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A new proof of the Bott periodicity theorem

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Abstract

We give a simplification of the proof of the Bott periodicity theorem presented by Aguilar and Prieto. These methods are extended to provide a new proof of the real Bott periodicity theorem. The loop spaces of the groups O and U are identified by considering the fibers of explicit quasifibrations with contractible total spaces. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In [1], Aguilar and Prieto gave a new proof of the complex Bott periodicity theorem based on ideas of McDuff [4]. The idea of the proof is to use an appropriate restriction of the exponential map to construct an explicit quasifibration with base space U and contractible total space. The fiber of this map is seen to be $BU \times \mathbb{Z}$. This proof is compelling because it is more elementary and simpler than previous proofs. In this paper we present a streamlined version of the proof by Aguilar and Prieto, which is simplified by the introduction of coordinate free vector space notation and a more convenient filtration for application of the Dold–Thom theorem. These methods are then extended to prove the real Bott periodicity theorem.

2. Preliminaries

We shall review the necessary facts about quasifibrations that will be used in the proof of the Bott periodicity theorem, as well as prove a technical result on the behavior of the

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classical groups under linear isometries. The latter will be essential for our applications of the Dold–Thom theorem. A surjective map $p: X \to Y$ is a quasifibration if for every $y \in Y$ and $x \in p^{-1}(y)$, the natural map

$$\pi_i\bigl(X, p^{-1}(y); x\bigr) \to \pi_i(Y; y)$$

is an isomorphism for every i. It follows immediately that if F is a fiber of p, then there is a long exact homotopy sequence associated to p.

If X is contractible, we obtain a map of the quasifibration sequence to the path space fibration.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[d] \\ \Omega Y \arrow[r] & PY \arrow[r] & Y \end{tikzcd}$$

It follows from the long exact homotopy sequences and the five lemma that $F \simeq \Omega Y$.

The definition of a quasifibration does not lend itself to easy verification. The following theorem of Dold and Thom [2] gives a more practical program. Recall that for a map $p: X \to Y$, a subset $S \subseteq Y$ is said to be *distinguished* if for every open $U \subseteq S$, the map $p^{-1}(U) \to U$ is a quasifibration.

Theorem 2.1. *Suppose $p: X \to Y$ is a surjection. Suppose that X is endowed with an increasing filtration $\{F_i Y\}$, such that the following conditions hold.*

(1) *$F_n Y - F_{n-1} Y$ is distinguished for every n.*

(2) *For every n there exists a neighborhood N of $F_{n-1}Y$ in $F_n Y$ and a deformation $h: N \times I \to N$ such that $h_0 = \mathrm{Id}$ and $h_1(N) \subseteq F_{n-1}Y$.*

(3) *This deformation is covered by a deformation $H: p^{-1}(N) \times I \to p^{-1}(N)$ such that $H_0 = \mathrm{Id}$, and for every $y \in N$, the induced map*

$$H_1: p^{-1}(y) \to p^{-1}\bigl(h_1(y)\bigr)$$

is a weak homotopy equivalence.

Then p is a quasifibration.

Let Λ be $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and let $\mathcal{I}(W, V)$ denote the space of linear isometries from W to V, where W and V are (possibly countably infinite dimensional) inner product spaces over Λ topologized as the unions of their finite dimensional subspaces. Let $G(W)$ be $O(W)$, $U(W)$, or $Sp(W)$, where $G(W)$ is the space finite type linear automorphisms of W. We define a continuous map

$$\Gamma_{W,V}: \mathcal{I}(W, V) \to \mathrm{Map}\bigl(G(W), G(V)\bigr).$$

Writing $\Gamma_{W,V}(\phi) = \phi_*$, if $X \in G(W)$, then $\phi_*(X): V \to V$ is defined by

$$\phi_*(X) = \phi X \phi^{-1} \oplus I_{\phi(W)^\perp}$$

under the orthogonal decomposition $V = \phi(W) \oplus \phi(W)^{\perp}$. Let $\mathcal{U}$ and $\mathcal{V}$ be countably infinite dimensional Λ inner product spaces. In [3, II.1.5] it is proven that $\mathcal{I}(\mathcal{U}, \mathcal{V})$ is contractible. So we have the following lemmas.

