}$ is given by

$$g_n(w) = \frac{(w-1)(w-3)\dots(w-(2n-1))}{2^n n!}.$$

(Compare with [Ada74, Prop. 17.6(i)].)

From this we deduce a basis of the image of the map

$$\mathrm{bu}_*\mathrm{bu} \hookrightarrow \mathrm{KU}_*\mathrm{KU}.$$

In [Ada74, p. 358] it is shown that this image is the ring

$$\frac{\mathrm{bu}_*\mathrm{bu}}{v_1\text{-tor}} = (\mathrm{KU}_*\mathrm{bu} \cap \mathbb{Q}[u,v])_{\mathrm{AF}\geq 0},$$

where $\mathrm{AF} \geq 0$ means the elements of Adams filtration ≥ 0. Since the elements 2, u, and v have Adams filtration 1, this image is equivalently described as

$$\frac{\mathrm{bu}_*\mathrm{bu}}{v_1\text{-tor}} = \mathrm{KU}_*\mathrm{bu} \cap \mathbb{Z}_{(2)}[u/2, v/2].$$

To compute a basis for this image we need to calculate the Adams filtration of the elements of this basis $\{g_n(w)\}$. Since w has Adams filtration 0 we need only compute the 2-divisibility of the denominators of the functions $g_n(w)$. As usual in this subject, for an integer $k \in \mathbb{Z}$ let $\nu_2(k)$ be the largest power of 2 that divides k and let $\alpha(k)$ be the number of 1's in the binary expansion of k. Then

$$\nu_2(n!) = n - \alpha(n)$$

and so

$$\mathrm{AF}(g_n) = \alpha(n) - 2n.$$

The following is a list of the Adams filtration of the first few basis elements:

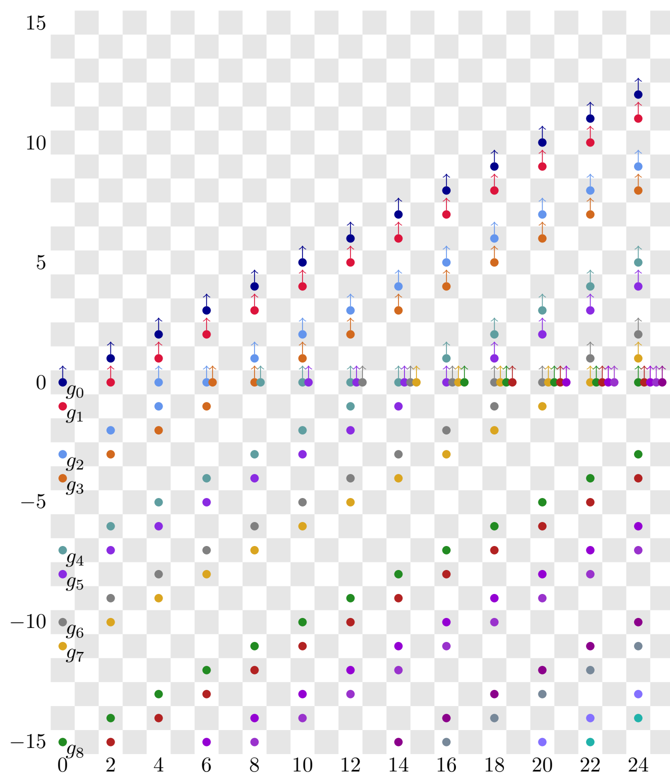
n	binary	$\mathrm{AF}(g_n)$
0	0	0
1	1	-1
2	10	-3
3	11	-4
4	100	-7
5	101	-8
6	110	-10
7	111	-11
8	1000	-15

It follows (compare with [Ada74, Prop. 17.6(ii)]) that the image of $\mathrm{bu}_*\mathrm{bu}$ in $\mathrm{KU}_*\mathrm{KU}$ is the free module:

$$\frac{\mathrm{bu}_*\mathrm{bu}}{v_1\text{-tor}} = \mathbb{Z}_{(2)}\{2^{\max(0,2n-m-\alpha(n))}u^m g_n(w) \;:\; n \geq 0, m \geq n\}.$$

The Adams chart in Figure 2.3 illustrates how the description of $\mathrm{bu}_*\mathrm{bu}$ given above along with the Mahler basis can be used to identify $\mathrm{bu}_*\mathrm{bu}$ as a bu_*-module inside of $\mathrm{KU}_*\mathrm{KU}$.

FIGURE 2.1. $\mathrm{bu}_*\mathrm{bu}$



2.4. **The cooperations of KO and bo.** Adams and Switzer computed $\mathrm{KO}_*\mathrm{KO}$ along simlar lines [Ada74, Sec. II.17]. There is an arithmetic square

$$\begin{tikzcd} \mathrm{KO} \wedge \mathrm{KO} \arrow[r] \arrow[d] & (\mathrm{KO} \wedge \mathrm{KO})_2^\wedge \arrow[d] \\ (\mathrm{KO} \wedge \mathrm{KO})_{\mathbb{Q}} \arrow[r] & ((\mathrm{KO} \wedge \mathrm{KO})_2^\wedge)_{\mathbb{Q}}. \end{tikzcd}$$

This results in a pullback when applying π_*:

$$\begin{tikzcd} \mathrm{KO}_*\mathrm{KO} \arrow[r] \arrow[d] & \mathrm{Map}^c(\mathbb{Z}_2^\times/\{\pm 1\}, \pi_*\mathrm{KO}_2^\wedge) \arrow[d] \\ \mathbb{Q}[u^{\pm 2}, v^{\pm 2}] \arrow[r] & \mathrm{Map}^c(\mathbb{Z}_2^\times/\{\pm 1\}, \mathbb{Q}_2[u^{\pm 2}]). \end{tikzcd}$$

(One can use the fact that $\mathrm{KU}_2^\wedge$ is a $K(1)$-local C_2-Galois extension of $\mathrm{KO}_2^\wedge$ to identify the upper right hand corner of the above pullback.) Continuing to let $w = v/u$, the bottom map in the above square is given by

$$f(u^2, v^2) = u^{2n} f(1, w^2) \mapsto \left([\lambda] \mapsto u^{2n} f(1, \lambda^2)\right).$$

We therefore deduce that $\mathrm{KO}_*\mathrm{KO} = \mathrm{KO}_* \otimes_{\mathrm{KO}_0} \mathrm{KO}_0\mathrm{KO}$, with

$$\mathrm{KO}_0\mathrm{KO} = \{f(w^2) \in \mathbb{Q}[w^{\pm 2}] \ : \ f(\lambda^2) \in \mathbb{Z}_2^\times, \text{for all } [\lambda] \in \mathbb{Z}_2^\times/\{\pm 1\}\}.$$

Again, $\mathrm{KO}_*\mathrm{bo}$ is similarly determined: since bo and KO are $K(1)$-locally equivalent, applying π_* to the arithmetic square yields a pullback square with the same terms on the right hand edge:

$$\begin{tikzcd}
\mathrm{KO}_*\mathrm{bo} \arrow[r] \arrow[d] & \mathrm{Map}^c(\mathbb{Z}_2^\times/\{\pm 1\}, \pi_*\mathrm{KO}_2^\wedge) \arrow[d] \\
\mathbb{Q}[u^{\pm 2}, v^2] \arrow[r] & \mathrm{Map}^c(\mathbb{Z}_2^\times/\{\pm 1\}, \mathbb{Q}_2[u^{\pm 2}]).
\end{tikzcd}$$

We therefore deduce that $\mathrm{KO}_*\mathrm{bo} = \mathrm{KO}_* \otimes_{\mathrm{KO}_0} \mathrm{KO}_0\mathrm{bo}$, with

$$\mathrm{KO}_0\mathrm{bo} = \{f(w^2) \in \mathbb{Q}[w^2] \ : \ f(\lambda^2) \in \mathbb{Z}_2, \text{for all } [\lambda] \in \mathbb{Z}_2^\times/\{\pm 1\}\}.$$

To produce a basis of this space of functions we use the q-Mahler bases developed in [Con00]. First note that there is an exponential isomorphism

$$\mathbb{Z}_2 \xrightarrow{\cong} \mathbb{Z}_2^\times/\{\pm 1\} : k \mapsto [3^k].$$

Taking $w = 3^k$, we have $w^2 = 9^k$, or in other words, the functions $f(w^2)$ that we are concerned with can be regarded as functions on $2\mathbb{Z}_2$. They take the form

$$f(9^k) : 2\mathbb{Z}_2 \cong 1 + 8\mathbb{Z}_2 \longrightarrow \mathbb{Z}_2,$$

where $1 + 8\mathbb{Z}_2 \subset \mathbb{Z}_2^\times$ is the image of $2\mathbb{Z}_2$ under the isomorphism given by 3^k.

To apply the q-Mahler basis of [Con00] with $q = 9$ it is important that $|9-1|_2 < 1$. The q-Mahler basis is a basis for numerical polynomials with domain restricted to $2\mathbb{Z}_2$. In the notation of [Con00] we have that

$$f(9^k) = \sum_n c_n \binom{k}{n}_9 \ c_n \in \mathbb{Z}_{(2)},$$

where

$$\binom{k}{n}_9 = \frac{(9^k-1)(9^k-9)\cdots(9^k-9^{n-1})}{(9^n-1)(9^n-9)\cdots(9^n-9^{n-1})}.$$

Let us set

$$f_n(w^2) = \frac{(w^2-1)(w^2-9)\cdots(w^2-9^{n-1})}{(9^n-1)(9^n-9)\cdots(9^n-9^{n-1})}.$$

Then

$$f(w^2) = \sum_n c_n f_n(w^2) \ c_n \in \mathbb{Z}_{(2)}.$$

We deduce that a basis for $\mathrm{KO}_0\mathrm{bo}$ is given by the set $\{f_n(w^2)\}_{n\geq 0}$.

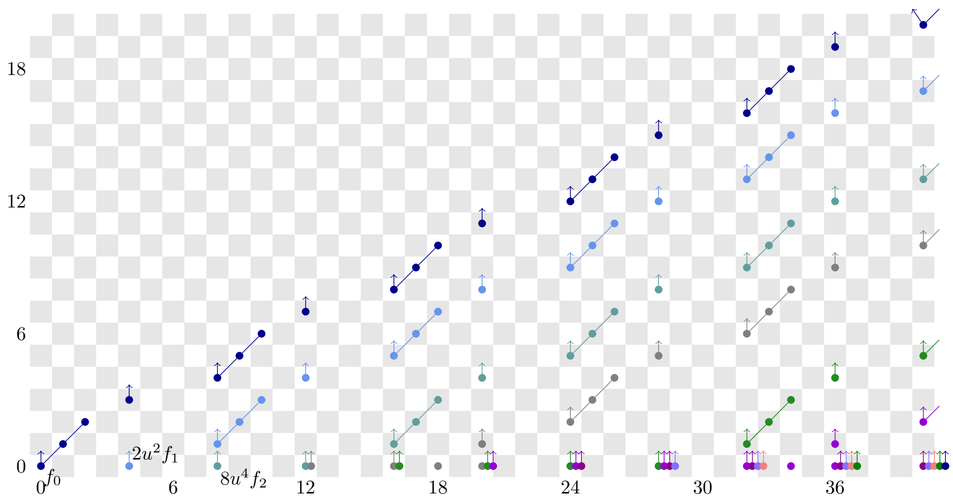
As in the KU-case, it turns out that the image of $\mathrm{bo}_*\mathrm{bo}$ in $\mathrm{KO}_*\mathrm{KO}$ is given by

$$\frac{\mathrm{bo}_*\mathrm{bo}}{v_1\text{-tor}} = (\mathrm{KO}_*\mathrm{bo} \cap \mathbb{Q}[u^2, v^2])_{\mathrm{AF}\geq 0}.$$

In order to compute a basis for this we once again need to know the Adams filtration of f_n. One can show that

$$\begin{aligned}
\nu_2((9^n-1)(9^n-9)\cdots(9^n-9^{n-1})) &= \nu_2(n!) + 3n \\
&= 4n - \alpha(n).
\end{aligned}$$

FIGURE 2.2. $\mathrm{bo}_*\mathrm{bo}$



It follows that we have

$$\begin{aligned}\frac{\mathrm{bo}_*\mathrm{bo}}{v_1\text{-tor}} &= \mathbb{Z}_{(2)}\{2^{\max(0,4n-2m-\alpha(n))}u^{2m}f_n(w) \,:\, n \geq 0,\ m \geq n,\ m \equiv 0 \mod 2\} \\ &\oplus \mathbb{Z}_{(2)}\{2^{\max(0,4n-2m-1-\alpha(n))}2u^{2m}f_n(w) \,:\, n \geq 0,\ m \geq n,\ m \equiv 0 \mod 2\} \\ &\oplus \mathbb{Z}/2\left\{u^{2m}f_n(w)\eta^i \,:\, \begin{array}{l} n \geq 0,\ m \geq n,\ m \equiv 0 \mod 2, \\ i \in \{1,2\},\ \alpha(n) - 4n + 2m + i \geq 0 \end{array}\right\}.\end{aligned}$$

Here is a list of the Adams filtration of the first several elements in the q-Mahler basis:

n	f_n in terms of g_i	$\mathrm{AF}(f_n)$
0	g_0	0
1	$g_2 + g_1$	-3
2	$\frac{1}{15}g_4 + \frac{2}{15}g_3 + \frac{1}{15}g_2$	-7

With this information we can now give an Adams chart of $\mathrm{bo}_*\mathrm{bo}$.

2.5. **Calculation of the image of** $\mathrm{bo}_*\mathrm{H}\mathbb{Z}_i$ **in** $\mathrm{KO}_*\mathrm{KO}$**.** We now compute the image (on the level of Adams E_∞-terms) of the composite

$$\mathrm{bo}_*\mathrm{H}\mathbb{Z}_i \to \mathrm{bo}_*\mathrm{bo} \to \mathrm{KO}_*\mathrm{KO}.$$

Since $v_1^{-1}\mathrm{bo}_*\Sigma^{4i}\mathrm{H}\mathbb{Z}_i \cong \mathrm{KO}_*$, it suffices to determine the image of the generator

$$e_{4i} \in \mathrm{bo}_{4i}(\Sigma^{4i}\mathrm{H}\mathbb{Z}_i).$$

Because the maps

$$\mathrm{bo} \wedge \Sigma^{4i}\mathrm{H}\mathbb{Z}_i \to \mathrm{bo} \wedge \mathrm{bo}$$

are constructed to be bo-module maps, everything else is determined by 2 and $v_1 = u$-multiplication. Consider the diagram induced by the maps $\mathrm{bo} \to \mathrm{bu}$,

$\mathrm{bu} \to \mathrm{H}\mathbb{F}_2$, and $\mathrm{BP} \to \mathrm{bu}$:

$$\begin{tikzcd}
\mathrm{bo} \wedge \Sigma^{4i}\mathrm{H}\mathbb{Z}_i \arrow[r] \arrow[d] & \mathrm{bo} \wedge \mathrm{bo} \arrow[r] \arrow[d] & \mathrm{bu} \wedge \mathrm{bu} \arrow[d] & \mathrm{BP}_*\mathrm{BP} \arrow[l] \arrow[dl] \\
\mathrm{H}\mathbb{F}_2 \wedge \Sigma^{4i}\mathrm{H}\mathbb{Z}_i \arrow[r] & \mathrm{H}\mathbb{F}_2 \wedge \mathrm{bo} \arrow[r] & \mathrm{H}\mathbb{F}_2 \wedge \mathrm{H}\mathbb{F}_2.
\end{tikzcd}$$

On the level of homotopy groups the bottom row of the above diagram takes the form

$$\mathbb{F}_2\{\bar{\xi}_1^{4i}, \ldots\} \hookrightarrow \mathbb{F}_2[\bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_3, \ldots] \hookrightarrow \mathbb{F}_2[\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \ldots].$$

Since we have

$$\begin{aligned} \mathrm{bo}_*\Sigma^{4i}\mathrm{H}\mathbb{Z}_i &\to (\mathrm{H}\mathbb{F}_2)_*\Sigma^{4i}\mathrm{H}\mathbb{Z}_i \\ e_{4i} &\mapsto \bar{\xi}_1^{4i}, \end{aligned}$$

it suffices to find an element $b_i \in \mathrm{bo}_{4i}\mathrm{bo}$ such that

$$\begin{aligned} \mathrm{bo}_*\mathrm{bo} &\to (\mathrm{H}\mathbb{F}_2)_*\mathrm{bo} \\ b_i &\mapsto \bar{\xi}_1^{4i}. \end{aligned}$$

Clearly we can take $b_0 = 1 \in \mathrm{bo}_0\mathrm{bo}$. Note that we have

$$\begin{aligned} \mathrm{BP}_*\mathrm{BP} &\to (\mathrm{H}\mathbb{F}_2)_*\mathrm{H}\mathbb{F}_2 \\ t_1 &\mapsto \bar{\xi}_1^2. \end{aligned}$$

From the equation

$$\eta_R(v_1) = v_1 + 2t_1$$

we deduce that we have

$$\begin{aligned} \mathrm{BP}_*\mathrm{BP} &\to \mathrm{bu}_*\mathrm{bu} \\ t_1 &\mapsto \frac{v-u}{2} = ug_1(w). \end{aligned}$$

Thus we deduce that

$$\begin{aligned} \mathrm{bu}_*\mathrm{bu} &\to (\mathrm{H}\mathbb{F}_2)_*\mathrm{H}\mathbb{F}_2 \\ \frac{v-u}{2} &\mapsto \bar{\xi}_1^2 \end{aligned}$$

and thus

$$\begin{aligned} \mathrm{bu}_*\mathrm{bu} &\to (\mathrm{H}\mathbb{F}_2)_*\mathrm{H}\mathbb{F}_2 \\ \left(\frac{v^2-u^2}{2}\right)^i &\mapsto \bar{\xi}_1^{4i}. \end{aligned}$$

Since

$$2^{2i-\alpha(i)}u^{2i}f_i(w) \cong \left(\frac{v^2-u^2}{2}\right)^i \quad \text{modulo terms of higher AF}$$

we see that we have

$$\begin{aligned} \mathrm{bo}_*\mathrm{bo} &\to (\mathrm{H}\mathbb{F}_2)_*\mathrm{bo} \\ 2^{2i-\alpha(i)}u^{2i}f_i(w) &\mapsto \bar{\xi}_1^{4i}. \end{aligned}$$

We therefore can take

$$b_i = 2^{2i-\alpha(i)}u^{2i}f_i(w).$$

We have therefore arrived the following well-known theorem (see [LM87, Cor. 2.5(a)]).

Theorem 2.6. The image of the map

$$\frac{\mathrm{Ext}(\mathrm{bo} \wedge \Sigma^{4i}\underline{\mathrm{H}\mathbb{Z}_i})}{v_1\text{-tors}} \to \frac{\mathrm{Ext}(\mathrm{bo} \wedge bo)}{v_1\text{-tors}}$$

is the submodule

$$\begin{aligned} &\mathbb{F}_2[v_0]\{v_0^{\max(0,4i-2m-\alpha(i))}u^{2m}f_i(w) \;:\; m \geq i,\, m \equiv 0 \mod 2\} \\ &\oplus \mathbb{F}_2[v_0]\{v_0^{\max(0,4i-2m-1-\alpha(i))}v_0u^{2m}f_i(w) \;:\; m \geq i,\, m \equiv 0 \mod 2\} \\ &\oplus \mathbb{F}_2\left\{u^{2m}f_i(w)\eta^j \;:\; \begin{array}{l} m \geq i,\, m \equiv 0 \mod 2, \\ j \in \{1,2\},\, \alpha(i) - 4i + 2m + j \geq 0 \end{array}\right\}. \end{aligned}$$

Remark 2.7. These are the colors in Figure 2.4.