Lemma 2.2. *Let $\phi, \phi' \in \mathcal{I}(\mathcal{U}, \mathcal{V})$. Then the induced maps*

$$\phi_*, \phi'_* : G(\mathcal{U}) \to G(\mathcal{V})$$

are homotopic.

Lemma 2.3. *Let $\phi \in \mathcal{I}(\mathcal{U}, \mathcal{V})$. Then ϕ_* is a homotopy equivalence.*

3. Complex Bott periodicity

The existence of the fiber sequence

$$U \to EU \to BU$$

yields that $\Omega BU \simeq U$. We aim to prove the following theorem, which implies that $\Omega^2 BU \simeq BU \times \mathbb{Z}$.

Theorem 3.1. *Let U denote the infinite unitary group. There exists a quasifibration sequence*

$$BU \times \mathbb{Z} \to E \to U$$

such that E is contractible. Consequently, $\Omega U \simeq BU \times \mathbb{Z}$.

Fix a complex infinite dimensional inner product space $\mathcal{U} \cong \mathbb{C}^{\infty}$. For $W \subset \mathcal{U}$, a finite dimensional complex subspace, let $U(W \oplus W)$ denote complex linear isometries of $W \oplus W$. If $V \subseteq W$, then there is a natural map $i_{V,W} : U(V \oplus V) \to U(W \oplus W)$ given by

$$i_{V,W}(X) = X \oplus I_{(W-V)\oplus(W-V)},$$

where $W - V$ denotes the orthogonal complement of V in W. Then

$$U = \varinjlim_W U(W \oplus W),$$

where the colimit is taken over all finite dimensional subspaces $W \subset \mathcal{U}$.

Let $H(W \oplus W)$ denote the hermitian linear transformations of $W \oplus W$. Define

$$E(W) = \bigl\{A \mid \sigma(A) \subseteq I = [0,1]\bigr\} \subseteq H(W \oplus W),$$

where $\sigma(A)$ is the spectrum of A. Define

$$p_W : E(W) \to U(W \oplus W)$$

by $p_W(A) = \exp(2\pi \mathrm{i} A)$. Analogous to U, define a map $E(V) \to E(W)$ for $V \subseteq W$ by sending A to $A \oplus \pi_{(W-V)\oplus 0}$. Here, π_Y denotes orthogonal projection onto the subspace Y.

It will become apparent that this map is defined so that, upon stabilization, the fibers are $BU \times \mathbb{Z}$. Then the following diagram commutes, since $\mathrm{e}^{2\pi\mathrm{i}} = \mathrm{e}^0$.

$$\begin{tikzcd} E(V) \arrow[r] \arrow[d, "p_V"'] & E(W) \arrow[d, "p_W"] \\ U(V \oplus V) \arrow[r] & U(W \oplus W) \end{tikzcd}$$

So taking colimits we obtain

$$p : E \to U,$$

where $E = \varinjlim_W E(W)$. We claim that this map is a quasifibration. E is clearly contractible, by the contracting homotopy $h_t(A) = tA$.

To fix notation define

$$BU_n(Y) = \{V \mid V \subseteq Y,\ \dim_{\mathbb{C}} V = n\}$$

for any $Y \subset \mathcal{U} \oplus \mathcal{U}$. For $V \subseteq W \subset \mathcal{U}$, there is a natural map $BU_n(V \oplus V) \to BU_m(W \oplus W)$ given by sending V' to $V' \oplus ((W - V) \oplus 0)$. Letting $BU(Y) = \coprod_n BU_n(Y)$, define $BU \times \mathbb{Z} = \varinjlim_W BU(W \oplus W)$.

Lemma 3.2. *Let $X \in U(W \oplus W)$. Then $p_W^{-1}(X) \cong BU(\ker(X - I))$.*

Proof. Define $\phi : p_W^{-1}(X) \to BU(\ker(X - I))$ by $\phi(A) = \ker(A - I)$. In order for this map to make sense, we must verify that $\ker(A - I) \subseteq \ker(X - I)$. Suppose $Av = v$. Then

$$Xv = \exp(2\pi\mathrm{i}A)v = \sum_n \frac{(2\pi\mathrm{i})^n}{n!} A^n v = \mathrm{e}^{2\pi\mathrm{i}} v = v$$