2.6. **The embedding into $\prod$ KO.** Finally we consider the maps of KO-algebras given by the composite

$$\widetilde{\psi}^{3^k} : \mathrm{KO} \wedge \mathrm{KO} \xrightarrow{1\wedge\psi^{3^k}} \mathrm{KO} \wedge \mathrm{KO} \xrightarrow{\mu} \mathrm{KO}.$$

These result in a map of KO-algebras

$$\mathrm{KO} \wedge \mathrm{KO} \xrightarrow{\Pi\,\widetilde{\psi}^{3^k}} \prod_{k\in\mathbb{Z}} \mathrm{KO}.$$

Remark 2.8. The map above has a modular interpretation. Let

$$\mathrm{Spec}(\mathbb{Z})//(\mathbb{Z}/2) \to \mathcal{M}_{fg}$$

pick out $\hat{\mathbb{G}}_m$ with the action of $[-1]$. Then the derived global sections of $\mathrm{Spec}(\mathbb{Z})//(\mathbb{Z}/2)$ are KO. The spectrum $\mathrm{KO} \wedge \mathrm{KO}$ is the global sections of the pullback

$$(\mathrm{Spec}(\mathbb{Z}) \times_{\mathcal{M}_{fg}} \mathrm{Spec}(\mathbb{Z}))//(\mathbb{Z}/2 \times \mathbb{Z}/2).$$

For $k \in \mathbb{Z}$ we may consider the map of stacks

$$\mathrm{Spec}(\mathbb{Z})//(\mathbb{Z}/2) \to (\mathrm{Spec}(\mathbb{Z}) \times_{\mathcal{M}_{fg}} \mathrm{Spec}(\mathbb{Z}))//(\mathbb{Z}/2 \times \mathbb{Z}/2)$$

sending $\hat{\mathbb{G}}_m$ to the object $[3^k] : \hat{\mathbb{G}}_m \to \hat{\mathbb{G}}_m$. As k varies this induces the map $\prod \widetilde{\psi}^{3^k}$.

Proposition 2.9. The map

$$\mathrm{KO}_*\mathrm{KO} \xrightarrow{\Pi\,\widetilde{\psi}^{3^k}} \prod_{k\in\mathbb{Z}} \mathrm{KO}_*$$

is an injection.

Proof. Consider the diagram

$$\begin{tikzcd} \mathrm{KO}_*\mathrm{KO} \arrow[r, "\Pi\,\widetilde{\psi}^{3^k}"] \arrow[d] & \prod_{k\in\mathbb{Z}} \mathrm{KO}_* \arrow[d] \\ (\mathrm{KO}_*\mathrm{KO})^\wedge_2 \arrow[r, "\Pi\,\widetilde{\psi}^{3^k}"] \arrow[d, equal] & \prod_{k\in\mathbb{Z}} (\mathrm{KO}_*)^\wedge_2 \arrow[d, equal] \\ \mathrm{Map}^c(\mathbb{Z}_2^\times/\{\pm 1\}, (\mathrm{KO}_*)^\wedge_2) \arrow[r] & \mathrm{Map}(3^{\mathbb{Z}}, (\mathrm{KO}_*)^\wedge_2), \end{tikzcd}$$

where the bottom horizontal map is the map induced from the inclusion of groups

$$3^{\mathbb{Z}} \hookrightarrow \mathbb{Z}_2^{\times}/\{\pm 1\}.$$

The vertical maps are injections, since

$$\bigcap_i 2^i \mathrm{KO}_*\mathrm{KO} = 0, \quad \text{and} \quad \bigcap_{\mathrm{i}} 2^{\mathrm{i}}\mathrm{KO}_* = 0.$$

The bottom horizontal map is an injection since $3^{\mathbb{Z}}$ is dense in $\mathbb{Z}_2^{\times}/\{\pm 1\}$. The result follows. □

We began by investigating the wedge decomposition

$$\bigvee_i \mathrm{bo} \wedge \Sigma^{4i}\mathrm{H}\mathbb{Z}_i \xrightarrow{\simeq} \mathrm{bo} \wedge \mathrm{bo}.$$

We end this section by explaining how the map

$$\mathrm{KO} \wedge \mathrm{KO} \xrightarrow{\prod \widetilde{\psi}^{3^k}} \prod_{k \in \mathbb{Z}} \mathrm{KO}$$

is compatible with the Brown-Gitler decomposition.

Proposition 2.10. The composites

$$\mathrm{bo} \wedge \mathrm{H}\mathbb{Z}_i \to \mathrm{bo} \wedge \mathrm{bo} \to \mathrm{KO} \wedge \mathrm{KO} \xrightarrow{\widetilde{\psi}^{3^i}} \mathrm{KO}$$

are equivalences after inverting v_1.

Proof. This follows from the fact that $f_i(9^i) = 1$. □

Remark 2.11. In fact, the matrix representing the composite

$$\bigvee_i \mathrm{bo} \wedge \mathrm{H}\mathbb{Z}_i \to \mathrm{bo} \wedge \mathrm{bo} \to \mathrm{KO} \wedge \mathrm{KO} \xrightarrow{\prod \widetilde{\psi}^{3^k}} \prod_{k \in \mathbb{Z}} \mathrm{KO}$$

is upper triangular, as we have

$$f_i(9^k) = \begin{cases} 0, & k < i, \\ 1, & k = i. \end{cases}$$

3. Recollections on topological modular forms

3.1. **Generalities.** The remainder of this paper is concerned with determining as much information as we can about the cooperations in the homology theory tmf based on connective topological modular forms, following our guiding example of bo. Even more than in the bo case, other players will come up. First of all, we will extensively use the periodic spectrum TMF, which is the analogue of KO. In particular, we will use that this form TMF of topological modular forms arises as the global sections of the Goerss-Hopkins-Miller sheaf of ring spectra $\mathcal{O}^{top}$ on the moduli stack of smooth elliptic curves $\mathcal{M}$. As the associated homotopy sheaves are

$$\pi_k \mathcal{O}^{top} = \begin{cases} \omega^{\otimes k/2}, & \text{if k is even}, \\ 0, & \text{if k is odd}, \end{cases}$$

there is a descent spectral sequence

$$H^s(\mathcal{M}, \omega^{\otimes t}) \Rightarrow \pi_{2t-s}TMF.$$

Morally, the connective tmf should arise as global sections of an analogous sheaf on the moduli stack of all cubic curves (i.e. allowing nodal and cuspidal singularities); however, this has not been formally carried out. Nevertheless, tmf can be constructed as an E_∞ ring spectrum from TMF as a result of the gap in the homotopy of a third, non-connective and non-periodic, version of topological modular forms associated to the compactification of $\mathcal{M}$.

Rationally, every smooth elliptic curve C/S is locally isomorphic to a cubic of the form

$$y^2 = x^3 - 27c_4x - 54c_6,$$

with the discriminant $\Delta = c_4^3 - c_6^2$ invertible. Here c_i is a section of the line bundle $\omega^{\otimes i}$ over the étale map $S \to \mathcal{M}$ classifying C. This translates to the fact that $\mathcal{M}_{\mathbb{Q}} \cong \operatorname{Proj} \mathbb{Q}[c_4, c_6][\Delta^{-1}]$, which in turn implies that $(TMF_*)_{\mathbb{Q}} = \mathbb{Q}[c_4, c_6][\Delta^{-1}]$. The connective version has $(tmf_*)_{\mathbb{Q}} = \mathbb{Q}[c_4, c_6]$.

Topological modular forms are, of course, not complex orientable, and just like in the case of bo, we will need the aid of a related orientable spectrum. The periodic TMF admits ring maps to several families of orientable (as well as non-orientable) spectra which come from the theory of elliptic curves. Namely, an elliptic curve C is an abelian group scheme so in particular it has a subgroup scheme $C[n]$ of points of order n for any positive integer n. When n is invertible, $C[n]$ is locally isomorphic to the constant group $(\mathbb{Z}/n)^2$. Rooted in this fact are the various additional structures that one can assign to an elliptic curve. In this work we will be concerned with two types, the so-called $\Gamma_1(n)$ and $\Gamma_0(n)$ level structures.

A $\Gamma_1(n)$ level structure on an elliptic curve C is a specification of a point P of (exact) order n on C, whereas a $\Gamma_0(n)$ level structure is a specification of a cyclic subgroup H of C of order n. The corresponding moduli problems are denoted $\mathcal{M}_1(n)$ and $\mathcal{M}_0(n)$. Assigning to the pair (C, P) the pair (C, H_P) where H_P is the subgroup of C generated by P determines an étale map of moduli stacks

$$g : \mathcal{M}_1(n) \to \mathcal{M}_0(n).$$

Moreover, there are two morphisms

$$f, q : \mathcal{M}_0(n) \to \mathcal{M}[1/n]$$

which are étale; f forgets the level structure whereas q quotients C by the level structure subgroup. Composing with g we obtain analogous maps from $\mathcal{M}_1(n)$. We can take sections of $\mathcal{O}^{top}$ over the forgetful maps and obtain ring spectra $TMF_1(n)$ and $TMF_0(n)$, ring maps $TMF[1/n] \to TMF_0(n) \to TMF_1(n)$ as well as maps of descent spectral sequences

$$\begin{tikzcd} H^*(\mathcal{M}[1/n], \omega^{\otimes *}) \arrow[r, Rightarrow] \arrow[d] & \pi_*TMF[1/n] \arrow[d] \\ H^*(\mathcal{M}_?(n), \omega^*) \arrow[r, Rightarrow] & \pi_*TMF_?(n), \end{tikzcd}$$

obtained by pulling back. In particular, for any odd integer n we have such a situation 2-locally.

We use the ring map $f : TMF[1/n] \to TMF_0(n)$ induced by the forgetful $f : \mathcal{M}_0(n) \to \mathcal{M}[1/n]$ to equip $TMF_0(n)$ with a $TMF[1/n]$-module structure. With this convention, the map $q : TMF[1/n] \to TMF_0(n)$ induced by the quotient map on the moduli stacks does not respect the $TMF[1/n]$-module structure. However, one can uniquely extend q to

$$\begin{tikzcd} TMF[1/n] \arrow[r, "q"] \arrow[d] & TMF_0(n) \\ TMF[1/n] \wedge TMF[1/n]. \arrow[ur, dashed, "\Psi_n"'] & \end{tikzcd} \tag{3.1}$$

Another way to define Ψ_n is as the composition of $f \wedge q$ with the multiplication on $TMF_0(n)$.

Finally, we will be interested in the morphism

$$\phi_{[n]} : \mathcal{M}[1/n] \to \mathcal{M}[1/n].$$

This is the étale map induced by the multiplication-by-n isogeny on an elliptic curve, and the induced map $\phi_{[n]} : \mathrm{TMF}[1/n] \to \mathrm{TMF}[1/n]$ can be thought of as an "Adams operation" on $\mathrm{TMF}[1/n]$.

In Section 6 below, we will make heavy use of the maps Ψ_3 and Ψ_5. Their usefulness is due to the relative ease with which their behavior on non-torsion homotopy groups can be computed.

3.2. **Details on $tmf_1(3)$ as $BP\langle 2\rangle$.** The significance of bu in the computation of bo_*bo was that at the prime 2, bu is a truncated Brown-Peterson spectrum $BP\langle 1\rangle$ with a ring map $bo \to bu$ which upon $K(1)$-localization becomes the inclusion of homotopy fixed points $(KU\hat{_2})^{hC_2} \to KU\hat{_2}$ and in particular, the image of $KO\hat{_2} \to KU\hat{_2}$ in homotopy is describable as certain invariant elements. By work of Lawson-Naumann [LN12], we know that there is a 2-primary form of $BP\langle 2\rangle$ obtained from topological modular forms; this will be our analogue of bu in the tmf-cooperations case.

Lawson-Naumann study the (2-local) compactification of the moduli stack $\mathcal{M}_1(3)$. Given an elliptic curve C (over a 2-local base), it is locally isomorphic to a Weierstrass curve of the form

$$y^2 + a_1xy + a_3y = x^3 + a_4x + a_6.$$

A point $P = (r, s)$ of order 3 is an inflection point of such a curve; transforming the curve so that the given point P is moved to have coordinates $(0, 0)$ puts C in the form

$$y^2 + a_1xy + a_3y = x^3. \tag{3.2}$$

This is the universal equation of an elliptic curve together with a $\Gamma_1(3)$ level structure. The discriminant of this curve is $\Delta = (a_1^3 - 27a_3)a_3^3$, and $\mathcal{M}_1(3) \simeq \operatorname{Proj} \mathbb{Z}_{(2)}[a_1, a_3][\Delta^{-1}]$. Consequently, $\pi_*TMF_1(3) = \mathbb{Z}_{(2)}[a_1, a_3][\Delta^{-1}]$. Lawson-Naumann show that the compactification $\bar{\mathcal{M}}_1(3) \simeq \operatorname{Proj} \mathbb{Z}_{(2)}[a_1, a_3]$ also admits a sheaf of E_∞-ring spectra, giving rise to a non-connective and non-periodic spectrum $Tmf_1(3)$ with a gap in its homotopy allowing to take a connective cover $tmf_1(3)$ which is an E_∞ ring spectrum with

$$\pi_* tmf_1(3) = \mathbb{Z}_{(2)}[a_1, a_3].$$

This spectrum is complex oriented such that the composition of graded rings

$$\mathbb{Z}_{(2)}[v_1, v_2] \subset BP_* \to (MU_{(2)})_* \to tmf_1(3)_*$$

is an isomorphism [LN12, Theorem 1.1], where the v_i are Hazewinkel generators. Of course, the map $BP_* \to tmf_1(3)_*$ classifies the p-typicalization of the formal group associated to the curve (3.2), which starts as [Sil86, IV.2], [**?**].

$$\begin{aligned} F(X,Y) = X + Y - a_1XY - 2a_3X^3Y - 3a_3X^2Y^2 + -2a_3XY^3 \\ - 2a_1a_3X^4Y - a_1a_3X^3Y^2 - a_1a_3X^2Y^3 - 2a_1a_3XY^4 + O(X,Y)^6, \end{aligned}$$

We used Sage to compute the logarithm of this formal group law, from which we read off the coefficients l_i [Rav86, A2.1.27] in front of X^{2^i} as

$$\begin{aligned} l_1 &= \frac{a_1}{2}, \qquad l_2 = \frac{a_1^3 + 2a_3}{4}, \\ l_3 &= \frac{a_1^7 + 30a_1^4a_3 + 30a_1a_3^2}{8} \dots \end{aligned}$$

Now the formula [Rav86, A2.1.1] $pl_n = \sum_{0 \le i < n} l_i v_{n-i}^{2^i}$ (in which l_0 is understood to be 1) allows us to recursively compute the map $BP_* \to tmf_1(3)_*$. For the first few values of n, we have that

$$v_1 \mapsto a_1 \qquad v_2 \mapsto a_3 \qquad v_3 \mapsto 7a_1a_3(a_1^3 + a_3) \dots$$

We can do even more with this orientation of $tmf_1(3)$, as

$$BP_*BP \to tmf_1(3)_*tmf_1(3)$$

is a morphism of Hopf algebroids.

Recall that $BP_*BP = \mathbb{Z}_{(2)}[v_1, v_2, \dots][t_1, t_2, \dots]$ with v_i and t_i in degree $2(2^i - 1)$ and right unit $\eta_R : BP_* \to BP_*BP$ determined by the fact [Rav86, A2.1.27] that

$$\eta_R(l_n) = \sum_{0 \le i \le n} l_i t_{n-i}^{2^i}$$

with $l_0 = t_0 = 1$ by convention. On the other hand,

$$tmf_1(3)_*tmf_1(3)_{\mathbb{Q}} = \mathbb{Q}[a_1, a_3, \bar{a}_1, \bar{a}_3]$$

and the right unit $tmf_1(3)_* \to tmf_1(3)_*tmf_1(3)$ sends a_i to $\bar{a}_i$. With computer aid from Sage and/or Magma, we can recursively compute the images of each t_i in

$tmf_1(3)_* tmf_1(3)$; as an example, we include here the first three values

$$t_1 \mapsto \frac{1}{2}(\bar{a}_1 - a_1),$$
$$t_2 \mapsto \frac{1}{8}(4\bar{a}_3 + 2\bar{a}_1^3 - a_1\bar{a}_1^2 + 2a_1^2\bar{a}_1 - 4a_3 - 3a_1^3), \text{ and}$$
$$\begin{aligned} t_3 \mapsto \frac{1}{128}(&480\bar{a}_1\bar{a}_3^2 - 16a_1\bar{a}_3^2 + 480\bar{a}_1^4\bar{a}_3 - 16a_1\bar{a}_1^3\bar{a}_3 + 8a_1^2\bar{a}_1^2\bar{a}_3 - 16a_1^3\bar{a}_1\bar{a}_3 \\ &+ 32a_1a_3\bar{a}_3 + 24a_1^4\bar{a}_3 + 16\bar{a}_1^7 - 4a_1\bar{a}_1^6 + 4a_1^2\bar{a}_1^5 - 4a_3\bar{a}_1^4 - 11a_1^3\bar{a}_1^4 + 32a_1a_3\bar{a}_1^3 \\ &+ 24a_1^4\bar{a}_1^3 - 32a_1^2a_3\bar{a}_1^2 - 22a_1^5\bar{a}_1^2 + 32a_1^3a_3\bar{a}_1 + 20a_1^6\bar{a}_1 - 496a_1a_3^2 - 508a_1^4a_3 - 27a_1^7)\end{aligned}$$

and rather than urging the reader to analyze the terms, we simply point out the exponential increase of their number. What will allow us to simplify and make sense of these expressions is using the Adams filtration in 3.4 below.

3.3. **The relationship between $TMF_1(3)$ and TMF and their connective versions.** As we mentioned already, the forgetful map $f : \mathcal{M}_1(3) \to \mathcal{M}$ is étale; moreover, $f^*\omega = \omega$. As a consequence, we have a Čech descent spectral sequence

$$E_1 = H^p(\mathcal{M}_1(3)^{\times_{\mathcal{M}}(q+1)}, \omega^*) \Rightarrow H^{p+q}(\mathcal{M}, \omega^*),$$

giving in particular that the modular forms $H^0(\mathcal{M}, \omega^*)$ can be computed as the equalizer of the diagram

$$H^0(\mathcal{M}_1(3), \omega^*) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} H^0(\mathcal{M}_1(3) \times_{\mathcal{M}} \mathcal{M}_1(3), \omega^*), \tag{3.3}$$

in which p_1 and p_2 are the left and right projection maps. The interpretation is that the $\mathcal{M}$-modular forms MF_* are precisely the invariant $\mathcal{M}_1(3)$-modular forms.

To be more explicit, note that $\mathcal{M}_1(3) \times_{\mathcal{M}} \mathcal{M}_1(3)$ classifies tuples $((C,P),(C',P'),\varphi)$ of elliptic curves with a point of order 3 and an isomorphism $\varphi : C \to C'$ of elliptic curves which does not need to preserve the level structures. This data is locally given by

$$\begin{aligned} C: &\quad y^2 + a_1xy + a_3y = x^3 \\ C': &\quad y^2 + a_1'xy + a_3'y = x^3 \\ \varphi: &\quad x \mapsto u^{-2}x + r \qquad\qquad y \mapsto u^{-3}y + u^{-2}sx + t, \end{aligned} \tag{3.4}$$

such that the following relations hold

$$\begin{aligned} &sa_1 - 3r + s^2 = 0 \\ &sa_3 + (t + rs)a_1 - 3r^2 + 2st = 0 \\ &r^3 - ta_3 - t^2 - rta_1 = 0. \end{aligned} \tag{3.5}$$

(Note: For more details on this presentation of $\mathcal{M}_1(3)$, see the beginning of [Sto, §4]; the relations follow from the general transformation formulas in [Sil86, III.1] by observing that the coefficients a_{even} must remain zero.)

Hence, the diagram (3.3) becomes

$$\mathbb{Z}_{(2)}[a_1, a_3] \rightrightarrows \mathbb{Z}_{(2)}[a_1, a_3][u^{\pm 1}, r, s, t]/(\sim)$$

(where $\sim$ denotes the relations (3.5)) with p_1 being the obvious inclusion and p_2 determined by

$$\begin{aligned} a_1 &\mapsto u(a_1 + 2s) \\ a_3 &\mapsto u^3(a_3 + ra_1 + 2t). \end{aligned}$$

which is in fact a Hopf algebroid representing $\mathcal{M}_{(2)}$. Note that we do not need to localize at 2 but only to invert 3 to obtain this presentation.

As a consequence of this discussion we can explicitly compute that the modular forms MF_* are the subring of $MF_1(3)_*$ generated by

$$c_4 = a_1^4 - 24a_1a_3, \qquad c_6 = a_1^6 + 36a_1^3a_3 - 216a_3^2, \qquad \text{and} \qquad \Delta = (a_1^3 - 27a_3)a_3^3, \tag{3.6}$$

which in particular determines the map $TMF_* \to TMF_1(3)_*$ on non-torsion elements.

3.4. **Adams filtrations.** The maps $BP_* \to tmf_1(3)_*$ and $BP_*BP \to tmf_1(3)_*tmf_1(3)$ respect the Adams filtration (henceforth AF), which allows us to determine the AF in the right hand sides. Recall that

$$AF(v_i) = 1, \qquad i \geq 0$$

where as usual, $v_0 = 2$. Consequently, $AF(a_1) = AF(a_3) = 1$, which in turn implies via (3.6) that $AF(c_4) = 4$, $AF(c_6) = 5$, $AF(\Delta) = 4$. More precisely, modulo higher Adams filtration we have

$$c_4 \sim a_4, \qquad c_6 \sim 216a_3^2 \sim 8a_3^2, \qquad \Delta \sim a_3^4.$$

Note that the Adams filtration of each t_i is zero.