so $v \in \ker(X - I)$. Suppose the spectral decomposition of X is

$$X = \pi_V + \sum_i \lambda_i \pi_{V_i},$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$, $\lambda_i \neq 1$, and $\pi_{V'}$ denotes orthogonal projection onto the subspace V' of $W \oplus W$. Note that $V = \ker(X - I)$ and since X is unitary, $|\lambda_i| = 1$ for all i and $W \oplus W = V \oplus \bigoplus_i V_i$. Suppose that $A \in p_W^{-1}(X)$. Then A, being hermitian, possesses a spectral decomposition

$$A = \pi_{V'} + 0 \cdot \pi_{V''} + \sum_i \mu_i \pi_{W_i},$$

where $W \oplus W = V' \oplus V'' \oplus \bigoplus_i W_i$. Since

$$\exp(2\pi\mathrm{i}A) = \pi_{V' \oplus V''} + \sum_i \mathrm{e}^{2\pi\mathrm{i}\mu_i} \pi_{W_i} = X$$

we conclude that $V' \oplus V'' = V$, $V_i = W_i$, and the eigenvalues μ_i are uniquely determined by the non-unital eigenvalues λ_i of X. It is clear then that $\phi(A) = V'$ possesses a continuous inverse $\psi : BU(V) \to p_W^{-1}(X)$ given by

$$\psi(V') = \pi_{V'} + \sum_i \mu_i \pi_{V_i}. \qquad \square$$

We shall now prove that $p : E \to U$ is a quasifibration. Define a filtration of U by letting

$$F_n U = \{X \mid \dim_{\mathbb{C}}(\ker(X - I)^{\perp}) \leqslant n\} \subseteq U.$$

Let $B_n = F_n U - F_{n-1} U$. The following lemma proves that B_n is distinguished.

Lemma 3.3. *$p^{-1}(B_n) \to B_n$ is a Serre fibration.*

Proof. Suppose we are presented with the following commutative diagram.

$$\begin{tikzcd}
\{0\} \times I^k \arrow[r, "\alpha"] \arrow[d] & p^{-1}(B_n) \arrow[d] \\
I^{k+1} \arrow[r, "\beta"'] & B_n
\end{tikzcd}$$

We wish to give a lift of this diagram. By compactness, there exists a finite dimensional $W \subset \mathcal{U}$ such that the diagram factors as

$$\begin{tikzcd}
\{0\} \times I^k \arrow[r, "\alpha'"] \arrow[d] & E(W) \cap p^{-1}(B_n) \arrow[r] \arrow[d] & p^{-1}(B_n) \arrow[d] \\
I^{k+1} \arrow[r, "\beta'"'] & U(W \oplus W) \cap B_n \arrow[r] & B_n
\end{tikzcd}$$

Now, let $A(0, t_1, \ldots, t_k) = \alpha'(t_1, \ldots, t_k)$ and $X(t_0, \ldots, t_k) = \beta'(t_0, \ldots, t_k)$. Then we may write spectral decompositions, for $t \in I^k, I^{k+1}$, respectively, as

$$\begin{aligned}
A(t) &= \pi_{W_0(t)} + \sum_l \mu_l(t) \pi_{W_l(t)}, \\
X(t) &= \pi_{V_0(t)} + \sum_l \lambda_l(t) \pi_{V_l(t)},
\end{aligned}$$

where $e^{2\pi i \mu_l(t)} = \lambda_l(t)$, $W_0(t) \subseteq V_0(t)$, and $W_l(t) = V_l(t)$ for all $t \in I^k$. Consider, for an n-dimensional complex subspace W of $\mathcal{U}$, the homogeneous space

$$\begin{aligned}
\mathrm{Perp}_{i,j}(W \oplus W) &= \{(V', V'') \mid V', V'' \subseteq W \oplus W,\ V' \perp V'', \\
&\qquad \dim_{\mathbb{C}} V' = i,\ \dim_{\mathbb{C}} V'' = j\} \\
&\cong U_{2n}/U_i \times U_j \times U_{2n-(i+j)}.
\end{aligned}$$

There is a natural mapping

$$P : \mathrm{Perp}_{i,j}(W \oplus W) \to BU_{i+j}(W \oplus W)$$

given by $P(V', V'') = V' \oplus V''$. Under the isomorphism $BU_{i+j}(W \oplus W) \cong U_{2n}/U_{i+j} \times U_{2n-(i+j)}$, we see that P is the natural projection map, and therefore is a fibration. Let $\alpha'' : I^k \to \mathrm{Perp}_{i,j}(W \oplus W)$ where $i = \dim W_0(0)$ and $j = \dim(V_0(0) - W_0(0))$ be given by $\alpha''(t) = (W_0(t), V_0(t) - W_0(t))$, and let $\beta'' : I^{k+1} \to BU_{i+j}(W \oplus W)$ be given by