3.5. **Supersingular elliptic curves and $K(2)$-localizations.** At the prime 2, there is a unique isomorphism class of supersingular elliptic curve; one representative is the Weierstrass curve

$$C: \qquad y^2 + y = x^3$$

over $\mathbb{F}_2$. Recall that a supersingular elliptic curve is one whose formal completion at the identity section $\hat{C}$ is a formal group of height two.[1] Under the natural map $\mathcal{M} \to \mathcal{M}_{fg}$ from the moduli stack of elliptic curves to the one of formal groups sending an elliptic curve to its formal completion at the identity section, the supersingular elliptic curves (in fixed characteristic) are sent to the (unique up to isomorphism, by Cartier's theorem) formal group of height two in that characteristic.

Let $\mathcal{M}^{ss}$ denote a formal neighborhood of the supersingular point C of $\mathcal{M}$, and let $\hat{\mathcal{H}}(2)$ denote a formal neighborhood of the characteristic 2 point of height two of $\mathcal{M}_{fg}$. Formal completion yields a map $\mathcal{M}^{ss} \to \hat{\mathcal{H}}(2)$ which is used to explicitly describe the $K(2)$-localization of TMF (or equivalently, tmf) in terms of Morava E-theory.

[1] As opposed to an ordinary elliptic curve whose formal completion has height one. These two are the only options.

The formal stack $\hat{\mathcal{H}}(2)$ has a pro-Galois cover by Spf $\mathbb{W}(\mathbb{F}_4)[[u_1]]$ for the big Morava Stabilizer group $\mathbb{G}$. The Goerss-Hopkins-Miller theorem implies in particular that this quotient description of $\hat{\mathcal{H}}(2)$ has a derived version, namely the stack Spf $E_2//\mathbb{G}_2$, where E_2 is a Lubin-Tate spectrum of height two. As we are working with elliptic curves, we take the Lubin-Tate spectrum associated to the formal group $\hat{C}$ over $\mathbb{F}_2$, and $\mathbb{G}_2 = \mathrm{Aut}_{\mathbb{F}_2}(\hat{C})$.

Let G denote the automorphism group of C; it is a finite group of order 48 given as an extension of the binary tetrahedral group with the Galois group of $\mathbb{F}_4/\mathbb{F}_2$. Then G embeds in $\mathbb{G}_2$ as a maximal finite subgroup and Spf E_2 is a Galois cover $\mathcal{M}^{ss}$ for the group G. In particular, taking sections of the structure sheaf $\mathcal{O}^{top}$ over $\mathcal{M}^{ss}$ gives the $K(2)$-localization of TMF which is equivalent to E_2^{hG}. Moreover, we have $K(2)$-local equivalences

$$(TMF \wedge TMF)_{K(2)} \simeq \mathrm{Hom}^c(\mathbb{G}_2/G, E_2)^{hG} \simeq \prod_{x \in G\backslash \mathbb{G}_2/G} E_2^{h(G\cap xGx^{-1})}.$$

The decomposition on the right hand side is interesting though we will not pursue it further in this work. The interested reader is referred to Peter Wear's explicit calculation of the double coset in [**?**].

4. The Adams spectral sequence for $\mathrm{tmf}_*\mathrm{tmf}$ and bo-Brown-Gitler modules

4.1. **Brown-Gitler modules.** (Mod 2) Brown-Gitler spectra were introduced in [BG73] to study obstructions to immersing manifolds, but immediately found use in studying the stable homotopy groups of spheres [Mah77], [Coh81] and many other places. As discussed in Section 2, Mahowald, Milgram, and others have used integral Brown-Gitler modules/spectra to decompose the ring of cooperations of bo [Mah81], [Mil75], and much of the work of Davis, Mahowald, and Rezk on tmf-resolutions has been based on the use of bo-Brown-Gitler spectra [MR09],[DM10], [BHHM08]. In this section we recapitulate and extend this latter body of work.

Generalizing the discussion of Section 2, we consider the subalgebra of of the dual Steenrod algebra

$$(A//A(i))_* = \mathbb{F}_2[\bar{\xi}_1^{2^{i+1}}, \bar{\xi}_2^{2^i}, \dots, \bar{\xi}_{i+1}^2, \bar{\xi}_{i+2}, \dots].$$

We have

$$\begin{aligned} H_*\mathrm{H}\mathbb{F}_2 &\cong A_*, \\ H_*\mathrm{H}\mathbb{Z} &\cong (A//A(0))_*, \\ H_*\mathrm{bo} &\cong (A//A(1))_*, \\ H_*\mathrm{tmf} &\cong (A//A(2))_*. \end{aligned}$$

The algebra $(A//A(i))_*$ admits an increasing filtration by defining $wt(\bar{\xi}_i) = 2^{i-1}$; then every element has filtration divisible by 2^{i+1}. The Brown-Gitler submodule $N_i(j)$ is defined to be the subspace spanned by all monomials of weight less than or equal to $2^{i+1}j$, which is also an A_*-subcomodule.

The modules $N_{-1}(j)$ through $N_1(j)$ are known to be realizable by the mod-2 (classical), integral, and bo-Brown-Gitler spectra respectively, and are usually denoted by $(H\mathbb{F}_2)_j$, $H\mathbb{Z}_j$, and bo_j, since we have

$$\begin{aligned} \mathrm{H}\mathbb{F}_2 &\simeq \varinjlim (\mathrm{H}\mathbb{F}_2)_j \\ \mathrm{H}\mathbb{Z} &\simeq \varinjlim \mathrm{H}\mathbb{Z}_j \\ \mathrm{bo} &\simeq \varinjlim \mathrm{bo}_j \end{aligned}$$

For clarifying notation we shall continue the convention we adopted in Section 2 and use underline notation to refer to the corresponding sub-comodules of the dual Steenrod algebra, so that we have

$$\begin{aligned} (\underline{\mathrm{H}\mathbb{F}_2})_j &:= H_*(\mathrm{H}\mathbb{F}_2)_j = N_{-1}(j) \\ \underline{\mathrm{H}\mathbb{Z}}_j &:= H_*\mathrm{H}\mathbb{Z}_j = N_0(j) \\ \underline{\mathrm{bo}}_j &:= H_*\mathrm{bo}_j = N_1(j) \end{aligned}$$

It is not known if tmf-Brown-Gitler spectra tmf_j exist in general, though we will still define

$$\underline{\mathrm{tmf}}_j := N_2(j).$$

The spectrum $N_3(1)$ is not realizible, by the Hopf-invariant one theorem.

There are algebraic splittings of $A(i)_*$-comodules:

$$(A//A(i))_* \cong \bigoplus_j \Sigma^{2^{i+1}j} N_{i-1}(j).$$

This splitting is given by the sum of maps:

$$\begin{aligned} \Sigma^{2^{j+1}} N_{i-1}(j) &\to (A//A(i))_* \\ \bar{\xi}_1^{i_1}\bar{\xi}_2^{i_2}\cdots &\mapsto \bar{\xi}_1^{a}\bar{\xi}_2^{i_1}\bar{\xi}_3^{i_2}\cdots \end{aligned} \tag{4.1}$$

where the exponent a above is chosen such that the monomial has weight $2^{i+1}j$. It follows that there are algebraic splittings

$$\mathrm{Ext}(\mathrm{H}\mathbb{Z} \wedge \mathrm{H}\mathbb{Z}) \cong \bigoplus \mathrm{Ext}(\Sigma^{2j}(\mathrm{H}\mathbb{F}_2)_j), \tag{4.2}$$

$$\mathrm{Ext}(\mathrm{bo} \wedge \mathrm{bo}) \cong \bigoplus \mathrm{Ext}(\Sigma^{4j}\mathrm{H}\mathbb{Z}_j), \tag{4.3}$$

$$\mathrm{Ext}(\mathrm{tmf} \wedge \mathrm{tmf}) \cong \bigoplus \mathrm{Ext}(\Sigma^{8j}\mathrm{bo}_j). \tag{4.4}$$

These algebraic splittings can be realized topologically for $i \le 1$ [Mah81]:

$$\begin{aligned} \mathrm{H}\mathbb{Z} \wedge \mathrm{H}\mathbb{Z} &\simeq \bigvee_j \Sigma^{2j}\mathrm{H}\mathbb{Z} \wedge (\mathrm{H}\mathbb{F}_2)_j, \\ \mathrm{bo} \wedge \mathrm{bo} &\simeq \bigvee_j \Sigma^{4j}\mathrm{bo} \wedge \mathrm{H}\mathbb{Z}_j. \end{aligned}$$

However, the corresponding splitting was shown by Davis, Mahowald, and Rezk [MR09], [DM10] to fail for tmf:

$$\mathrm{tmf} \wedge \mathrm{tmf} \not\simeq \bigvee_j \Sigma^{8j}\mathrm{tmf} \wedge \mathrm{bo}_j.$$

Indeed, they observe that in $\mathrm{tmf} \wedge \mathrm{tmf}$ the homology summands

$$\Sigma^8\mathrm{tmf} \wedge \mathrm{bo}_1, \quad \text{and} \quad \Sigma^{16}\mathrm{tmf} \wedge \mathrm{bo}_2$$

are attached non-trivially. We shall see in Section 7 that our methods recover this fact.

4.2. **Rational calculations.** Note that we have

$$\mathrm{tmf}_*\mathrm{tmf}_{\mathbb{Q}} \cong \mathbb{Q}[c_4, c_6, \bar{c}_4, \bar{c}_6].$$

Consider the (collapsing) v_0-inverted ASS

$$\bigoplus_i v_0^{-1} \operatorname{Ext}_{A(2)_*}(\Sigma^{8i}\underline{\mathrm{bo}}_i) \Rightarrow \mathrm{tmf}_*\mathrm{tmf} \otimes \mathbb{Q}_2.$$

In this section we explain the decomposition imposed on the E_∞-term of this spectral sequence from the decomposition on the E_2-term. In particular, given a torsion-free element $x \in \mathrm{tmf}_*\mathrm{tmf}$, this will allow us to determine which bo-Brown-Gitler module supports it in the E_2-term of the ASS for $\mathrm{tmf} \wedge \mathrm{tmf}$.

Recall from Section 3 that $\mathrm{tmf}_1(3) \simeq \mathrm{BP}\langle 2 \rangle$. In particular, we have

$$H^*(\mathrm{tmf}_1(3)) \cong A//E[Q_0, Q_1, Q_2].$$

We begin by studying the map between v_0-inverted ASS's induced by the map $\mathrm{tmf} \to \mathrm{tmf}_1(3)$.

$$\begin{tikzcd}
v_0^{-1} \operatorname{Ext}^{*,*}_{A(2)_*}(\mathbb{F}_2) \arrow[r, Rightarrow] \arrow[d] & \pi_*\mathrm{tmf} \otimes \mathbb{Q}_2 \arrow[d] \\
v_0^{-1} \operatorname{Ext}^{*,*}_{E[Q_0,Q_1,Q_2]_*}(\mathbb{F}_2) \arrow[r, Rightarrow] & \pi_*\mathrm{tmf}_1(3) \otimes \mathbb{Q}_2
\end{tikzcd}$$

We have

$$v_0^{-1} \operatorname{Ext}^{*,*}_{E[Q_0,Q_1,Q_2]_*}(\mathbb{F}_2) \cong \mathbb{F}_2[v_0^{\pm 1}, v_1, v_2]$$

where the v_i's have $(t-s, s)$ bidegrees:

$$\begin{aligned}
|v_0| &= (0, 1) \\
|v_1| &= (2, 1) \\
|v_2| &= (6, 1)
\end{aligned}$$

Recall from Section 3 that $\pi_*\mathrm{tmf}_1(3)_{\mathbb{Q}} = \mathbb{Q}[a_1, a_3]$, and that

$$\begin{aligned}
v_1 &= [a_1], \\
v_2 &= [a_3].
\end{aligned}$$

We of course have $\pi_*\mathrm{tmf}_{\mathbb{Q}} = \mathbb{Q}[c_4, c_6]$, with corresponding localized Adams E_2-term

$$v_0^{-1} \operatorname{Ext}^{*,*}_{A(2)_*}(\mathbb{F}_2) \cong \mathbb{F}_2[v_0^{\pm 1}, c_4, c_6]$$

where the $[c_i]$'s have $(t-s, s)$ bidegrees:

$$\begin{aligned}
|[c_4]| &= (8, 4) \\
|[c_6]| &= (12, 5)
\end{aligned}$$

Recall also from Section 3 that the formulas for c_4 and c_6 in terms of a_1 and a_3 imply that the map of E_2-terms of spectral sequences above is injective, and is given by

$$\begin{aligned} [c_4] &\mapsto [a_1^4], \\ [c_6] &\mapsto [8a_3^2]. \end{aligned} \tag{4.5}$$

Corresponding to the isomorphism

$$\pi_* \mathrm{tmf}_{\mathbb{Q}} \cong \mathrm{H}\mathbb{Q}_* \mathrm{tmf}$$

there is an isomorphism of localized Adams E_2-terms

$$v_0^{-1} \operatorname{Ext}_{A(2)}(\mathbb{F}_2) \cong v_0^{-1} \operatorname{Ext}_{A(0)}((A//A(2))_*).$$

Since the decomposition

$$A//A(2)_* \cong \bigoplus_j \Sigma^{8j} \underline{\mathrm{bo}}_j$$

is a decomposition of $A(2)_*$-comodules, it is in particular a decomposition of $A(0)_*$-comodules, and there is therefore a decomposition

$$v_0^{-1} \operatorname{Ext}_{A(2)_*}(\mathbb{F}_2) \cong \bigoplus_j v_0^{-1} \operatorname{Ext}_{A(0)_*}(\Sigma^{8j} \underline{\mathrm{bo}}_j) \tag{4.6}$$

Proposition 4.7. Under the decomposition (4.6), we have

$$\begin{aligned} v_0^{-1} \operatorname{Ext}_{A(0)_*}(\Sigma^{8j} \underline{\mathrm{bo}}_j) &= \mathbb{F}_2[v_0^{\pm 1}]\{[c_4^{i_1} c_6^{i_2}] \ : \ i_1 + i_2 = j\} \\ &\subset v_0^{-1} \operatorname{Ext}_{A(2)_*}(\mathbb{F}_2). \end{aligned}$$

Proof. Statement (2) of the proof of Lemma 2.5 implies that we have

$$v_0^{-1} \operatorname{Ext}_{A(0)_*}(\underline{\mathrm{bo}}_j) \cong \mathbb{F}_2[v_0^{\pm 1}]\{\bar{\xi}_1^{4i} \ : \ 0 \le i \le j\}.$$

Using the map (4.1), we deduce that we have

$$\begin{aligned} v_0^{-1} \operatorname{Ext}_{A(0)_*}(\Sigma^{8j} \underline{\mathrm{bo}}_j) &\cong \mathbb{F}_2[v_0^{\pm 1}]\{\bar{\xi}_1^{8i_1} \bar{\xi}_2^{4i_2} \ : \ i_1 + i_2 = j\} \\ &\subset \operatorname{Ext}_{A(0)_*}((A//A(2))_*). \end{aligned}$$

Consider the diagram:

$$\begin{tikzcd} H_* \mathrm{tmf} \arrow[r] & H_* \mathrm{tmf}_1(3) & \mathrm{BP}_* \mathrm{BP} \arrow[l] \arrow[d] \\ \mathrm{H}\mathbb{Z}_* \mathrm{tmf} \arrow[r] \arrow[u] \arrow[d] & \mathrm{H}\mathbb{Z}_* \mathrm{tmf}_1(3) \arrow[u] \arrow[d] & \mathrm{tmf}_1(3)_* \mathrm{tmf}_1(3) \arrow[l] \arrow[d] \\ \mathrm{H}\mathbb{Q}_* \mathrm{tmf} \arrow[r] & \mathrm{H}\mathbb{Q}_* \mathrm{tmf}_1(3) & \mathrm{tmf}_1(3)_* \mathrm{tmf}_1(3)_{\mathbb{Q}} \arrow[l] \end{tikzcd} \tag{4.8}$$

The map

$$\mathrm{BP}_* \mathrm{BP} \to H_* \mathrm{tmf}_1(3) \cong \mathbb{F}_2[\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_3^2, \bar{\xi}_4, \ldots]$$

sends t_i to $\bar{\xi}_i^2$. Thus the elements

$$\begin{aligned} \bar{\xi}_1^{8i_1} \bar{\xi}_2^{4i_2} &\in H_* \mathrm{tmf} \\ t_1^{4i_1} t_2^{2i_2} &\in \mathrm{BP}_* \mathrm{BP} \end{aligned}$$

have the same image in $H_*\mathrm{tmf}_1(3)$. However, using the formulas of Section 3, we deduce that the images of t_1 and t_2 in

$$\mathrm{tmf}_1(3)_*\mathrm{tmf}_1(3)_{\mathbb{Q}} = \mathbb{Q}[a_1, a_3, \bar{a}_1, \bar{a}_3]$$

are given by

$$\begin{aligned} t_1 &\mapsto (\bar{a}_1 + a_1)/2, \\ t_2 &\mapsto (4\bar{a}_3 - a_1\bar{a}_1^2 - 4a_3 - a_1^3)/8 + \text{terms of higher Adams filtration}. \end{aligned}$$

Since the map

$$\mathrm{tmf}_1(3)_*\mathrm{tmf}_1(3)_{\mathbb{Q}} \to \mathrm{H}\mathbb{Q}_*\mathrm{tmf}_1(3) = \mathbb{Q}[a_1, a_3]$$

of Diagram (4.8) sends $\bar{a}_i$ to a_i and a_i to zero, we deduce that the image of t_1 and t_2 in $\mathrm{H}\mathbb{Q}_*\mathrm{tmf}_1(3)$ is

$$\begin{aligned} t_1 &\mapsto a_1/2, \\ t_2 &\mapsto a_3/2 + \text{terms of higher Adams filtration}. \end{aligned}$$

It follows that under the map of v_0-localized ASS's induced by the map $\mathrm{tmf} \to \mathrm{tmf}_1(3)$:

$$v_0^{-1}\operatorname{Ext}_{A(2)_*}(\mathbb{F}_2) \to v_0^{-1}\operatorname{Ext}_{E[Q_0,Q_1,Q_2]_*}(\mathbb{F}_2)$$

we have

$$\bar{\xi}_1^{8i_1}\bar{\xi}_2^{4i_2} \mapsto [a_1/2]^{4i_1}[a_3/2]^{2i_2}.$$

Therefore, by (4.5), we have (in $v_0^{-1}\operatorname{Ext}_{A(0)_*}((A//A(2))_*)$)

$$\bar{\xi}_1^{8i_1}\bar{\xi}_2^{4i_2} = [c_4/16]^{i_1}[c_6/32]^{i_2}$$

and the result follows. □

Corresponding to the Künneth isomorphism for $\mathrm{H}\mathbb{Q}$, there is an isomorphism

$$v_0^{-1}\operatorname{Ext}_{A(0)_*}(M \otimes N) \cong v_0^{-1}\operatorname{Ext}_{A(0)_*}(M) \otimes_{\mathbb{F}_2[v_0^{\pm 1}]} \operatorname{Ext}_{A(0)_*}(N).$$

In particular, since the maps

$$v_0^{-1}\operatorname{Ext}(\mathrm{tmf} \wedge \Sigma^{8j}\mathrm{bo}_j) \to v_0^{-1}\operatorname{Ext}(\mathrm{tmf} \wedge \mathrm{tmf})$$

can be identified with the maps

$$\begin{aligned} &v_0^{-1}\operatorname{Ext}_{A(0)_*}((A//A(2))_*) \otimes_{\mathbb{F}_2[v_0^{\pm 1}]} v_0^{-1}\operatorname{Ext}_{A(0)_*}(\Sigma^{8j}\underline{\mathrm{bo}}_j) \\ &\qquad \to v_0^{-1}\operatorname{Ext}_{A(0)_*}((A//A(2))_*) \otimes_{\mathbb{F}_2[v_0^{\pm 1}]} v_0^{-1}\operatorname{Ext}_{A(0)_*}((A//A(2))_*) \end{aligned}$$

we have the following corollary.

Corollary 4.9. The map

$$v_0^{-1}\operatorname{Ext}(\mathrm{tmf} \wedge \Sigma^{8j}\mathrm{bo}_j) \to v_0^{-1}\operatorname{Ext}(\mathrm{tmf} \wedge \mathrm{tmf})$$

obtained by localizing (4.4) is the cannonical inclusion

$$\mathbb{F}_2[v_0^{\pm 1}, [c_4], [c_6]]\{[\bar{c}_4]^{i_1}[\bar{c}_6]^{i_2} \ : \ i_1 + i_2 = j\} \hookrightarrow \mathbb{F}_2[v_0^{\pm 1}, [c_4], [c_6], [\bar{c}_4], [\bar{c}_6]].$$

4.3. **Exact sequences relating the bo-Brown-Gitler modules.** In order to proceed with integral calculations we use analogs of the short exact sequences of Section 2. Lemmas 7.1 and 7.2 from [BHHM08] state that there are short exact sequences

$$0 \to \Sigma^{8j}\underline{\mathrm{bo}}_j \to \underline{\mathrm{bo}}_{2j} \to (A(2)//A(1))_* \otimes \underline{\mathrm{tmf}}_{j-1} \to \Sigma^{8j+9}\underline{\mathrm{bo}}_{j-1} \to 0 \tag{4.10}$$

$$0 \to \Sigma^{8j}\underline{\mathrm{bo}}_j \otimes \underline{\mathrm{bo}}_1 \to \underline{\mathrm{bo}}_{2j+1} \to (A(2)//A(1))_* \otimes \underline{\mathrm{tmf}}_{j-1} \to 0 \tag{4.11}$$

of $A(2)_*$-comodules. These short exact sequences provide an inductive method of computing $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_j)$ in terms of $\mathrm{Ext}_{A(1)_*}$ computations and $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^i)$.

We briefly recall how the maps in the exact sequences (4.10) and (4.11) are defined. On the level of basis elements, the maps

$$\Sigma^{8j}\underline{\mathrm{bo}}_j \to \underline{\mathrm{bo}}_{2j}$$
$$\Sigma^{8j}\underline{\mathrm{bo}}_j \otimes \underline{\mathrm{bo}}_1 \to \underline{\mathrm{bo}}_{2j+1}$$

are given respectively by

$$\bar\xi_1^{4i_1}\bar\xi_2^{2i_2}\bar\xi_3^{i_3}\cdots \mapsto \bar\xi_1^{a}\bar\xi_2^{4i_1}\bar\xi_3^{2i_2}\bar\xi_4^{i_3}\cdots,$$
$$\bar\xi_1^{4i_1}\bar\xi_2^{2i_2}\bar\xi_3^{i_3}\cdots \otimes \{1, \bar\xi_1^4, \bar\xi_2^2, \bar\xi_3\} \mapsto (\bar\xi_1^{a}\bar\xi_2^{4i_1}\bar\xi_3^{2i_2}\bar\xi_4^{i_3}\cdots)\cdot\{1, \bar\xi_1^4, \bar\xi_2^2, \bar\xi_3\}$$

where a is taken to be $8j - wt(\bar\xi_2^{4i_1}\bar\xi_3^{2i_2}\bar\xi_4^{i_3}\cdots)$. The maps

$$\underline{\mathrm{bo}}_{2j} \to (A(2)//A(1))_* \otimes \underline{\mathrm{tmf}}_{j-1}, \tag{4.12}$$

$$\underline{\mathrm{bo}}_{2j+1} \to (A(2)//A(1))_* \otimes \underline{\mathrm{tmf}}_{j-1} \tag{4.13}$$

are given by

$$\bar\xi_1^{8i_1+4\epsilon_1}\bar\xi_2^{4i_2+2\epsilon_2}\bar\xi_3^{2i_3+\epsilon_3}\bar\xi_4^{i_4}\cdots \mapsto$$
$$\begin{cases} \bar\xi_1^{8i_1}\bar\xi_2^{4i_2}\bar\xi_3^{2i_3}\bar\xi_4^{i_4}\cdots \otimes \bar\xi_1^{4\epsilon_1}\bar\xi_2^{2\epsilon_2}\bar\xi_3^{\epsilon_3}, & wt(\bar\xi_1^{8i_1}\bar\xi_2^{4i_2}\bar\xi_3^{2i_3}\bar\xi_4^{i_4}\cdots) \le 8j-8, \\ 0, & \text{otherwise,} \end{cases}$$

where $\epsilon_s \in \{0,1\}$. The only change from the integral Brown-Gitler case is that while the map (4.13) is surjective, the map (4.12) is not. The cokernel is spanned by the submodule

$$\mathbb{F}_2\{\bar\xi_1^4\bar\xi_2^2\bar\xi_3\} \otimes \Sigma^{8j-8}\underline{\mathrm{bo}}_{j-1} \subset (A(2)//A(1))_* \otimes \underline{\mathrm{tmf}}_{j-1}.$$

We therefore have an exact sequence

$$\underline{\mathrm{bo}}_{2j} \to (A(2)//A(1))_* \otimes \underline{\mathrm{tmf}}_{j-1} \to \Sigma^{8j+9}\underline{\mathrm{bo}}_{j-1} \to 0$$

We give some low dimensional examples. We shall use the shorthand

$$M \Leftarrow \bigoplus M_i[k_i]$$

to denote the existence of a spectral sequence

$$\bigoplus \mathrm{Ext}_{A(2)_*}^{s-k_i, t+k_i}(M_i) \Rightarrow \mathrm{Ext}_{A(2)_*}^{s,t}(M).$$

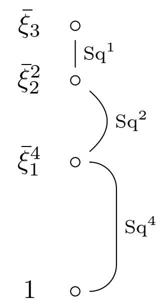
In the notation above, we shall abbreviate $M_i[0]$ as M_i. We have:

(4.14)

$$
\begin{aligned}
\Sigma^{16}\underline{\mathrm{bo}}_2 &\Leftarrow \Sigma^{16}(A(2)//A(1))_* \oplus \Sigma^{24}\underline{\mathrm{bo}}_1 \oplus \Sigma^{32}\mathbb{F}_2[1] \\
\Sigma^{24}\underline{\mathrm{bo}}_3 &\Leftarrow \Sigma^{24}(A(2)//A(1))_* \oplus \Sigma^{32}\underline{\mathrm{bo}}_1^2 \\
\Sigma^{32}\underline{\mathrm{bo}}_4 &\Leftarrow (A(2)//A(1))_* \otimes \left(\Sigma^{32}\underline{\mathrm{tmf}}_1 \oplus \Sigma^{48}\mathbb{F}_2\right) \oplus \Sigma^{56}\underline{\mathrm{bo}}_1 \oplus \Sigma^{56}\underline{\mathrm{bo}}_1[1] \oplus \Sigma^{64}\mathbb{F}_2[1] \\
\Sigma^{40}\underline{\mathrm{bo}}_5 &\Leftarrow (A(2)//A(1))_* \otimes \left(\Sigma^{40}\underline{\mathrm{tmf}}_1 \oplus \Sigma^{56}\underline{\mathrm{bo}}_1\right) \oplus \Sigma^{64}\underline{\mathrm{bo}}_1^2 \oplus \Sigma^{72}\underline{\mathrm{bo}}_1[1] \\
\Sigma^{48}\underline{\mathrm{bo}}_6 &\Leftarrow (A(2)//A(1))_* \otimes \left(\Sigma^{48}\underline{\mathrm{tmf}}_2 \oplus \Sigma^{72}\mathbb{F}_2 \oplus \Sigma^{80}\mathbb{F}_2[1]\right) \\
&\qquad \oplus \Sigma^{80}\underline{\mathrm{bo}}_1^2 \oplus \Sigma^{88}\underline{\mathrm{bo}}_1[1] \oplus \Sigma^{96}\mathbb{F}_2[2] \\
\Sigma^{56}\underline{\mathrm{bo}}_7 &\Leftarrow (A(2)//A(1))_* \otimes \left(\Sigma^{56}\underline{\mathrm{tmf}}_2 \oplus \Sigma^{80}\underline{\mathrm{bo}}_1\right) \oplus \Sigma^{88}\underline{\mathrm{bo}}_1^3 \\
\Sigma^{64}\underline{\mathrm{bo}}_8 &\Leftarrow (A(2)//A(1))_* \otimes \left(\Sigma^{64}\underline{\mathrm{tmf}}_3 \oplus \Sigma^{96}\underline{\mathrm{tmf}}_1 \oplus \Sigma^{112}\mathbb{F}_2 \oplus \Sigma^{104}\mathbb{F}_2[1]\right) \\
&\qquad \oplus \Sigma^{112}\underline{\mathrm{bo}}_1^2[1] \oplus \Sigma^{120}\underline{\mathrm{bo}}_1 \oplus \Sigma^{120}\underline{\mathrm{bo}}_1[1] \oplus \Sigma^{128}\mathbb{F}_2[1]
\end{aligned}
$$

In practice, these spectral sequences seem to tend to collapse. In fact, in the range computed explicitly in this paper, there are no differentials in these spectral sequences, and the authors have not yet encountered any differentials in these spectral sequences. These spectral sequences do collapse with v_0-inverted, for dimensional reasons.

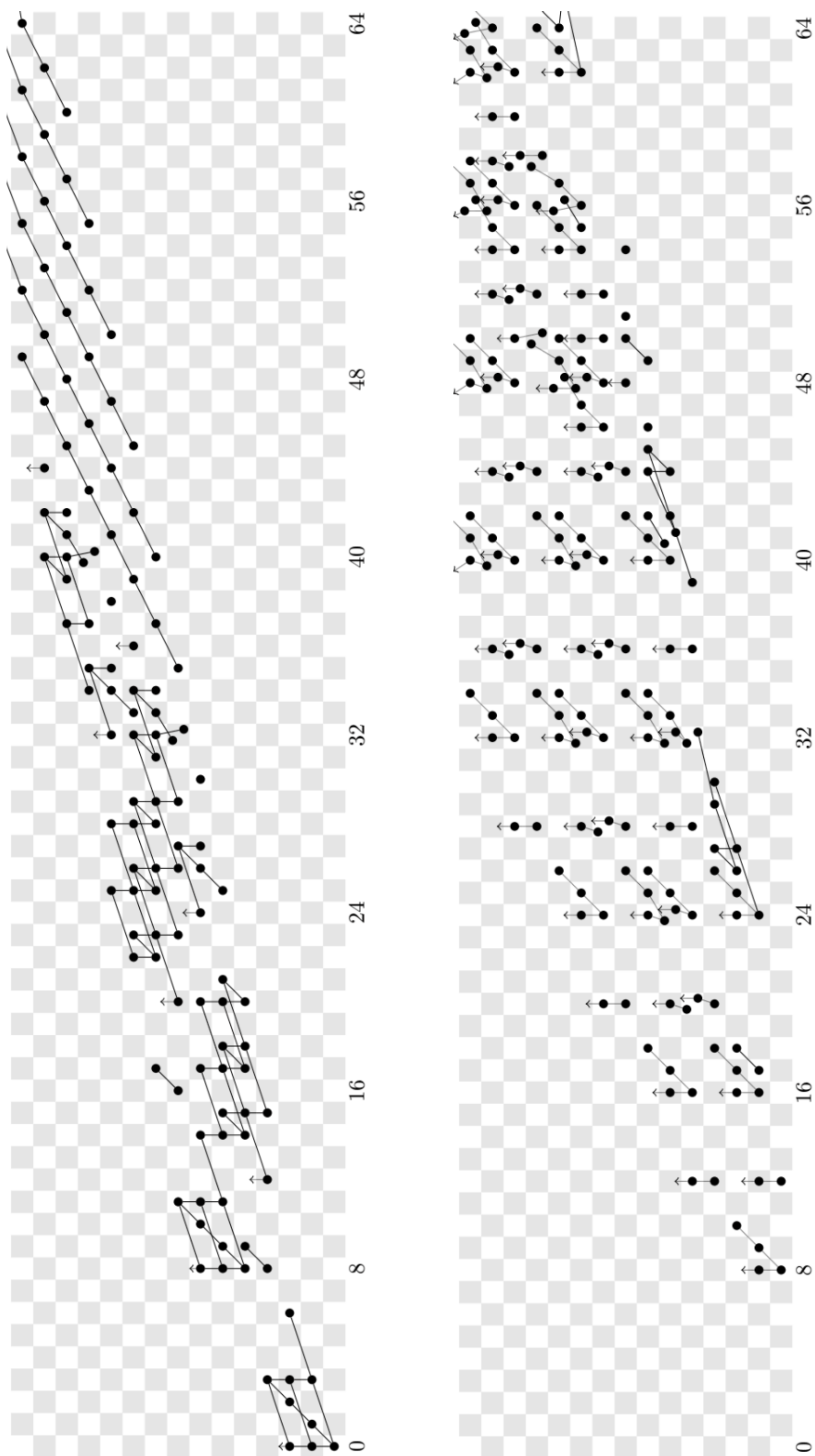
In principle the exact sequences (4.10), (4.11) allow one to inductively compute $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_j)$ given $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k})$, where $\underline{\mathrm{bo}}_1$ is depicted below.



The problem is that, unlike the $A(1)$-case, we do not have a closed form computation of $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k})$. These computations for $k \leq 3$ appeared in [BHHM08] (the cases of $k = 0, 1$ appeared elsewhere). We include in Figures **??** through **??** the charts for $\Sigma^{8j}\underline{\mathrm{bo}}_j$, for $0 \leq j \leq 6$, as well as $\Sigma^8\underline{\mathrm{bo}}_1^2$ in dimensions ≤ 64.

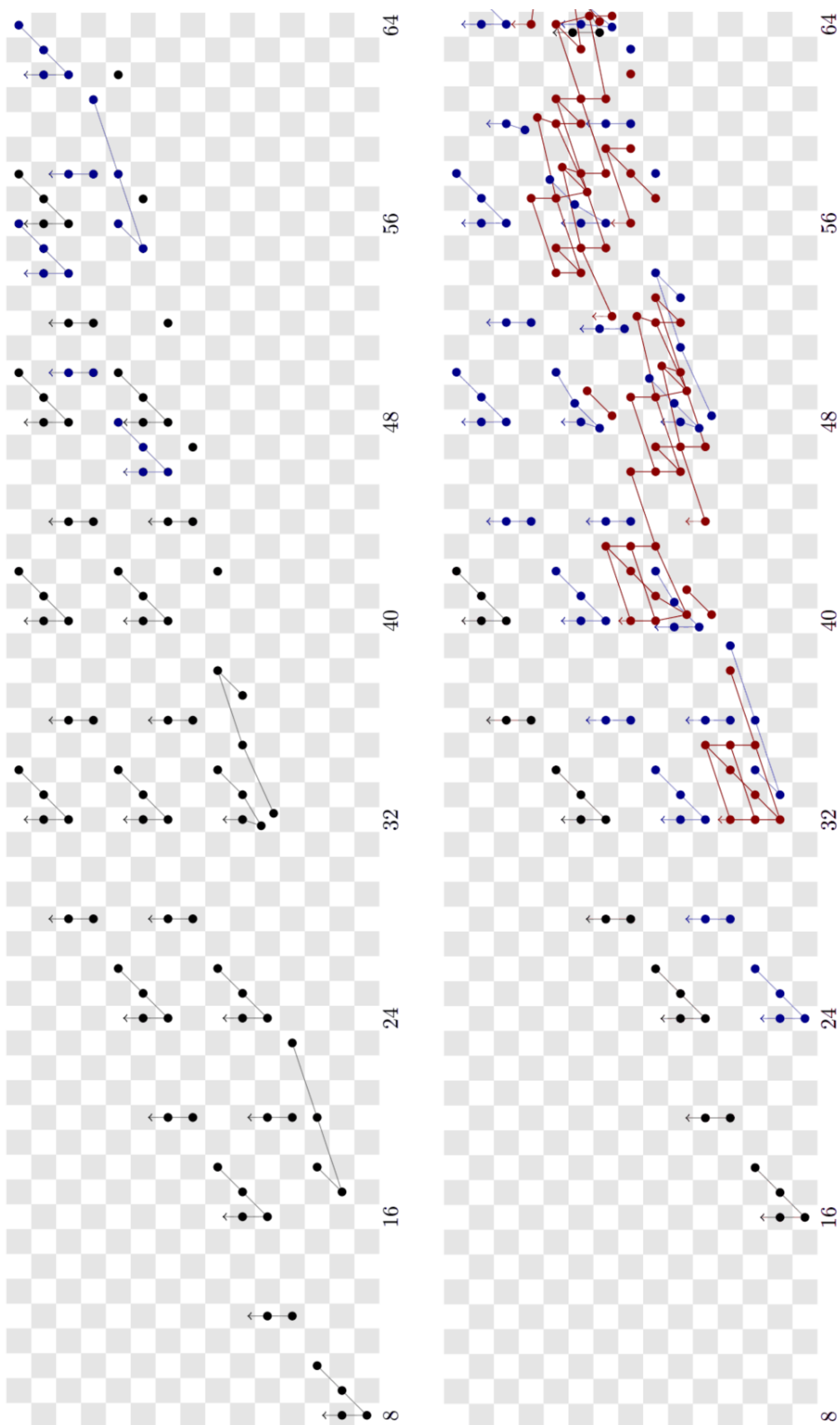
4.4. **Rational behavior of the exact sequences.** We finish this section with a discussion on how to identify the generators of $\frac{\mathrm{Ext}_{A(2)_*}(\Sigma^{8j}\underline{\mathrm{bo}}_j)}{v_0-tors}$. On one hand, the inclusion

$$
\begin{tikzcd}
\frac{\mathrm{Ext}_{A(2)_*}(\Sigma^{8j}\underline{\mathrm{bo}}_j)}{v_0-tors} \arrow[r, hook] & v_0^{-1}\mathrm{Ext}_{A(2)_*}(\Sigma^{8j}\underline{\mathrm{bo}}_j) \arrow[r, equal] \arrow[d, hook] & \mathbb{F}_2[v_0^{\pm 1}, [c_4], [c_6]]\{\bar{\xi}^{8i_1}\bar{\zeta}^{4i_2} \ : \ i_1 + i_2 = j\} \\
& v_0^{-1}\mathrm{Ext}_{A(2)_*}((A//A(2))_*) &
\end{tikzcd}
$$



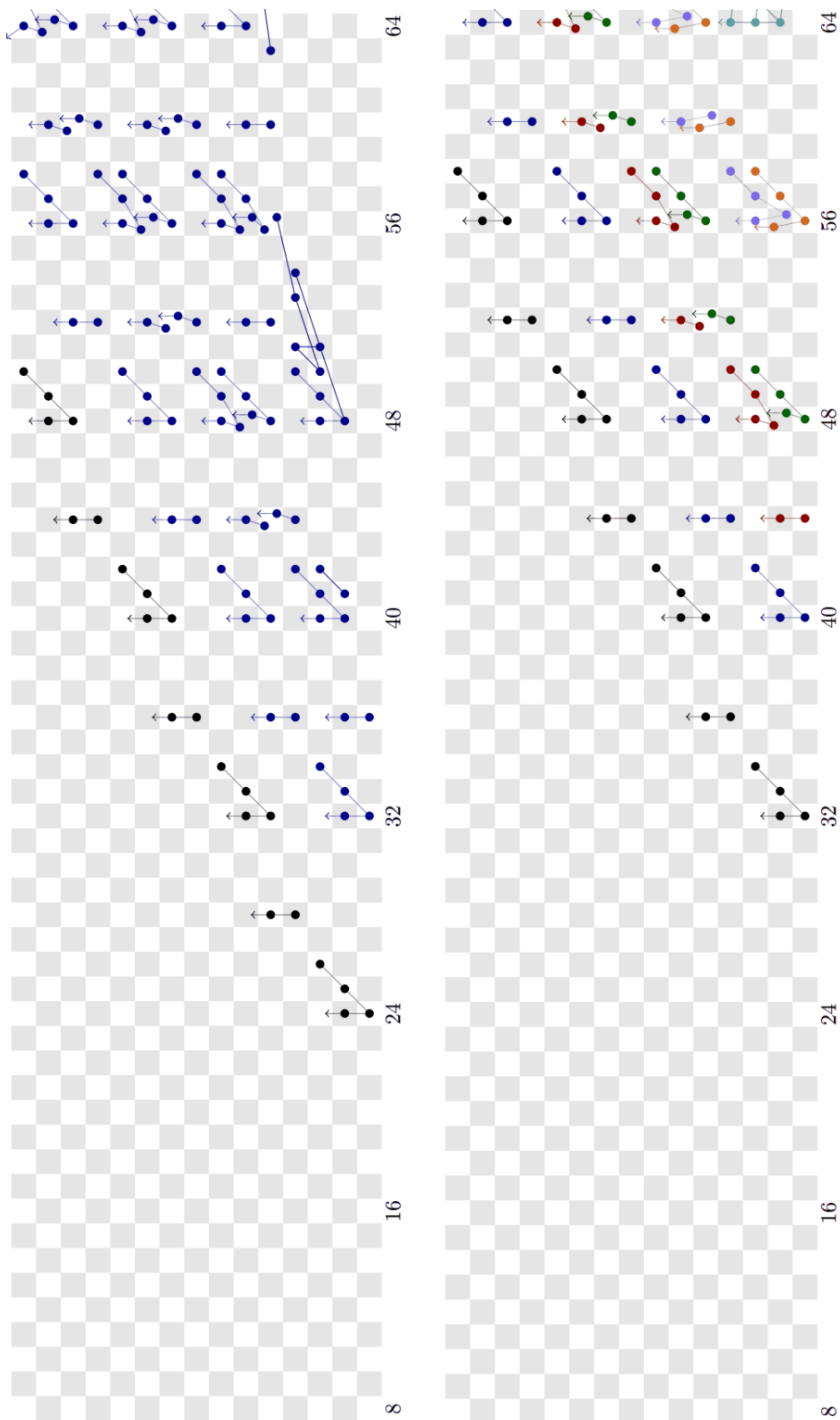
(A) $\underline{\mathrm{bo}}_0$ (B) $\Sigma^8\underline{\mathrm{bo}}_1^2$