$\beta''(t) = V_0(t)$. Our filtration is defined so that these maps make sense. Then, since P is a fibration, there exists a lift ω'' making the diagram below commute.

$$\begin{tikzcd}
\{0\} \times I^k \arrow[r, "\alpha''"] \arrow[d] & \mathrm{Perp}_{i,j}(W \oplus W) \arrow[d, "P"] \\
I^{k+1} \arrow[r, "\beta''"'] \arrow[ur, "\omega''"] & BU_{i+j}(W \oplus W)
\end{tikzcd}$$

Let $\mu_l(t) \in (0,1)$ be the unique solutions to $\mathrm{e}^{2\pi \mathrm{i}\mu_l(t)} = \lambda_l(t)$, and write $\omega''(t) = (W_0'(t), V_0(t) - W_0'(t))$. Then letting $\omega' : I^{k+1} \to E(W) \cap p^{-1}(B_n)$ be defined by

$$\omega'(t) = \pi_{W_0'(t)} + \sum_l \mu_l(t)\pi_{V_l(t)}$$

we obtain a lift to our original diagram. □

Define

$$\overline{BU}_{V,W} = \varinjlim_{W' \geqslant W} BU\big(V \oplus \big(W' - W\big) \oplus \big(W' - W\big)\big)$$

for W finite dimensional and $V \subseteq W \oplus W$. It is clear that $\overline{BU}_{V,W} \cong BU \times \mathbb{Z}$, by a (non-canonical) choice of isometry $V \oplus W^{\perp} \oplus W^{\perp} \cong \mathcal{U} \oplus \mathcal{U}$. Then if $X \in U(W \oplus W)$, $p^{-1}(X) \cong \overline{BU}_{\ker(X-I),W}$.

Define a neighborhood N_n of $F_{n-1}U$ in F_nU to be

$$N_n = \{X \in F_nU\colon \dim_{\mathbb{C}} \mathrm{Eig}_{\mathrm{e}^{2\pi \mathrm{i}[1/3,2/3]}} X < n\} \subseteq F_nU,$$

where $\mathrm{Eig}_S X$ is the direct sum of the eigenspaces of X corresponding to eigenvalues in S. In other words, N_n is simply the space of unitary matrices with "extra eigenvalues" in a neighborhood of 1 that we shall deform to 1, pushing the matrix into $F_{n-1}U$. Let $f : I \to I$ be defined by

$$f(x) = \begin{cases} 1, & x \geqslant \frac{2}{3}, \\ 3x - 1, & \frac{1}{3} \leqslant x \leqslant \frac{2}{3}, \\ 0, & x \leqslant \frac{1}{3}. \end{cases}$$

Clearly $f \simeq \mathrm{Id}$ rel ∂I. Let H be such a homotopy. Then, since H fixes ∂I, there exists an $h : S^1 \times I \to S^1$ such that the following diagram commutes.

$$\begin{tikzcd}
I \arrow[r, "H_t"] \arrow[d, "\mathrm{e}^{2\pi \mathrm{i}(\cdot)}"'] & I \arrow[d, "\mathrm{e}^{2\pi \mathrm{i}(\cdot)}"] \\
S^1 \arrow[r, "h_t"'] & S^1
\end{tikzcd}$$

Then for $A \in E$, where $A = \sum_i \mu_i \pi_{W_i}$, define a new hermitian matrix $H_t(A)$ where for $t \in I$,

$$H_t(A) = \sum_i H_t(\mu_i)\pi_{W_i}.$$

We may similarly define $h_t : U \to U$. Observe that the $h_t : N_n \to N_n$ satisfy $h_1 = \mathrm{Id}$ and $h_0(N_n) \subseteq F_{n-1}U$. Furthermore, h_t is covered by $H_t : p^{-1}(N_n) \to p^{-1}(N_n)$.

Consider the induced map on fibers $H_0 : p^{-1}(X) \to p^{-1}(h_0(X))$. We need only prove that this map is a weak equivalence to complete the proof that p is a quasifibration. This follows from the following lemma.