FIGURE 4.1



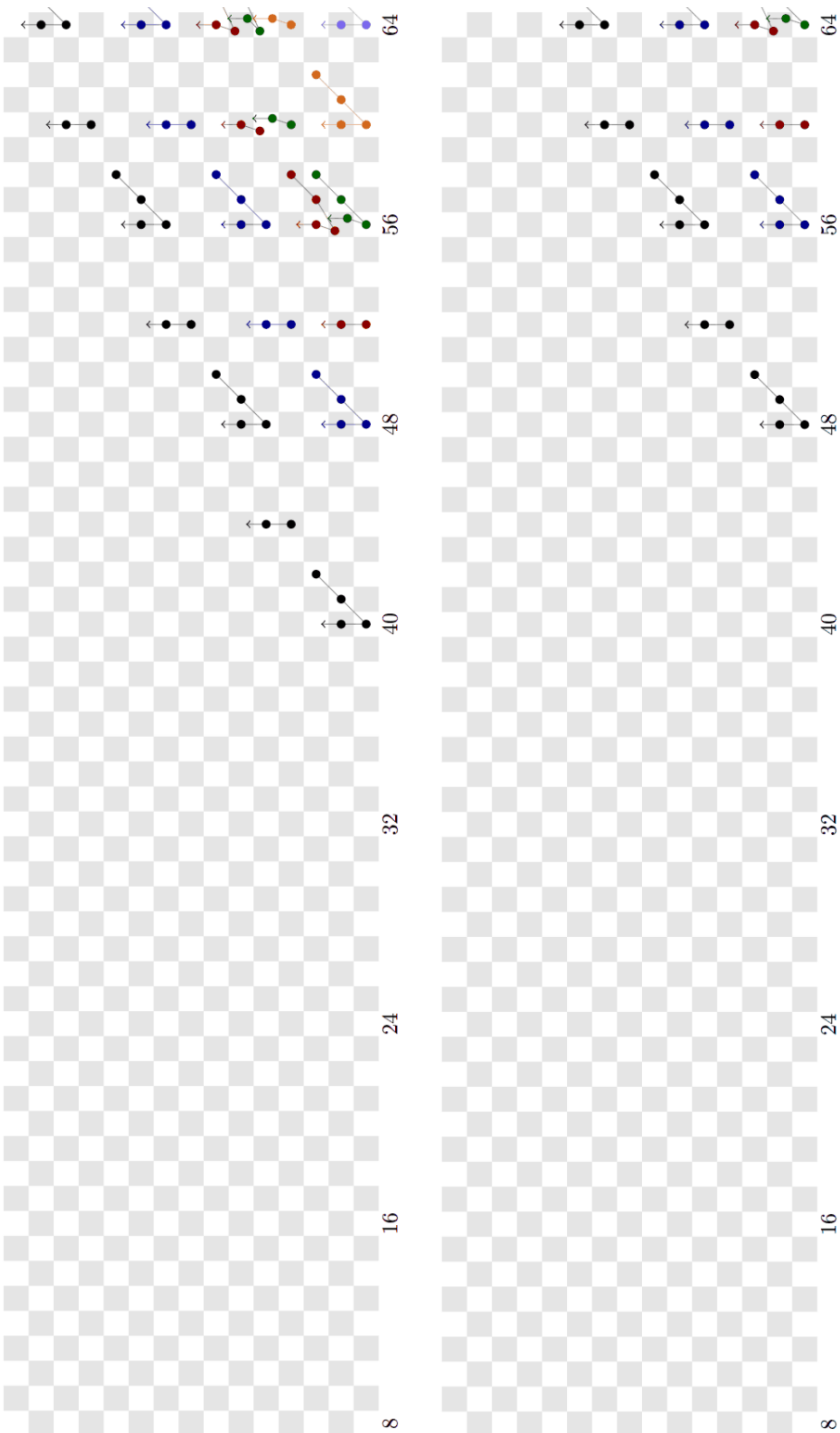
(A) $\Sigma^8\underline{\mathrm{bo}}_1$ (B) $\Sigma^{16}\underline{\mathrm{bo}}_2$

FIGURE 4.2



(A) $\Sigma^{24}\underline{\mathrm{bo}}_3$ (B) $\Sigma^{32}\underline{\mathrm{bo}}_4$

FIGURE 4.3



(A) $\Sigma^{40}\underline{\mathrm{bo}}_5$ (B) $\Sigma^{48}\underline{\mathrm{bo}}_6$

FIGURE 4.4

discussed in Section 4.2 informs us that the h_0-towers of $\mathrm{Ext}_{A(2)_*}(\Sigma^{8j}\underline{\mathrm{bo}}_j)$ are all generated by

$$h_0^k[c_4]^p[c_6]^q\bar{\xi}_1^{8i_1}\bar{\xi}_2^{4i_2}$$

for appropriate (possibly negative) values of k depending on i_1, i_2, p, and q.

The problem lies in that the terms

$$v_0^{-1}\,\mathrm{Ext}_{A(2)}(\Sigma^{16j}(A(2)//A(1))_*\otimes\underline{\mathrm{tmf}}_{j-1})\subset\mathrm{Ext}_{A(2)_*}(\Sigma^{16j}\mathrm{bo}_{2j}), \tag{4.15}$$

$$v_0^{-1}\,\mathrm{Ext}_{A(2)}(\Sigma^{16j+8}(A(2)//A(1))_*\otimes\underline{\mathrm{tmf}}_{j-1})\subset\mathrm{Ext}_{A(2)_*}(\Sigma^{16j+8}\mathrm{bo}_{2j+1}) \tag{4.16}$$

in the short exact sequences (4.10), (4.11) are not free over $\mathbb{F}_2[v_0^{\pm1},[c_4],[c_6]]$ (however, they are free over $\mathbb{F}_2[v_0^{\pm1},[c_4]]$).

We therefore instead identify the generators of $v_0^{-1}\,\mathrm{Ext}_{A(2)_*}((A//A(2))_*)$ corresponding to the generators of (4.15) and (4.16) as modules over $\mathbb{F}_2[v_0^{\pm1},[c_4]]$, as well as those generators coming (inductively) from

$$v_0^{-1}\,\mathrm{Ext}_{A(2)_*}(\Sigma^{24j}\underline{\mathrm{bo}}_j)\subset v_0^{-1}\,\mathrm{Ext}_{A(2)_*}(\Sigma^{16j}\underline{\mathrm{bo}}_{2j}), \tag{4.17}$$

$$v_0^{-1}\,\mathrm{Ext}_{A(2)_*}(\Sigma^{24j+8}\underline{\mathrm{bo}}_j\otimes\underline{\mathrm{bo}}_1)\subset v_0^{-1}\,\mathrm{Ext}_{A(2)_*}(\Sigma^{16j+8}\underline{\mathrm{bo}}_{2j+1}). \tag{4.18}$$

in the following two lemmas, whose proofs are immediate from the definitions of the maps in (4.10), (4.11).

Lemma 4.19. The summands (4.15) (respectively (4.16)) are generated, as modules over $\mathbb{F}_2[v_0^{\pm1},[c_4]]$, by the elements

$$\bar{\xi}_1^a\bar{\xi}_2^{8i_1}\bar{\xi}_3^{4i_3},\ \bar{\xi}_1^{a-8}\bar{\xi}_2^{8i_1+4}\bar{\xi}_3^{4i_3}\in(A//A(2))_*$$

with $i_1+i_2\le j-1$ and $a=16j-8i_1-8i_2$ (respectively $a=16j+8-8i_1-8i_2$).

Lemma 4.20. Suppose inductively (via the exact sequences (4.10),(4.11)) that the summand

$$v_0^{-1}\,\mathrm{Ext}_{A(2)_*}(\Sigma^{8j}\underline{\mathrm{bo}}_j)\subset v_0^{-1}\,\mathrm{Ext}_{A(2)_*}((A//A(2))_*)$$

is generated by generators of the form

$$\{\bar{\xi}_1^{i_1}\bar{\xi}_2^{i_2}\ldots\}.$$

Then the summand (4.17) is generated by

$$\{\bar{\xi}_2^{i_1}\bar{\xi}_3^{i_2}\cdots\}$$

and the summand (4.18) is generated by

$$\{\bar{\xi}_2^{i_1}\bar{\xi}_3^{i_2}\cdots\}\cdot\{\bar{\xi}_1^8,\bar{\xi}_2^4\}.$$

The remaining term

$$v_0^{-1}\,\mathrm{Ext}_{A(2)_*}(\Sigma^{24j+8}\underline{\mathrm{bo}}_{j-1}[1])\subset v_0^{-1}\,\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_{2j}) \tag{4.21}$$

coming from (4.10) is handled by the following lemma.

Lemma 4.22. Consider the summand

$$v_0^{-1}\,\mathrm{Ext}_{A(1)_*}(\Sigma^{24j-8}\underline{\mathrm{bo}}_{j-1})\subset v_0^{-1}\,\mathrm{Ext}_{A(1)_*}(\Sigma^{16j}\underline{\mathrm{tmf}}_{j-1})\subset v_0^{-1}\,\mathrm{Ext}_{A(2)_*}(\Sigma^{16j}\mathrm{bo}_{2j})$$

generated as a module over $\mathbb{F}_2[v_0^{\pm1},[c_4]]$ by the generators

$$\bar{\xi}_1^{16}\bar{\xi}_2^{8i_1}\bar{\xi}_3^{4i_2},\ \bar{\xi}_1^{8}\bar{\xi}_2^{8i_1+4}\bar{\xi}_3^{4i_2}\in(A//A(2))_*$$

with $i_1 + i_2 = j - 1$. Let x_i $(0 \le i \le j-1)$ be the generator of the summand (4.21), as a module over $\mathbb{F}_2[v_0^{\pm 1}, [c_4], [c_6]]$ corresponding to the generator $\bar{\xi}_1^{4i} \in \underline{\mathrm{bo}}_{j-1}$. The we have

$$[c_6]\bar{\xi}_1^8 \bar{\xi}_2^{8i_1+4} \bar{\xi}_3^{4i_2} = v_0^4 x_{i_2} + \cdots$$

in $v_0^{-1}\operatorname{Ext}_{A(2)_*}(\Sigma^{16j}\underline{\mathrm{bo}}_{2j})$, where the additional terms not listed above all come from the summand

$$v_0^{-1}\operatorname{Ext}_{A(2)_*}(\Sigma^{24j}\underline{\mathrm{bo}}_j) \subset v_0^{-1}\operatorname{Ext}_{A(2)_*}(\Sigma^{16j}\underline{\mathrm{bo}}_{2j}).$$

Proof. This follows from the definition of the last map in (4.10), together with the fact that with v_0-inverted, the cell $\bar{\xi}_1^4\bar{\xi}_2^2\bar{\xi}_3 \in (A(2)//A(1))_*$ attaches to the cell $\bar{\xi}_1^4$ with attaching map $[c_6]/v_0^4$. □

Lemmas 4.19, 4.20, and 4.22 give an inductive method of identifying a collection of generators for $v_0^{-1}\operatorname{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_j)$ which are compatible with the exact sequences (4.10), (4.11). We tabulate these below for the decompositions arising from the spectral sequences (4.14). For those summands of the form $(A(2)//A(1))_* \otimes -$ these are generators over $\mathbb{F}_2[v_0^{\pm}1, [c_4]]$, for the other summands these are generators over $\mathbb{F}_2[v_0, [c_4], [c_6]]$:

$$\begin{array}{rrl}
\underline{\mathrm{bo}}_0 : & \mathbb{F}_2 : & 1 \\
\Sigma^8\underline{\mathrm{bo}}_1 : & \Sigma^8\underline{\mathrm{bo}}_1 : & \bar{\xi}_1^8, \bar{\xi}_2^4 \\
\Sigma^{16}\underline{\mathrm{bo}}_2 : & \Sigma^{16}(A(2)//A(1))_* : & \bar{\xi}_1^{16}, \bar{\xi}_1^8\bar{\xi}_2^4 \\
 & \Sigma^{24}\underline{\mathrm{bo}}_1 : & \bar{\xi}_2^8, \bar{\xi}_3^4 \\
 & \Sigma^{32}\mathbb{F}_2[1] : & v_0^{-4}[c_6]\bar{\xi}_1^8\bar{\xi}_2^4 + \cdots \\
\Sigma^{24}\underline{\mathrm{bo}}_3 : & \Sigma^{24}(A(2)//A(1))_* : & \bar{\xi}_1^{24}, \bar{\xi}_1^{16}\bar{\xi}_2^4 \\
 & \Sigma^{32}\underline{\mathrm{bo}}_1^2 : & \{\bar{\xi}_2^8, \bar{\xi}_3^4\} \cdot \{\bar{\xi}_1^8, \bar{\xi}_2^4\} \\
\Sigma^{32}\underline{\mathrm{bo}}_4 : & \Sigma^{32}(A(2)//A(1))_* \otimes \underline{\mathrm{tmf}}_1 : & \bar{\xi}_1^3 2, \bar{\xi}_1^{24}\bar{\xi}_2^4, \bar{\xi}_1^{16}\bar{\xi}_2^8, \bar{\xi}_1^8\bar{\xi}_2^{12}, \bar{\xi}_1^{16}\bar{\xi}_3^4, \bar{\xi}_1^8\bar{\xi}_2^4\bar{\xi}_3^4 \\
 & \Sigma^{48}(A(2)//A(1))_* : & \bar{\xi}_2^{16}, \bar{\xi}_2^8\bar{\xi}_3^4 \\
 & \Sigma^{56}\underline{\mathrm{bo}}_1 : & \bar{\xi}_3^8, \bar{\xi}_4^4 \\
 & \Sigma^{64}\mathbb{F}_2[1] : & v_0^{-4}[c_6]\bar{\xi}_2^8\bar{\xi}_3^4 + \cdots \\
 & \Sigma^{56}\underline{\mathrm{bo}}_1[1] : & v_0^{-4}[c_6]\bar{\xi}_1^8\bar{\xi}_2^{12} + \cdots, v_0^{-4}[c_6]\bar{\xi}_1^8\bar{\xi}_2^4\bar{\xi}_3^4 + \cdots \\
\Sigma^{40}\underline{\mathrm{bo}}_5 : & \Sigma^{40}(A(2)//A(1))_* \otimes \underline{\mathrm{tmf}}_1 : & \bar{\xi}_1^{40}, \bar{\xi}_1^{32}\bar{\xi}_2^4, \bar{\xi}_1^{24}\bar{\xi}_2^8, \bar{\xi}_1^{16}\bar{\xi}_2^{12}, \bar{\xi}_1^{24}\bar{\xi}_3^4, \bar{\xi}_1^{16}\bar{\xi}_2^4\bar{\xi}_3^4 \\
 & \Sigma^{56}(A(2)//A(1))_* \otimes \underline{\mathrm{bo}}_1 : & \{\bar{\xi}_2^{16}, \bar{\xi}_2^8\bar{\xi}_3^4\} \cdot \{\bar{\xi}_1^8, \bar{\xi}_2^4\} \\
 & \Sigma^{64}\underline{\mathrm{bo}}_1^2 : & \{\bar{\xi}_3^8, \bar{\xi}_4^4\} \cdot \{\bar{\xi}_1^8, \bar{\xi}_2^4\} \\
 & \Sigma^{72}\underline{\mathrm{bo}}_1[1] : & \{v_0^{-4}[c_6]\bar{\xi}_2^8\bar{\xi}_3^4 + \cdots\} \cdot \{\bar{\xi}_1^8, \bar{\xi}_2^4\} \\
\Sigma^{48}\underline{\mathrm{bo}}_6 : & \Sigma^{48}(A(2)//A(1))_* \otimes \underline{\mathrm{tmf}}_2 : & \bar{\xi}_1^{48}, \bar{\xi}_1^{40}\bar{\xi}_2^4, \bar{\xi}_1^{32}\bar{\xi}_2^8, \bar{\xi}_1^{24}\bar{\xi}_2^{12}, \bar{\xi}_1^{32}\bar{\xi}_3^4, \bar{\xi}_1^{24}\bar{\xi}_2^4\bar{\xi}_3^4, \\
 & & \quad \bar{\xi}_1^{16}\bar{\xi}_2^{16}, \bar{\xi}_1^8\bar{\xi}_2^{20}, \bar{\xi}_1^{16}\bar{\xi}_2^8\bar{\xi}_3^4, \bar{\xi}_1^8\bar{\xi}_2^{12}\bar{\xi}_3^4, \bar{\xi}_1^{16}\bar{\xi}_3^8, \bar{\xi}_1^8\bar{\xi}_2^4\bar{\xi}_3^8 \\
 & \Sigma^{72}(A(2)//A(1))_* : & \bar{\xi}_2^{24}, \bar{\xi}_2^{16}\bar{\xi}_3^4 \\
 & \Sigma^{80}\underline{\mathrm{bo}}_1^2 : & \{\bar{\xi}_3^8, \bar{\xi}_4^4\} \cdot \{\bar{\xi}_2^8, \bar{\xi}_3^4\} \\
 & \Sigma^{80}\underline{\mathrm{bo}}_2[1] & v_0^{-4}[c_6]\bar{\xi}_1^8\bar{\xi}_2^{20} + \cdots, v_0^{-4}[c_6]\bar{\xi}_1^8\bar{\xi}_2^{12}\bar{\xi}_3^4 + \cdots,
\end{array}$$

$$\begin{array}{rrl}
 & & v_0^{-4}[c_6]\bar{\xi}_1^8\bar{\xi}_2^4\bar{\xi}_3^8 + \cdots \\
\Sigma^{56}\underline{\mathrm{bo}}_7: & \Sigma^{56}(A(2)//A(1))_* \otimes \underline{\mathrm{tmf}}_2: & \bar{\xi}_1^{56}, \bar{\xi}_1^{48}\bar{\xi}_2^4, \bar{\xi}_1^{40}\bar{\xi}_2^8, \bar{\xi}_1^{32}\bar{\xi}_2^{12}, \bar{\xi}_1^{40}\bar{\xi}_3^4, \bar{\xi}_1^{32}\bar{\xi}_2^4\bar{\xi}_3^4, \\
 & & \quad \bar{\xi}_1^{24}\bar{\xi}_2^{16}, \bar{\xi}_1^{16}\bar{\xi}_2^{20}, \bar{\xi}_1^{24}\bar{\xi}_2^8\bar{\xi}_3^4, \bar{\xi}_1^{16}\bar{\xi}_2^{12}\bar{\xi}_3^4, \bar{\xi}_1^{24}\bar{\xi}_3^8, \bar{\xi}_1^{16}\bar{\xi}_2^4\bar{\xi}_3^8 \\
 & \Sigma^{80}(A(2)//A(1))_* \otimes \underline{\mathrm{bo}}_1: & \{\bar{\xi}_2^{24}, \bar{\xi}_2^{16}\bar{\xi}_3^4\} \cdot \{\bar{\xi}_1^8, \bar{\xi}_2^4\} \\
 & \Sigma^{88}\underline{\mathrm{bo}}_1^3: & \{\bar{\xi}_3^8, \bar{\xi}_4^4\} \cdot \{\bar{\xi}_2^8, \bar{\xi}_3^4\} \cdot \{\bar{\xi}_1^8, \bar{\xi}_2^4\} \\
\Sigma^{64}\underline{\mathrm{bo}}_8: & \Sigma^{64}(A(2)//A(1))_* \otimes \underline{\mathrm{tmf}}_3: & \bar{\xi}_1^{64}, \bar{\xi}_1^{56}\bar{\xi}_2^4, \bar{\xi}_1^{48}\bar{\xi}_2^8, \bar{\xi}_1^{40}\bar{\xi}_2^{12}, \bar{\xi}_1^{48}\bar{\xi}_3^4, \bar{\xi}_1^{40}\bar{\xi}_2^4\bar{\xi}_3^4, \\
 & & \quad \bar{\xi}_1^{32}\bar{\xi}_2^{16}, \bar{\xi}_1^{24}\bar{\xi}_2^{20}, \bar{\xi}_1^{32}\bar{\xi}_2^8\bar{\xi}_3^4, \bar{\xi}_1^{24}\bar{\xi}_2^{12}\bar{\xi}_3^4, \bar{\xi}_1^{32}\bar{\xi}_3^8, \bar{\xi}_1^{24}\bar{\xi}_2^4\bar{\xi}_3^8, \\
 & & \quad \bar{\xi}_1^{16}\bar{\xi}_2^{24}, \bar{\xi}_1^8\bar{\xi}_2^{28}, \bar{\xi}_1^{16}\bar{\xi}_2^{16}\bar{\xi}_3^4, \bar{\xi}_1^8\bar{\xi}_2^{20}\bar{\xi}_3^4, \bar{\xi}_1^{16}\bar{\xi}_2^8\bar{\xi}_3^8, \bar{\xi}_1^8\bar{\xi}_2^{12}\bar{\xi}_3^8, \\
 & & \quad \bar{\xi}_1^{16}\bar{\xi}_3^{12}, \bar{\xi}_1^8\bar{\xi}_2^4\bar{\xi}_3^{12} \\
 & \Sigma^{96}(A(2)//A(1))_* \otimes \underline{\mathrm{tmf}}_1: & \bar{\xi}_2^3 2, \bar{\xi}_2^{24}\bar{\xi}_3^4, \bar{\xi}_2^{16}\bar{\xi}_3^8, \bar{\xi}_2^8\bar{\xi}_3^{12}, \bar{\xi}_2^{16}\bar{\xi}_4^4, \bar{\xi}_2^8\bar{\xi}_3^4\bar{\xi}_4^4 \\
 & \Sigma^{112}(A(2)//A(1))_*: & \bar{\xi}_3^{16}, \bar{\xi}_3^8\bar{\xi}_4^4 \\
 & \Sigma^{120}\underline{\mathrm{bo}}_1: & \bar{\xi}_4^8, \bar{\xi}_5^4 \\
 & \Sigma^{128}\mathbb{F}_2[1]: & v_0^{-4}[c_6]\bar{\xi}_3^8\bar{\xi}_4^4 + \cdots \\
 & \Sigma^{120}\underline{\mathrm{bo}}_1[1]: & v_0^{-4}[c_6]\bar{\xi}_2^8\bar{\xi}_3^{12} + \cdots, v_0^{-4}[c_6]\bar{\xi}_2^8\bar{\xi}_3^4\bar{\xi}_4^4 + \cdots \\
 & \Sigma^{104}\underline{\mathrm{bo}}_3[1]: & v_0^{-4}[c_6]\bar{\xi}_1^8\bar{\xi}_2^{28} + \cdots, v_0^{-4}[c_6]\bar{\xi}_1^8\bar{\xi}_2^{20}\bar{\xi}_3^4 + \cdots, \\
 & & \quad v_0^{-4}[c_6]\bar{\xi}_1^8\bar{\xi}_2^{12}\bar{\xi}_3^8 + \cdots
\end{array}$$

4.5. **Identification of the integral lattice.** Having constructed useful bases of the summands

$$v_0^{-1}\operatorname{Ext}_{A(2)_*}(\Sigma^{8j}\mathrm{bo}_j) \subset v_0^{-1}\operatorname{Ext}_{A(2)_*}(A//A(2)_*)$$

it remains to understand the lattices

$$\frac{\operatorname{Ext}_{A(2)_*}(\Sigma^{8j}\mathrm{bo}_j)}{v_0 - tors} \subset v_0^{-1}\operatorname{Ext}_{A(2)_*}(\Sigma^{8j}\mathrm{bo}_j)$$

This can accomplished inductively; the rational generators we identified in the last section are compatible with the exact sequences (4.10), (4.11), and $\frac{\operatorname{Ext}_{A(2)_*}}{v_0 - tors}$ of the terms in these exact sequences are determined by the $\frac{\operatorname{Ext}_{A(1)_*}}{v_0 - tors}$ computations of Section 2, and knowledge of

$$\frac{\operatorname{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^k)}{v_0 - tors}.$$

Unfortunately the latter requires explicit computation for each k, and hence does not yield a general answer.

Nevertheless, in this section we will give some lemmas which provide convenient criteria for identifying the i so that given a rational generator $x \in (A//A(2))_*$ (as in the previous section) we have

$$v_0^i x \in \frac{\operatorname{Ext}_{A(2)*}((A//A(2))_*)}{v_0 - tors} \subset v_0^{-1}\operatorname{Ext}_{A(2)*}((A//A(2))_*).$$

We first must clarify what we actually mean by "rational generator". The generators identified in the last section originate from the exact sequences (4.10), (4.15) from the generators of $v_0^{-1}\operatorname{Ext}_{A(2)_*}(M)$ where M is given by

$$\begin{aligned} &\text{Case 1:} \quad M = \underline{\mathrm{bo}}_1^k \\ &\text{Case 2:} \quad M = (A(2)//A(1))_* \otimes \underline{\mathrm{tmf}}_j \end{aligned}$$

In Case 1, the generators x of $v_0^{-1}\operatorname{Ext}_{A(2)_*}(M)$ are generators as a module over $\mathbb{F}_2[v_0^{\pm 1}, [c_4]]$ using the isomorphisms

$$\begin{aligned} & v_0^{-1}\operatorname{Ext}_{A(2)_*}((A(2)//A(1))_* \otimes \underline{\mathrm{tmf}}_j) \\ & \quad \cong v_0^{-1}\operatorname{Ext}_{A(1)_*}(\underline{\mathrm{tmf}}_j) \\ & \quad \cong v_0^{-1}\operatorname{Ext}_{A_*}((A//A(1))_* \otimes \underline{\mathrm{tmf}}_j) \\ & \quad \xrightarrow[\cong]{\alpha} v_0^{-1}\operatorname{Ext}_{A(0)_*}((A//A(1))_* \otimes \underline{\mathrm{tmf}}_j) \\ & \quad \cong v_0^{-1}\operatorname{Ext}_{A(0)_*}((A//A(1))_*) \otimes_{\mathbb{F}_2[v_0^{\pm 1}]} v_0^{-1}\operatorname{Ext}_{A(0)_*}(\underline{\mathrm{tmf}}_j) \\ & \quad \cong \mathbb{F}_2[v_0^{\pm 1}, [c_4]]\{1, \bar{\xi}_1^4\} \otimes_{\mathbb{F}_2} \mathbb{F}_2\{\bar{\xi}_1^{8i_1}\bar{\xi}_2^{4i_2} \ : \ i_1 + i_2 \le j\}. \end{aligned} \tag{4.23}$$

The rational generators in this case correspond to the generators

$$x = \bar{\xi}_1^{4\epsilon} \otimes \bar{\xi}_1^{8i_1}\bar{\xi}_2^{4i_2}.$$

In Case 2, the generators x of $v_0^{-1}\operatorname{Ext}_{A(2)_*}(M)$ are generators as a module over $\mathbb{F}_2[v_0^{\pm 1}, [c_4], [c_6]]$, using the isomorphisms

$$\begin{aligned} & v_0^{-1}\operatorname{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^k) \\ & \quad \cong v_0^{-1}\operatorname{Ext}_{A_*}((A//A(2))_* \otimes \underline{\mathrm{bo}}_1^k) \\ & \quad \xrightarrow[\cong]{\alpha} v_0^{-1}\operatorname{Ext}_{A(0)_*}((A//A(2))_* \otimes \underline{\mathrm{bo}}_1^k) \\ & \quad \cong v_0^{-1}\operatorname{Ext}_{A(0)_*}((A//A(2))_*) \otimes_{\mathbb{F}_2[v_0^{\pm 1}]} v_0^{-1}\operatorname{Ext}_{A(0)_*}(\underline{\mathrm{bo}}_1^k) \\ & \quad \cong \mathbb{F}_2[v_0^{\pm 1}, [c_4], [c_6]] \otimes_{\mathbb{F}_2} \mathbb{F}_2\{1, \bar{\xi}_1^4\}^{\otimes k}. \end{aligned} \tag{4.24}$$

The rational generators in this case correspond to the generators

$$x \in \{1, \bar{\xi}_1^4\}^{\otimes k}.$$

In either case, since the maps α in both (4.23) and (4.24) arise from surjections of cobar complexes

$$C^*_{A_*}(N) \to C^*_{A(0)_*}(N)$$

induced from the surjection

$$A_* \to A(0)_*.$$

Thus a term $v_0^i x \in C^*_{A(0)_*}(N)$ representing an element in $v_0^{-1}\operatorname{Ext}_{A(0)_*}(N)$ corresponds (for i sufficiently large) to a term $[\bar{\xi}_1]^i x + \cdots \in C^*_{A_*}(N)$. Then we have determined an element of the integral lattice

$$\left[[\bar{\xi}_1]^i x + \cdots\right] \in \frac{\operatorname{Ext}_{A_*}(N)}{v_0 - tors} \subset v_0^{-1}\operatorname{Ext}_{A_*}(N).$$

Lemma 4.25. Suppose that the $A(2)_*$-coaction on $x \in (A//A(2))_*$ satisfies

$$\psi(x) = \bar{\xi}_1^4 \otimes y + \text{terms in lower dimension}$$

with y primitive, as in the following "cell diagram":

$$\left.\begin{array}{ll} x & \circ \\ & \\ & \\ y & \circ \end{array}\right\rbrace \mathrm{Sq}^4$$

Then

$$v_0^3 x \in \frac{\mathrm{Ext}_{A(2)*}((A//A(2))_*)}{v_0 - tors} \subset v_0^{-1}\,\mathrm{Ext}_{A(2)*}((A//A(2))_*)$$

and is represented by

$$[\bar{\xi}_1|\bar{\xi}_1|\bar{\xi}_1]x + \left([\bar{\xi}_1|\bar{\xi}_2|\bar{\xi}_2] + [\bar{\xi}_1|\bar{\xi}_1|\bar{\xi}_1^2\bar{\xi}_2] + [\bar{\xi}_1|\bar{\xi}_1\bar{\xi}_2|\bar{\xi}_1^2] + [\bar{\xi}_2|\bar{\xi}_1^2|\bar{\xi}_1^2]\right) y$$

in the cobar complex $C^*_{A(2)_*}((A//A(2))_*)$.

Proof. Since the cell complex depicted agrees with $A(2)//A(1)$ through dimension 4, $\mathrm{Ext}_{A(2)_*}$ of this comodule agrees with $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_2)$ through dimension 4. In particular, $v_0^3 x + \cdots$ generates $\frac{\mathrm{Ext}_{A(2)_*}(-)}{v_0 - tors}$ in this dimension. To determine the exact representing cocycle, we note that

$$[\bar{\xi}_1|\bar{\xi}_2|\bar{\xi}_2] + [\bar{\xi}_1|\bar{\xi}_1|\bar{\xi}_1^2\bar{\xi}_2] + [\bar{\xi}_1|\bar{\xi}_1\bar{\xi}_2|\bar{\xi}_1^2] + [\bar{\xi}_2|\bar{\xi}_1^2|\bar{\xi}_1^2]$$

kills $h_0^3 h_2$ in $\mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$. □

Example 4.26. A typical instance of a set of generators of $(A//A(2))_*$ satisfying the hypotheses of Lemma 4.25 is

$$\left.\begin{array}{ll} \bar{\xi}_i^4\alpha & \circ \\ & \\ & \\ \bar{\xi}_{i-1}^8\alpha & \circ \end{array}\right\rbrace \mathrm{Sq}^4$$

where $\alpha = \bar{\xi}_{i_1}^{8j_1}\bar{\xi}_{i_2}^{8j_2}\cdots$ is a monomial with exponents all divisible by 8.

The following corollary will be essential to relating the integral generators of Lemma 4.25 to 2-variable modular forms in Section 5.

Corollary 4.27. Suppose that x satisfies the hypotheses of Lemma 4.25. Then image of the corresponding integral generator

$$v_0^3 x + \cdots \in \mathrm{Ext}_{A(2)_*}((A//A(2)_*))$$

in $\mathrm{Ext}_{E[Q_0,Q_1,Q_2]_*}((A//E[Q_0,Q_1,Q_2])_*)$ is given by

$$v_0^3 x + v_0[a_1]^2 y.$$

Proof. Note that

$$E[Q_0,Q_1,Q_2]_* = \mathbb{F}_2[\bar{\xi}_1,\bar{\xi}_2,\bar{\xi}_3]/(\bar{\xi}_1^2,\bar{\xi}_2^2,\bar{\xi}_3^2).$$

Therefore the image of the integral generator of Lemma 4.25 under the map

$$C^*_{A(2)_*}((A//A(2))_*) \to C^*_{E[Q_0,Q_1,Q_2]_*}((A//E[Q_0,Q_1,Q_2])_*)$$

is

$$[\bar{\xi}_1|\bar{\xi}_1|\bar{\xi}_1]x + [\bar{\xi}_1|\bar{\xi}_2|\bar{\xi}_2]y$$

and this represents $v_0^3 x + v_0[a_1]^2 y$. $\square$

Similar arguments provide the following slight refinement.

Lemma 4.28. Suppose that the $A(2)_*$-coaction on $x \in (A//A(2))_*$ satisfies

$$\psi(x) = \bar{\xi}_1^4 \otimes y + \text{terms in lower dimension}$$

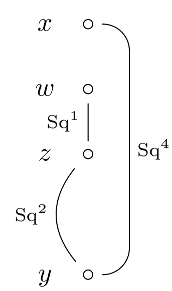
with y primitive, and that there exists w and z satisfying

$$\psi(z) = \bar{\xi}_1^2 y + \text{terms in lower dimension}$$

and

$$\psi(w) = \bar{\xi}_1 z + \bar{\xi}_2 y + \text{terms in lower dimension}$$

as in the following "cell diagram":



Then

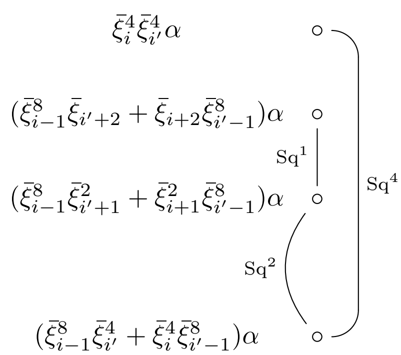
$$v_0 x \in \frac{\mathrm{Ext}_{A(2)*}((A//A(2))_*)}{v_0 - tors} \subset v_0^{-1}\,\mathrm{Ext}_{A(2)*}((A//A(2))_*)$$

and is represented by

$$[\bar{\xi}_1]x + [\bar{\xi}_1^2]w + \left([\bar{\xi}_1^3] + [\bar{\xi}_2]\right) z + [\bar{\xi}_1^2\bar{\xi}_2]y$$

in the cobar complex $C^*_{A(2)_*}((A//A(2))_*)$.

Example 4.29. A typical instance of a set of generators of $(A//A(2))_*$ satisfying the hypotheses of Lemma 4.28 is



where $\alpha = \bar{\xi}_{i_1}^{8j_1} \bar{\xi}_{i_2}^{8j_2} \cdots$ is a monomial with exponents all divisible by 8.

Corollary 4.30. Suppose that x satisfies the hypotheses of Lemma 4.28. Then image of the corresponding integral generator

$$v_0 x + \cdots \in \mathrm{Ext}_{A(2)_*}((A//A(2)_*))$$

in $\mathrm{Ext}_{E[Q_0,Q_1,Q_2]_*}((A//E[Q_0,Q_1,Q_2])_*)$ is given by

$$v_0 x + [a_1]z.$$

5. The image of $\mathrm{tmf}_*\mathrm{tmf}$ in $\mathrm{TMF}_*\mathrm{TMF}_{\mathbb{Q}}$: two variable modular forms

5.1. **Review of Laures' work on cooperations.** For $N > 1$, the spectrum $\mathrm{TMF}_1(N)$ is even periodic, with

$$\mathrm{TMF}_1(N)_{2*} \cong M_*(\Gamma_1(N))[\Delta^{-1}]_{\mathbb{Z}[1/N]}.$$

In particular, its homotopy is torsion-free. As a result, there is an embedding

$$\begin{aligned}\mathrm{TMF}_1(N)_{2*}\mathrm{TMF}_1(N) &\hookrightarrow \mathrm{TMF}_1(N)_{2*}\mathrm{TMF}_1(N)_{\mathbb{Q}} \\ &\cong M_*(\Gamma_1(N))[\Delta^{-1}]_{\mathbb{Q}} \otimes M_*(\Gamma_1(N))[\Delta^{-1}]_{\mathbb{Q}}.\end{aligned}$$

Consider the multivariat q-expansion map

$$M_*(\Gamma_1(N))[\Delta^{-1}]_{\mathbb{Q}} \otimes M_*(\Gamma_1(N))[\Delta^{-1}]_{\mathbb{Q}} \to \mathbb{Q}[q^{\pm 1}, \bar{q}^{\pm 1}].$$

In [Lau99, Thm. 2.10], Laures determines the image of $\mathrm{TMF}_1(N)_*\mathrm{TMF}_1(N)$ under this embedding.

Theorem 5.1 (Laures)**.** The multivariate q-expansion map gives a pullback

$$\begin{tikzcd}
\mathrm{TMF}_1(N)_*\mathrm{TMF}_1(N) \arrow[r] \arrow[d] & \mathrm{TMF}_1(N)_*\mathrm{TMF}_1(N)_{\mathbb{Q}} \arrow[d] \\
\mathbb{Z}[1/N][q^{\pm 1}, \bar{q}^{\pm 1}] \arrow[r] & \mathbb{Q}[q^{\pm 1}, \bar{q}^{\pm 1}]
\end{tikzcd}$$

Therefore, elements of $\mathrm{TMF}_1(N)_*\mathrm{TMF}_1(N)$ are given by sums

$$\sum_i f_i \otimes g_i \in M_*(\Gamma_1(N))[\Delta^{-1}]_{\mathbb{Q}} \otimes M_*(\Gamma_1(N))[\Delta^{-1}]_{\mathbb{Q}}$$

with

$$\sum_i f_i(q) \otimes g_i(q) \in \mathbb{Z}[1/N][q^{\pm 1}, \bar{q}^{\pm 1}].$$

We shall let $M_*^{2-var}(\Gamma_1(N))[\Delta^{-1}, \bar{\Delta}^{-1}]$ denote this ring of integral 2-variable modular forms (meromorphic at the cusps).

Remark 5.2. Laures' methods also apply to the case of $N = 1$ provided 6 is inverted to give an isomorphism

$$\mathrm{TMF}_*\mathrm{TMF}[1/6] \cong M_*^{2-var}(\Gamma(1))[1/6, \Delta^{-1}, \bar{\Delta}^{-1}].$$

5.2. **Representing** $\mathrm{TMF}_*\mathrm{TMF}_{(2)}/tors$ **with** 2**-variable modular forms.** We now turn to adapting Laures' persective to identify $\mathrm{TMF}_*\mathrm{TMF}_{(2)}/tors$. To do this, we use the descent spectral sequence for

$$\mathrm{TMF}_{(2)} \to \mathrm{TMF}_1(3)_{(2)}.$$

Let (B_*, Γ_{B_*}) denote the Hopf algebroid encoding descent from $\mathcal{M}_1(3)$ to $\mathcal{M}$, with

$$\begin{aligned} B_* &= \pi_*\mathrm{TMF}_1(3)_{(2)} = \mathbb{Z}_{(2)}[a_1, a_3, \Delta^{-1}] \\ \Gamma_{B_*} &= \pi_*\mathrm{TMF}_1(3) \wedge_{\mathrm{TMF}} \mathrm{TMF}_1(3)_{(2)} = B_*[r,s,t]/(\sim) \end{aligned}$$

(see Section 3) where $\sim$ denotes the relations (3.5). The Bousfield-Kan spectral sequence associated to the cosimplicial resolution

$$\mathrm{TMF}_{(2)} \to \mathrm{TMF}_1(3)_{(2)} \rightrightarrows \mathrm{TMF}_1(3)^{\wedge_{\mathrm{TMF}}2}_{(2)} \Rrightarrow \mathrm{TMF}_1(3)^{\wedge_{\mathrm{TMF}}3}_{(2)} \cdots$$

yields the descent spectral sequence

$$\mathrm{Ext}^{s,t}_{\Gamma_{B_*}}(B_*) \Rightarrow \pi_{t-s}\mathrm{TMF}_{(2)}.$$

We can use parallel methods to construct a descent spectral sequence for the extension

$$\mathrm{TMF} \wedge \mathrm{TMF}_{(2)} \to \mathrm{TMF}_1(3) \wedge \mathrm{TMF}_1(3)_{(2)}.$$

Let $(B_*^{(2)}, \Gamma_{B_*^{(2)}})$ denote the associated Hopf algebroid encoding descent, with

$$\begin{aligned} B_*^{(2)} &= \pi_*\mathrm{TMF}_1(3) \wedge \mathrm{TMF}_1(3)_{(2)}, \\ \Gamma_{B_*^{(2)}} &= \pi_*(\mathrm{TMF}_1(3)^{\wedge_{\mathrm{TMF}}2} \wedge \mathrm{TMF}_1(3)^{\wedge_{\mathrm{TMF}}2})_{(2)}. \end{aligned}$$

The Bousfield-Kan spectral sequence associated to the cosimplicial resolution

$$\mathrm{TMF}^{\wedge 2}_{(2)} \to \mathrm{TMF}_1(3)^{\wedge 2}_{(2)} \rightrightarrows \left(\mathrm{TMF}_1(3)^{\wedge_{\mathrm{TMF}}2}\right)^{\wedge 2}_{(2)} \Rrightarrow \left(\mathrm{TMF}_1(3)^{\wedge_{\mathrm{TMF}}3}\right)^{\wedge 2} \cdots$$

yields a descent spectral sequence

$$\mathrm{Ext}^{s,t}_{\Gamma_{B_*^{(2)}}}(B_*^{(2)}) \Rightarrow \mathrm{TMF}_{t-s}\mathrm{TMF}_{(2)}.$$

Lemma 5.3. The map induced from the edge homomorphism

$$\mathrm{TMF}_*\mathrm{TMF}_{(2)}/tors \to \mathrm{Ext}^{0,*}_{\Gamma^{(2)}_{B_*}}(B_*^{(2)})$$

is an injection.