Lemma 3.4. *Suppose $V \subseteq V' \subseteq W \oplus W$ and $V'' \subseteq V' - V$. Then the map $\overline{BU}_{V,W} \to \overline{BU}_{V',W}$ given by sending Y to $Y \oplus V''$ is a weak equivalence.*

Proof. If C is a pointed compact space, then the induced map on reduced K-theory

$$\widetilde{K}_{\mathbb{C}}(C) \cong \left[C, \overline{BU}_{V,W}\right] \to \left[C, \overline{BU}_{V',W}\right] \cong \widetilde{K}_{\mathbb{C}}(C)$$

is just the addition of a trivial bundle, so induces an isomorphism. In particular, letting $C = S^i$, we get an isomorphism of homotopy groups. □

4. Real Bott periodicity

The same methods used in the complex case lend themselves to computing the iterated loop spaces of BO as well.

Theorem 4.1. *The loops of BO may be identified as follows.*

$$\begin{aligned}
\Omega BO &\simeq O, \\
\Omega O &\simeq O/U, \\
\Omega O/U &\simeq U/Sp, \\
\Omega U/Sp &\simeq BSp \times \mathbb{Z}, \\
\Omega BSp &\simeq Sp, \\
\Omega Sp &\simeq Sp/U, \\
\Omega Sp/U &\simeq U/O, \\
\Omega U/O &\simeq BO \times \mathbb{Z}.
\end{aligned}$$

We shall prove this theorem one loop at a time by constructing quasifibrations with contractible total spaces. Note that $\Omega BO \simeq O$ and $\Omega BSp \simeq Sp$ are obvious.

4.1. $\Omega O \simeq O/U$

Let $\mathcal{U} \cong \mathbb{C}^\infty$ be an infinite dimensional complex inner product space. For finite dimensional complex $W \subset \mathcal{U}$, let $O(W)$ denote the real linear isometries of W. Define

$$E(W) = \left\{A \mid \sigma(A) \subseteq [-i, i]\right\} \subseteq \mathfrak{o}(W),$$

where $\mathfrak{o}(W)$ is the lie algebra of $O(W)$; it consists of skew symmetric real linear transformations. Observe that $E(W)$ is contractible. Define

$$p_W : E(W) \to O(W)$$

by $p_W(A) = -\exp(\pi A)$. If $V \subseteq W$ then we have maps $O(V) \to O(W)$ given by sending X to $X \oplus I_{W-V}$, and $E(V) \to E(W)$ given by sending A to $A \oplus i$ where i is viewed as a skew symmetric real transformation of $W - V$. Upon taking colimits over finite dimensional subspaces of $\mathcal{U}$, these maps yield a map $p : E \to O$. We claim this map is a quasifibration onto SO, with fiber O/U.

We need a convenient way to think about O/U. For any finite dimensional $W \subset \mathcal{U}$, let $CX(W)$ denote the space of complex structures on W, that is, the space of real linear isometries $J : W \to W$ such that $J^2 = -I$.

Proposition 4.2. *Let $W \subset \mathcal{U}$ be finite dimensional. Then $O/U(W) \cong CX(W)$.*

Proof. $O(W)$ acts transitively on $CX(W)$ by conjugation, with stabilizer $U(W)$. $\square$

The fiber of p is therefore identified in the following lemma.

Lemma 4.3. *For $X \in SO(W)$, $p_W^{-1}(X) \cong CX(\ker(X - I))$.*

Proof. Regarding $\mathfrak{o}(W) \subseteq \mathfrak{u}(W \otimes_{\mathbb{R}} \mathbb{C})$, we see that if $A \in p^{-1}(X)$ then

$$A = i\pi_{V'} - i\pi_{V''} + \sum_j \mu_j \pi_{W_j},$$

where $\mu_j \in (-i, i)$. If we regard $O(W) \subseteq U(W \otimes_{\mathbb{R}} \mathbb{C})$, then we may write

$$X = \pi_V + \sum_j \lambda_j \pi_{V_j},$$

where $\lambda_j \neq 1$. Thus, $V = V' \oplus V'' = \ker(X - I) \otimes_{\mathbb{R}} \mathbb{C}$, $V_j = W_j$ and μ_j is completely determined by λ_j for all j. We conclude that $A(\ker(X - I)) \subseteq \ker(X - I)$, and $A^2|_{\ker(X-I)} = -I_{\ker(X-I)}$. Therefore $A \in CX(\ker(X - I))$. Conversely, given $J \in CX(\ker(X - I))$, let $A = J + \sum_j \mu_j \pi_{V_j}$. Then $A \in p_W^{-1}(X)$. $\square$