Proof. This follows from the fact that the map

$$\mathrm{TMF} \wedge \mathrm{TMF}_{(2)} \to \mathrm{TMF} \wedge \mathrm{TMF}_{\mathbb{Q}}$$

induces a map of descent spectral sequences

$$\begin{tikzcd} \mathrm{Ext}^{s,t}_{\Gamma^{(2)}_{B_*}}(B_*^{(2)}) \arrow[r, Rightarrow] \arrow[d] & \mathrm{TMF}_{t-s}\mathrm{TMF}_{(2)} \arrow[d] \\ \mathrm{Ext}^{s,t}_{\Gamma^{(2)}_{B_*}}(B_*^{(2)} \otimes \mathbb{Q}) \arrow[r, Rightarrow] & \mathrm{TMF}_{t-s}\mathrm{TMF}_{\mathbb{Q}} \end{tikzcd}$$

and the rational spectral sequence is concentrated on the $s = 0$ line. $\square$

The significance of this homomorphism is that the target is the space of 2-integral two variable modular forms for $\Gamma(1)$.

Lemma 5.4. The 0-line of the descent spectral sequence for $\mathrm{TMF}_*\mathrm{TMF}_{(2)}$ may be identified with the space of 2-integral two variable modular forms of level 1 (meromorphic at the cusp):

$$\mathrm{Ext}^{0,2*}_{\Gamma_{B_*}^{(2)}}(B_*^{(2)}) = M_*^{2-var}(\Gamma(1))[\Delta^{-1}, \bar{\Delta}^{-1}]_{(2)}.$$

Proof. This follows from the composition of pullback squares

$$\begin{tikzcd}
\mathrm{Ext}^{0,*}_{\Gamma_{B_*^{(2)}}}(B_*^{(2)}) \arrow[r, hook] \arrow[d, hook] & \mathrm{Ext}^{0,*}_{\Gamma_{B_*^{(2)}}}(B_*^{(2)} \otimes \mathbb{Q}) \arrow[d, hook] \\
\mathrm{TMF}_1(3)_*\mathrm{TMF}_1(3)_{(2)} \arrow[r, hook] \arrow[d] & \mathrm{TMF}_1(3)_*\mathrm{TMF}_1(3)_{\mathbb{Q}} \arrow[d] \\
\mathbb{Z}_{(2)}[q^{\pm 1}, \bar{q}^{\pm 1}] \arrow[r] & \mathbb{Q}[q^{\pm 1}, \bar{q}^{\pm 1}]
\end{tikzcd}$$

The bottom square is a pullback by Theorem 5.1. Note that since $\mathrm{TMF}_1(3) \wedge_{\mathrm{TMF}} \mathrm{TMF}_1(3)$ is Landweber exact, $\Gamma_{B_*^{(2)}}$ is torsion-free. Thus an element of $B_*^{(2)}$ is $\Gamma_{B_*^{(2)}}$-primitive if and only if its image in $B_*^{(2)} \otimes \mathbb{Q}$ is. This shows that the top square is a pullback. □

5.3. **Representing** $\mathrm{tmf}_*\mathrm{tmf}_{(2)}/tors$ **with 2-variable modular forms.** Recall from Section 3 that the Adams filtration of c_4 is 4 and the Adams filtration of c_6 is 5. Regarding 2-variable modular forms as a subring

$$M_*^{2-var}(\Gamma(1))_{(2)} \subset \mathbb{Q}[c_4, c_6, \bar{c}_4, \bar{c}_6]$$

we shall denote $M_*^{2-var}(\Gamma(1))_{(2)}^{AF \geq 0}$ the subring of 2-variable modular forms with non-negative Adams filtration. The results of the previous section now easily give the following result.

Proposition 5.5. The composite induced by Lemmas 5.3 and 5.4

$$\mathrm{tmf}_{2*}\mathrm{tmf}_{(2)}/tors \to \mathrm{TMF}_{2*}\mathrm{TMF}_{(2)}/tors \hookrightarrow M_*^{2-var}(\Gamma(1))[\Delta^{-1}, \bar{\Delta}^{-1}]_{(2)}$$

induces an injection

$$\mathrm{tmf}_{2*}\mathrm{tmf}_{(2)}/tors \hookrightarrow M_*^{2-var}(\Gamma(1))_{(2)}^{AF \geq 0}$$

which is a rational isomorphism.

Proof. Consider the commutative cube

$$
\begin{tikzcd}
\mathrm{tmf}_{2*}\mathrm{tmf}_{(2)}/tors \arrow[rr] \arrow[dd, hook] \arrow[dr, dotted] & & \mathrm{TMF}_{2*}\mathrm{TMF}_{(2)}/tors \arrow[dd] \arrow[dr] & \\
& M_*^{2-var}(\Gamma(1))_{(2)} \arrow[rr] \arrow[dd] & & M_*^{2-var}(\Gamma(1))[\Delta^{-1}, \bar{\Delta}^{-1}]_{(2)} \arrow[dd] \\
\mathrm{tmf}_{2*}\mathrm{tmf}_{\mathbb{Q}} \arrow[rr] \arrow[dr, equal] & & \mathrm{TMF}_{2*}\mathrm{TMF}_{\mathbb{Q}} \arrow[dr, equal] & \\
& M_*^{2-var}(\Gamma(1))_{\mathbb{Q}} \arrow[rr] & & M_*^{2-var}(\Gamma(1))[\Delta^{-1}, \bar{\Delta}^{-1}]_{\mathbb{Q}}
\end{tikzcd}
$$

(The dotted arrow exists because the front face of the cube is a pullback.) The commutativity of the diagram, and the fact that rationally the top face is isomorphic to the bottom face, give an injection

$$\mathrm{tmf}_{2*}\mathrm{tmf}_{(2)}/tors \hookrightarrow M_*^{2-var}(\Gamma(1))_{(2)}$$

which is a rational isomorphism. Since all of the elements of the source have Adams filtration ≥ 0, this injection factors through the subring

$$\mathrm{tmf}_{2*}\mathrm{tmf}_{(2)}/tors \hookrightarrow M_*^{2-var}(\Gamma(1))_{(2)}^{AF\geq 0}.$$

$\square$

5.4. **Detecting 2-variable modular forms in the ASS.**

Definition 5.6. Suppose that we are given a class

$$x \in \mathrm{Ext}(\mathrm{tmf} \wedge \mathrm{tmf})$$

and a 2-variable modular form

$$f \in M_*^{2-var}(\Gamma(1))_{(2)}^{AF\geq 0}.$$

We shall say that x *detects* f if the image of x in $v_0^{-1}\,\mathrm{Ext}(\mathrm{tmf} \wedge \mathrm{tmf})$ detects the image of f in $M_*^{2-var}(\Gamma(1)) \otimes \mathbb{Q}_2$ in the localized ASS

$$v_0^{-1}\,\mathrm{Ext}(\mathrm{tmf} \wedge \mathrm{tmf}) \Rightarrow \mathrm{tmf}_*\mathrm{tmf} \otimes \mathbb{Q}_2 \cong M_*^{2-var}(\Gamma(1)) \otimes \mathbb{Q}_2.$$

Remark 5.7. Suppose x as above is a permanent cycle in the unlocalized ASS

$$\mathrm{Ext}(\mathrm{tmf} \wedge \mathrm{tmf}) \Rightarrow \mathrm{tmf}_*\mathrm{tmf}_2^\wedge,$$

and detects $\zeta \in \mathrm{tmf}_*\mathrm{tmf}_2^\wedge$, and let f be the image of ζ under the map

$$\mathrm{tmf}_*\mathrm{tmf}_2^\wedge \to [M_*^{2-var}(\Gamma(1))_2^\wedge]^{AF\geq 0}.$$

Then x detects f.

Given a 2-variable modular form $f \in M_*^{2-var}(\Gamma(1))_{(2)}$, let $f(a_i, \bar{a}_i)$ denote its image in

$$M_*^{2-var}(\Gamma_0(3)) \otimes \mathbb{Q}_2 \cong \mathbb{F}_2[a_1, a_3, \bar{a}_1, \bar{a}_3] \cong \mathrm{tmf}_1(3)_*\mathrm{tmf}_1(3) \otimes \mathbb{Q}_2,$$

and let

$$[f(a_i, \bar{a}_i)] \in v_0^{-1}\,\mathrm{Ext}(\mathrm{tmf}_1(3) \wedge \mathrm{tmf}_1(3)) \cong \mathbb{F}_2[v_0^{\pm 1}, [a_1], [a_3], [\bar{a}_1], [\bar{a}_3]]$$

denote the element which detects it in the (collapsing) v_0-localized ASS. Similarly, let $t_k(a_i, \bar{a}_i)$ denote the images of t_k in $\mathrm{tmf}_1(3)_*\mathrm{tmf}_1(3) \otimes \mathbb{Q}_2$ (as in Section 3), and let $[t_k(a_i, \bar{a}_i)]$ denote the elements of Ext which detect these images in the v_0-localized ASS for $\mathrm{tmf}_1(3)_*\mathrm{tmf}_1(3) \otimes \mathbb{Q}_2$.

The following key proposition gives a convenient criterion for determining when a particular element $x \in \mathrm{Ext}(\mathrm{tmf} \wedge \mathrm{tmf})$ detects a 2-variable modular form f.

Proposition 5.8. Suppose that we are given a cocycle

$$z = \sum_j z_j \bar{\xi}_1^{2k_{1,j}} \bar{\xi}_2^{2k_{2,j}} \cdots \in C^*_{A(2)_*}((A//A(2))_*)$$

(with $z_j \in C^*_{A(2)_*}(\mathbb{F}_2)$) which represents $[z] \in \mathrm{Ext}(\mathrm{tmf} \wedge \mathrm{tmf})$, and a 2-variable modular form

$$f \in M_*^{2-var}(\Gamma(1))_{(2)}^{AF \geq 0}.$$

The images $\bar{z}_j$ of the terms z_j in the cobar complex $C^*_{E[Q_0,Q_1,Q_2]_*}(\mathbb{F}_2)$ are cycles, which represent classes

$$[\bar{z}_j] \in \mathrm{Ext}_{E[Q_0,Q_1,Q_2]}(\mathbb{F}_2) = \mathbb{F}_2[v_0, [a_1], [a_3]].$$

If we have

$$[f(a_i, \bar{a}_i)] = \sum_j [z_j][t_1(a_i, \bar{a}_i)]^{k_{1,j}}[t_2(a_i, \bar{a}_i)]^{k_{2,j}} \cdots$$

then $[z]$ detects f.

Proof. Let $\bar{z} \in C^*_{E[Q_0,Q_1,Q_2]_*}((A//E[Q_0, Q_1, Q_2])_*)$ denote the image of z. We first note that since the map

$$M_*(\Gamma(1))^{2-var} \otimes \mathbb{Q}_2 = \mathrm{tmf}_*\mathrm{tmf} \otimes \mathbb{Q}_2 \to \mathrm{tmf}_1(3)_*\mathrm{tmf}_1(3) \otimes \mathbb{Q}_2 = M_*(\Gamma_1(3))^{2-var} \otimes \mathbb{Q}_2$$

is injective, and both $\mathrm{tmf} \wedge \mathrm{tmf}$ and $\mathrm{tmf}_1(3) \wedge \mathrm{tmf}_1(3)$ both have collapsing v_0-localized ASS's, with induced map on E_2-terms induced from the map

$$C^*_{A(2)_*}((A//A(2))_*) \to C^*_{E[Q_0,Q_1,Q_2]}((A//E[Q_0, Q_1, Q_2])_*)$$

that $[z]$ detects f if and only if $[\bar{z}]$ detects $f(a_i, \bar{a}_i)$. Thus it suffices to prove the latter.

Note that since the elements

$$\bar{\xi}_1^{2k_{1,j}} \bar{\xi}_2^{2k_{2,j}} \cdots \in (A//E[Q_0, Q_1, Q_2])_*$$

are $E[Q_0, Q_1, Q_2]_*$-primitive, it follows from the fact that z is a cocycle that the elements $\bar{z}_j$ are cocycles. The only thing left to check is that

$$[\bar{\xi}_1^{2k_{1,j}} \bar{\xi}_2^{2k_{2,j}} \cdots] = [t_1(a_i, \bar{a}_i)]^{k_{1,j}}[t_2(a_i, \bar{a}_i)]^{k_{2,j}} \cdots$$

in $\mathrm{Ext}_{E[Q_0,Q_1,Q_2]_*}((A//E[Q_0, Q_1, Q_2])_*)$. But this follows from the commutative diagram

$$\begin{tikzcd} BP_*BP \arrow[rr] \arrow[dr] & & H_*H \\ & H_*\mathrm{tmf}_1(3) \arrow[ur, hook] & \end{tikzcd}$$

together with the fact that under the top map, t_k is mapped to $\bar{\xi}_k^2$. $\square$

5.5. **Low dimensional computations of 2-variable modular forms.** Below is a table of generators of $\mathrm{Ext}(\mathrm{tmf} \wedge \mathrm{tmf})/tors$, as a module over $\mathbb{F}_2[h_0, [c_4]]$, through dimension 64, with 2-variable modular forms they detect. The columns of this table are:

- **dim:** dimension of the generator,
- **bo$_k$:** indicates generator lies in the summand $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_k)$ (see the charts in Section 4),
- **AF:** the Adams filtration of the generator,
- **cell:** the name of the image of the generator in $v_0^{-1}\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_k)$, in the sense of Section 4.4,
- **form:** a two-variable modular form which is detected by the generator in the v_0-localized ASS (where f_k are defined below).

The table below also gives a basis of $M_*^{2-var}(\Gamma(1))_{(2)}$ as a $\mathbb{Z}_{(2)}[c_4]$-module: in dimension $2k$, a form αg in the last column, with $\alpha \in \mathbb{Q}$ and g a monomial in $\mathbb{Z}[c_4, c_6, \Delta, f_k]$ not divisible by 2, corresponds to a generator g of $M_k^{2-var}(\Gamma(1))_{(2)}$.

TABLE 1. Table of generators of $\mathrm{Ext}(\mathrm{tmf} \wedge \mathrm{tmf})/tors$.

dim	bo$_k$	AF	cell	form
8	1	0	$\bar{\xi}_1^8$	f_1
12	1	3	$[8]\bar{\xi}_2^4$	$2f_2$
16	2	0	$\bar{\xi}_1^{16}$	f_1^2
20	1	3	$[c_6/4] \cdot \bar{\xi}_1^8$	$2f_3$
20	2	3	$[8]\bar{\xi}_1^8\bar{\xi}_2^4$	$2f_1f_2$
24	1	4	$[c_6/2] \cdot \bar{\xi}_2^4$	f_4
24	2	0	$\bar{\xi}_2^8$	f_5
24	3	0	$\bar{\xi}_1^{24}$	f_1^3
28	2	3	$[8]\bar{\xi}_3^4$	$2f_6$
28	3	3	$[8]\bar{\xi}_1^{16}\bar{\xi}_2^4$	$2f_1^2f_2$
32	1	4	$[\Delta]\bar{\xi}_1^8$	Δf_1
32	2	1	$[c_6/16] \cdot \bar{\xi}_1^8\bar{\xi}_2^4 + [c_4/8] \cdot \bar{\xi}_2^8$	f_9
32	3	0	$\bar{\xi}_1^8\bar{\xi}_2^8$	f_1f_5
32	4	0	$\bar{\xi}_1^{32}$	f_1^4
36	1	7	$[8\Delta]\bar{\xi}_2^4$	$2\Delta f_2$
36	2	3	$[c_6/4] \cdot \bar{\xi}_2^8$	$2f_7$
36	3	3	$[8]\bar{\xi}_2^{12}$	$2f_2f_5$
36	3	0	$\bar{\xi}_1^8\bar{\xi}_3^4 + \bar{\xi}_2^{12}$	f_{10}

36	4	3	$[8]\bar{\xi}_1^{24}\bar{\xi}_2^4$	$2f_1^3f_2$
40	2	4	$[c_6/2]\cdot\bar{\xi}_3^4$	f_8
40	3	1	$[2]\bar{\xi}_2^4\bar{\xi}_3^4$	f_{11}
40	4	0	$\bar{\xi}_1^{16}\bar{\xi}_2^8$	$f_1^2f_5$
40	5	0	$\bar{\xi}_1^{20}$	f_1^5
44	1	7	$[\Delta c_6/4]\cdot\bar{\xi}_1^8$	$2\Delta f_3$
44	2	7	$[c_6/4]([c_6/16]\cdot\bar{\xi}_1^8\bar{\xi}_2^4+[c_4/8]\cdot\bar{\xi}_2^8)$	$c_6f_9/4$
44	3	3	$[c_6/4]\cdot\bar{\xi}_1^8\bar{\xi}_2^8$	$2f_1f_7$
44	4	3	$[8]\bar{\xi}_1^8\bar{\xi}_2^{12}$	$2f_1f_2f_5$
44	4	0	$\bar{\xi}_1^{16}\bar{\xi}_3^4+\bar{\xi}_1^8\bar{\xi}_2^{12}$	$2f_{13}$
44	5	3	$[8]\bar{\xi}_1^{32}\bar{\xi}_2^4$	$2f_1^4f_2$
48	1	8	$[\Delta c_6/2]\cdot\bar{\xi}_2^4$	Δf_4
48	2	4	$[\Delta]\bar{\xi}_2^8$	Δf_5
48	3	4	$[c_6/2]\cdot\bar{\xi}_2^{12}$	f_2f_7
48	3	1	$[c_6/16]\cdot(\bar{\xi}_1^8\bar{\xi}_3^4+\bar{\xi}_2^{12})$	f_{14}
48	4	0	$\bar{\xi}_2^{16}$	f_5^2
48	4	1	$[2]\bar{\xi}_1^8\bar{\xi}_2^4\bar{\xi}_3^4$	f_1f_{11}
48	5	0	$\bar{\xi}_1^{24}\bar{\xi}_2^8$	$f_1^3f_5$
48	6	0	$\bar{\xi}_1^{48}$	f_1^6
52	2	7	$[8\Delta]\bar{\xi}_3^4$	$2\Delta f_6$
52	3	4	$[c_6/2]\cdot\bar{\xi}_2^4\bar{\xi}_3^4$	$2f_{15}$
52	4	3	$[8]\bar{\xi}_2^8\bar{\xi}_3^4$	$2f_5f_6$
52	5	3	$[8]\bar{\xi}_1^{16}\bar{\xi}_2^{12}$	$2f_1^2f_2f_5$
52	5	0	$\bar{\xi}_1^{24}\bar{\xi}_3^4+\bar{\xi}_1^{16}\bar{\xi}_2^{12}$	$2f_1f_{13}$
52	6	3	$[8]\bar{\xi}_1^{40}\bar{\xi}_2^4$	$2f_1^5f_2$
56	1	8	$[\Delta^2\bar{\xi}_1^8$	Δ^2f_1
56	2	8	$[\Delta]([c_6/2]\cdot\bar{\xi}_1^8\bar{\xi}_2^4+[c_4]\cdot\bar{\xi}_2^8)$	$8\Delta f_9$
56	3	4	$[\Delta]\bar{\xi}_1^8\bar{\xi}_2^8$	Δf_5f_1
56	4	1	$[c_6/16]\cdot\bar{\xi}_1^8\bar{\xi}_2^{12}+[c_4/8]\cdot\bar{\xi}_2^{16}$	f_5f_9
56	4	0	$\bar{\xi}_3^8$	f_{16}
56	5	0	$\bar{\xi}_1^8\bar{\xi}_2^{16}$	$f_1f_5^2$
56	5	1	$[2]\bar{\xi}_1^{16}\bar{\xi}_2^4\bar{\xi}_3^4$	$f_1^2f_{11}$
56	6	0	$\bar{\xi}_1^{32}\bar{\xi}_2^8$	$f_1^4f_5$
60	1	11	$[8\Delta^2]\cdot\bar{\xi}_2^4$	$2\Delta^2f_2$
60	2	7	$[\Delta c_6/4]\cdot\bar{\xi}_2^8$	$2\Delta f_7$

60	3	7	$[8\Delta]\bar{\xi}_2^{12}$	$2\Delta f_5 f_2$
60	3	4	$[\Delta](\bar{\xi}_1^8\bar{\xi}_3^4 + \bar{\xi}_2^{12})$	Δf_{10}
60	4	4	$[c_6/2]\cdot\bar{\xi}_1^8\bar{\xi}_2^4\bar{\xi}_3^4 + [c_4]\cdot\bar{\xi}_2^8\bar{\xi}_3^4$	$2f_6f_9$
60	4	3	$[8]\bar{\xi}_4^4$	$2f_{17}$
60	5	0	$\bar{\xi}_2^{20} + \bar{\xi}_1^8\bar{\xi}_2^8\bar{\xi}_3^4$	f_{18}
60	5	3	$[8]\bar{\xi}_1^8\bar{\xi}_2^8\bar{\xi}_3^4$	$2f_1f_5f_6$
60	6	3	$[8]\bar{\xi}_1^{24}\bar{\xi}_2^{12}$	$2f_1^3f_2f_5$
60	6	0	$\bar{\xi}_1^{32}\bar{\xi}_3^4$	$2f_1^2f_{13}$
60	7	3	$[8]\bar{\xi}_1^{48}\bar{\xi}_2^4$	$2f_1^6f_2$
64	2	8	$[\Delta c_6/2]\cdot\bar{\xi}_3^4$	Δf_8
64	3	5	$[2\Delta]\bar{\xi}_2^4\bar{\xi}_3^4$	Δf_{11}
64	4	2	$[c_6/16]\cdot\bar{\xi}_2^8\bar{\xi}_3^4 + [c_4/8]\cdot\bar{\xi}_3^8$	$f_9^{2/2}$
64	5	1	$[2]\bar{\xi}_2^{12}]\bar{\xi}_3^4$	$f_1f_5f_9$
64	5	0	$\bar{\xi}_1^8\bar{\xi}_3^8$	f_1f_{16}
64	6	0	$\bar{\xi}_1^{16}\bar{\xi}_2^{16}$	$f_5^2f_1^2$
64	6	1	$[2]\bar{\xi}_1^{24}\bar{\xi}_2^4\bar{\xi}_3^4$	$f_{11}f_1^3$
64	7	0	$\bar{\xi}_1^{40}\bar{\xi}_2^8$	$f_1^5f_5$
64	8	0	$\bar{\xi}_1^{64}$	f_1^8

The 2-variable modular forms $f_k \in M_*^{2-var}(\Gamma(1))_{(2)}$ in the above table are the generators of $M_*^{2-var}(\Gamma(1))_{(2)}$ as an $M_*(\Gamma(1))_{(2)}$-algebra in this range, and are defined as follows.

$$
\begin{aligned}
f_1 &:= (-\bar{c}_4 + c_4)/16\\
f_2 &:= (-\bar{c}_6 + c_6)/8\\
f_3 &:= (5f_1c_6 + 21f_2c_4)/8\\
f_4 &:= (5f_2c_6 + 21f_1c_4^2)/8\\
f_5 &:= (-f_1^2c_4 + f_2^2)/16\\
f_6 &:= (-c_4^2c_6 + c_4^2c_6 + 544f_2c_4^2 + 768f_3c_4 + 1792f_1f_2c_4)/2048\\
f_7 &:= (4f_2\Delta + f_5c_6 + 5f_2c_4^3 + 6f_3c_4^2 + 5f_1f_2c_4^2 + 7f_6c_4 + 4f_1^2f_2c_4)/8\\
f_8 &:= (4f_1c_4\Delta + f_6c_6 + 5f_1c_4^4 + 5f_1^2c_4^3 + 7f_5c_4^2 + 2f_4c_4^2 + 4f_1^3c_4^2)/8\\
f_9 &:= (32f_1\Delta + f_1f_2c_6 + 33f_1^2c_4^2 + 8f_5c_4 + 32f_4c_4 + 32f_1^3c_4)/64\\
f_{10} &:= (2f_2c_4^3 + f_1f_2c_4^2 + 2f_6c_4 + 3f_1^2f_2c_4 + f_1f_6 + f_2f_5)/4\\
f_{11} &:= (4f_1c_4\Delta + 11f_1^2c_4^3 + 34f_5c_4^2 + 28f_4c_4^2 + 23f_1^3c_4^2 + 4f_9c_4 + f_1f_5c_4 + 4f_1^4c_4\\
&\qquad + 4f_8 + f_2f_6)/8\\
f_{12} &:= (f_1f_5c_6 + 8f_2c_4^4 + 8f_3c_4^3 + 8f_1f_2c_4^3 + 8f_6c_4^2 + 8f_1^2f_2c_4^2 + f_2f_5c_4)/8
\end{aligned}
$$

$$\begin{aligned}
f_{13} := {} & (8f_3\Delta + 80f_2c_4^4 + 56f_3c_4^3 + 80f_1f_2c_4^3 + 76f_6c_4^2 + 55f_1^2f_2c_4^2 + 4f_{10}c_4 \\
& + 18f_2f_5c_4 + 11f_1^3f_2c_4 + 4f_{12} + f_1^2f_6 + f_1f_2f_5 + 4f_1^4f_2)/8 \\
f_{14} := {} & (21f_1c_4^2\Delta + 8f_5\Delta + 16f_4\Delta + 20f_1^3\Delta + f_{10}c_6 + 11f_1c_4^5 + 36f_1^2c_4^4 + 591f_5c_4^3 \\
& + 490f_4c_4^3 + 437f_1^3c_4^3 + 119f_9c_4^2 + 140f_1f_5c_4^2 + 75f_1^4c_4^2 + 10f_{11}c_4 + 11f_8c_4 \\
& + 32f_1^5c_4 + 8f_1f_2f_6)/16 \\
f_{15} := {} & (4f_6\Delta + f_1^2f_2\Delta + 76f_2c_4^5 + 54f_3c_4^4 + 90f_1f_2c_4^4 + 73f_6c_4^3 + 50f_1^2f_2c_4^3 + 3f_{10}c_4^2 \\
& + 8f_7c_4^2 + 20f_2f_5c_4^2 + 8f_1^3f_2c_4^2 + 7f_{12}c_4 + 4f_1f_2f_5c_4)/8 \\
f_{16} := {} & (2f_1\Delta^2 + 24f_1c_4^3\Delta + 9f_5c_4\Delta + 18f_4c_4\Delta + 4f_1^3c_4\Delta + 2f_9\Delta + f_1f_5\Delta \\
& + 36f_1^2c_4^5 + 480f_5c_4^4 + 402f_4c_4^4 + 359f_1^3c_4^4 + 94f_9c_4^3 + 112f_1f_5c_4^3 + 55f_1^4c_4^3 \\
& + 12f_{11}c_4^2 + 14f_8c_4^2 + 20f_1^5c_4^2 + 2f_{14}c_4 + 5f_2f_7c_4 + f_5^2c_4 + 4f_1^3f_5c_4 + f_1f_{14} \\
& + f_5f_9 + f_1f_2f_7)/2 \\
f_{17} := {} & (2f_2\Delta^2 + 22f_3c_4^2\Delta + 11f_6c_4\Delta + f_2f_5\Delta + 19f_9c_4^2c_6 + 682f_2c_4^6 + 480f_3c_4^5 \\
& + 768f_1f_2c_4^5 + 648f_6c_4^4 + 462f_1^2f_2c_4^4 + 30f_{10}c_4^3 + 63f_7c_4^3 + 185f_2f_5c_4^3 \\
& + 84f_1^3f_2c_4^3 + 12f_{13}c_4^2 + 27f_{12}c_4^2 + 29f_1f_2f_5c_4^2 + 16f_1^4f_2c_4^2 + 4f_{15}c_4 + 4f_5f_6c_4 \\
& + 2f_1^2f_2f_5c_4 + f_2f_{14} + f_6f_9)/2 \\
f_{18} := {} & (4f_2\Delta^2 + 168f_3c_4^2\Delta + 96f_6c_4\Delta + 8f_2f_5\Delta + 168f_9c_4^2c_6 + 5880f_2c_4^6 \\
& + 4140f_3c_4^5 + 6648f_1f_2c_4^5 + 5592f_6c_4^4 + 3980f_1^2f_2c_4^4 + 248f_{10}c_4^3 + 560f_7c_4^3 \\
& + 1586f_2f_5c_4^3 + 744f_1^3f_2c_4^3 + 112f_{13}c_4^2 + 220f_{12}c_4^2 + 265f_1f_2f_5c_4^2 \\
& + 136f_1^4f_2c_4^2 + 40f_{15}c_4 + 4f_1f_{13}c_4 + 34f_5f_6c_4 + 19f_1^2f_2f_5c_4 + 8f_1^5f_2c_4 \\
& + 4f_6f_9 + f_1f_5f_6 + f_2f_5^2)/4
\end{aligned}$$

We shall now indicate the methods used to generate Table 1, and make some remarks about its contents.

The short exact sequences of Section 4.3 give an inductive scheme for computing $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_k)$, and charts in that section display the computation through dimension 64. In Section 4.4, these short exact sequences are used to give an inductive scheme for identifying the generators of $v_0^{-1}\,\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_k)$, and appropriate multiples of these generators generate the image of $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_k)/tors$ in these localized Ext groups. These generators are listed in the fourth column of Table 1.

The two variable modular forms in the last column of Table 1 are detected by the generators in the fourth column, in the sense of the previous section. In each instance, if necessary, we use Corollary 4.27 or 4.30 to find the image of the generator in $\mathrm{Ext}(\mathrm{tmf}_1(3) \wedge \mathrm{tmf}_1(3))$ and then apply Proposition 5.8.

The 2-variable modular forms were generated by the following inductive method. Suppose inductively that we have generated a basis of $M_*^{2-var}(\Gamma(1))_{(2)}$ in dimension n and Adams filtration greater than s and suppose that we wish to generate a 2-variable modular form f in dimension n and Adams filtration s.

Step 1: Write an approximation (modulo higher Adams filtration) for f. This could either be generated using Proposition prop:detection, or it could be obtained by taking an appropriate product of 2-variable modular forms in lower degrees. Write this approximation as $g(q,\bar{q})/2^k$ where $g(q,\bar{q})$ is a 2-integral 2-variable modular form.

Step 2: Write $g(q,\bar{q})$ as a linear combination of 2-variable modular forms already produced mod 2:

$$g(q,\bar{q}) = \sum_i h_i(q,\bar{q}).$$

Step 3: Setting

$$g'(q,\bar{q}) = \frac{g(q,\bar{q}) + \sum_i h_i(q,\bar{q})}{2}.$$

the form $g'(q,\bar{q})/2^{k-1}$ is a better approximation for f.

Step 4: Repeat steps 2 and 3 until the denominator is completely eliminated.

We explain all of this by working it through some low degrees:

$\mathbf{f_1}$**:** The corresponding generator of $\mathrm{Ext}^{0,8}_{A(2)_*}(\Sigma^8\underline{\mathrm{bo}}_1)$ is $\bar{\xi}_1^8$. We compute

$$[t_1(a_i,\bar{a}_i)^4] = \left[\frac{\bar{a}_1^4 + a_1^4}{2^4}\right] = [\frac{-\bar{c}_4 + c_4}{2^4}].$$

We check that

$$f_1 := \frac{-\bar{c}_4 + c_4}{2^4}$$

has an integral q-expansion.

$\mathbf{2f_2}$**:** The corresponding generator of $\mathrm{Ext}^{3,15}_{A(2)_*}(\Sigma^8\underline{\mathrm{bo}}_1)$ is $[8]\bar{\xi}_2^4$. We compute (appealing to Corollary 4.27)

$$[8t_2(a_i,\bar{a}_i)^2 + 2a_1^2 t_1(a_i,\bar{a}_i)^4] = \left[2\bar{a}_3^2 + 2a_3^2\right] = [\frac{-\bar{c}_6 + c_6}{4}].$$

We check that $\frac{-\bar{c}_6+c_6}{4}$ has integral q-expansion. In fact the q-expansion is zero mod 2, so we set

$$f_2 := \frac{-\bar{c}_6 + c_6}{8}.$$

$\mathbf{f_1^2}$**:** The corresponding generator of $\mathrm{Ext}^{0,16}_{A(2)_*}(\Sigma^{16}\underline{\mathrm{bo}}_2)$ is $\bar{\xi}_1^{16}$. Since $\bar{\xi}_1^8$ detects f_1, $\bar{\xi}_1^{16}$ detects f_1^2.

$\mathbf{2f_1f_2}$**:** The corresponding generator of $\mathrm{Ext}^{3,23}_{A(2)_*}(\Sigma^{16}\underline{\mathrm{bo}}_2)$ is $\bar{\xi}_1^8\bar{\xi}_2^4$. Since $\bar{\xi}_1^8$ detects f_1 and $[8]\bar{\xi}_2^4$ detects $2f_2$, $[8]\bar{\xi}_1^8\bar{\xi}_2^4$ detects $2f_1f_2$.

$\mathbf{2f_3}$**:** The corresponding generator of $\mathrm{Ext}^{3,23}_{A(2)_*}(\Sigma^8\underline{\mathrm{bo}}_1)$ is $[c_6/4]\bar{\xi}_1^8$. Since $\bar{\xi}_1^8$ detects f_1, we begin with a leading term $c_6f_1/4$. This 2-variable modular form is not integral, but we find that

$$c_6(q)f_1(q,\bar{q}) + f_2(q,\bar{q})c_4(q) \equiv 0 \mod 4.$$

Therefore $[c_6/4]\bar{\xi}_1^8$ detects

$$\frac{c_6f_1 + f_2c_4}{4}.$$

In fact

$$5c_6(q)f_1(q,\bar{q}) + 21f_2(q,\bar{q})c_4(q) \equiv 0 \mod 8,$$

so we set

$$f_3 := \frac{5c_6f_1 + 21f_2c_4}{8}.$$

6. Approximating by level structures

Recall from §3 the maps

$$\Psi_n : \mathrm{TMF}[1/n] \wedge \mathrm{TMF}[1/n] \to \mathrm{TMF}_0(n)$$

and

$$\phi_{[n]} : \mathrm{TMF} \wedge \mathrm{TMF}[1/n] \to \mathrm{TMF} \wedge \mathrm{TMF}[1/n].$$

Here Ψ_n is induced by the forgetful and quotient maps $f, q : \mathcal{M}_0(n) \to \mathcal{M}[1/n]$, while $\phi_{[n]} = 1 \wedge [n]$ where $[n] : \mathrm{TMF}[1/n] \to \mathrm{TMF}[1/n]$ is the "Adams operation" associated to the multiplication by n isogeny on $\mathcal{M}[1/n]$. For reasons motivated by the conjectural $K(2)$-local behavior of these objects, we are interested in the composite map Ψ given as

$$\begin{tikzcd} \mathrm{tmf} \wedge \mathrm{tmf}_{(2)} \arrow[r, "\Psi"] \arrow[d] & \prod_{i\in\mathbb{Z}, j\geq 0} \mathrm{TMF}_0(3^j) \times \mathrm{TMF}_0(5^j) \\ \mathrm{TMF} \wedge \mathrm{TMF}_{(2)} \arrow[ur, "\psi"'] & \end{tikzcd}$$

where

$$\psi = \prod_{i\in\mathbb{Z}, j\geq 0} \Psi_{3^j}\phi_{[3^i]} \times \Psi_{5^j}\phi_{[5^i]}.$$

We will abuse notation and refer to the composite

$$\mathrm{tmf} \wedge \mathrm{tmf}_{(2)} \to \mathrm{TMF} \wedge \mathrm{TMF}_{(2)} \xrightarrow{\Psi_n} \mathrm{TMF}_0(n)$$

(for $(2, n) = 1$) as Ψ_n as well; these are the $i = 0$ factors of Ψ.

In order to study Ψ_n we consider the square

$$\begin{tikzcd} \mathrm{tmf}_*\mathrm{tmf}_{(2)} \arrow[r, "\pi_*\Psi_n"] \arrow[d] & \pi_*\mathrm{TMF}_0(n) \arrow[d] \\ M_*^{2-var}(\Gamma(1))_{(2)} \arrow[r, "\psi_n"'] & M_*(\Gamma_0(n)). \end{tikzcd}$$

Here the left-hand vertical map is the composite

$$\mathrm{tmf}_*\mathrm{tmf}_{(2)} \to \mathrm{tmf}_*\mathrm{tmf}_{(2)}/tors \hookrightarrow M_*^{2-var}(\Gamma(1))_{(2)}^{AF\geq 0} \hookrightarrow M_*^{2-var}(\Gamma(1))_{(2)},$$

and $M_*(\Gamma_0(n))$ is the ring of level $\Gamma_0(n)$-modular forms. The bottom horizontal map is also induced by f and q; if we consider a 2-variable modlar form as a polynomial $p(c_4, c_6, \bar{c}_4, \bar{c}_6)$, then $\psi_n(p) = p(f^*c_4, f^*c_6, q^*c_4, q^*c_6)$.

We are especially interested in the cases $n = 3, 5$. Recall from [MR09] (or [BO, §3.3]) that $M_*(\Gamma_0(3))$ has a convenient presentation as a subalgebra of $M_*(\Gamma_1(3))$. Indeed,

$M_*(\Gamma_1(3))_{(2)} = \mathbb{Z}_{(2)}[a_1, a_3, \Delta^{-1}]$ where $\Delta = a_3^3(a_1^3 - 27a_3)$, and $M_*(\Gamma_0(3))_{(2)}$ is the subring generated by a_1^2, a_1a_3, a_3^2, *i.e.*,

$$M_*(\Gamma_0(3))_{(2)} = \mathbb{Z}_{(2)}[a_1^2, a_1a_3, a_3^2, \Delta^{-1}].$$

Using the formulas from *loc. cit.*, we may compute

$$\begin{aligned} f^*(c_4) &= a_1^4 - 24a_1a_3, & q^*(c_4) &= a_1^4 + 216a_1a_3, \\ f^*(c_6) &= -a_1^6 + 36a_1^3a_3 - 216a_3^2, & q^*(c_6) &= -a_1^6 + 540a_1^3a_3 + 5832a_3^2. \end{aligned}$$

There are similar formulas for the $n = 5$ case which we recall from [BO, §3.4]. Here the ring of $\Gamma_0(5)$-modular forms takes the form

$$M_*(\Gamma_0(5))_{(2)} = \mathbb{Z}_{(2)}[b_2, b_4, \delta, \Delta^{-1}]/(b_4^2 = b_2^2\delta - 4\delta^2)$$

where $|b_2| = 2$ and $|b_4| = |\delta| = 4$. (These are the algebraic, rather than topological, degrees.) The discriminant takes the form

$$\Delta = 11\delta^3 + \delta^2 b_4,$$

and we have

$$\begin{aligned} f^*(c_4) &= b_2^2 - 12b_4 + 12\delta, & q^*(c_4) &= b_2^2 + 228b_4 + 492\delta, \\ f^*(c_6) &= -b_2^3 + 18b_2b_4 - 72b_2\delta, & q^*(c_6) &= -b_2^3 + 522b_2b_4 + 10008b_2\delta. \end{aligned}$$

7. Connective covers of $\mathrm{TMF}_0(3)$ and $\mathrm{TMF}_0(5)$

7.1. Motivation from $K(2)$-local theory.

7.2. Computation of Ψ_3 and Ψ_5 in low degrees.

7.3. Using level structures to detect differentials and hidden extensions in the ASS.

7.4. Connective covers of $\mathrm{TMF}_0(3)$ and $\mathrm{TMF}_0(5)$ in the tmf-resolution.

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