Define

$$\overline{O/U}_{V,W} = \lim_{\to} {}_{W' \geqslant W}\, O/U\big(V \oplus \big(W' - W\big)\big)$$

for $V \subseteq W \subset \mathcal{U}$ where W is a complex space and V is a real even dimensional subspace. Then it is clear that for $X \in SO(W)$, we have $p^{-1}(X) \cong \overline{(O/U)}_{\ker(X-I),W}$. Define a filtration on SO by letting

$$F_n SO = \big\{X \in SO\colon \dim_{\mathbb{R}} \ker(X - I)^{\perp} \leqslant 2n\big\}.$$

We wish to show that $B_n = F_n SO - F_{n-1} SO$ is distinguished. Observe that B_n is the set of $X \in SO$ such that $\dim \ker(X - I)^{\perp} = 2n$. We claim that $p^{-1}(B_n) \to B_n$ is actually a Serre fibration. The proof of this is completely analogous to the proof of Lemma 3.3; it amounts to observing that the natural map $O_m/U_n \times O_{m-2n} \to O_m/O_{2n} \times O_{m-2n}$ is a fibration.

We define a neighborhood N_n of $F_{n-1} SO$ in $F_n SO$ by

$$N_n = \{X \mid \dim_{\mathbb{R}} \mathrm{Eig}_{e^{2\pi i[1/4,3/4]}} X < 2n\} \subseteq F_n SO.$$

Let $f:[-i,i]\to[-i,i]$ be defined by

$$f(x)=\begin{cases}-i, & \mathrm{Im}(x)<-\frac{1}{2},\\ 2x, & -\frac{1}{2}\leqslant \mathrm{Im}(x)\leqslant\frac{1}{2},\\ i, & \mathrm{Im}(x)>\frac{1}{2}.\end{cases}$$

Then $f\simeq \mathrm{Id}\,\mathrm{rel}\{-i,i\}$. Let H be such a homotopy, and let $h:S^1\times I\to S^1$ be such that the following diagram commutes for all $t\in I$.

$$\begin{tikzcd}
{[-i,i]} \arrow[r, "H_t"] \arrow[d, "-\mathrm{e}^{\pi(\cdot)}"'] & {[-i,i]} \arrow[d, "-\mathrm{e}^{\pi(\cdot)}"] \\
S^1 \arrow[r, "h_t"'] & S^1
\end{tikzcd}$$

Then H and h induce the deformations of N_n into $F_{n-1}SO$ as required in the Dold–Thom theorem. The fact that H_0 induces weak equivalences on fibers follows from the following lemma, which is proved by Lemma 2.3.

Lemma 4.4. *Let $V\subseteq V'$ be even dimensional real subspaces of a finite dimensional complex space $W\subset\mathcal{U}$. Then the map $f:\overline{O/U}_{V,W}\to\overline{O/U}_{V',W}$ given by sending A to $A\oplus J$ for some fixed complex structure J on $V'-V$ is a homotopy equivalence.*

4.2. $\Omega O/U\simeq U/Sp$

Let $\mathcal{U}\cong\mathbb{H}^\infty$ be an infinite dimensional quaternionic inner product space. For finite dimensional $W\subset\mathcal{U}$, $O(W)$ is the space of real linear isometries of W, and $U(W)$ is the space of complex linear isometries of W. Then $O/U=\varinjlim_W O/U(W)$. Define

$$E(W)=\bigl\{A\mid A\text{ is conjugate linear and }\sigma(A)\subseteq[-i,i]\bigr\}\subseteq\mathfrak{o}(W).$$

Note that $\mathfrak{u}(W)^\perp\subseteq\mathfrak{o}(W)$ is the collection of all skew symmetric conjugate linear transformations of W. This implies that every coset $[X]\in SO/U(W)$ has a representative $X\in O(W)$ such that $X=\exp(A)$ for some skew symmetric conjugate linear transformation A. Also observe that $E(W)$ is contractible. Define

$$p_W:E(W)\to O/U(W)$$

by $p_W(A)=i\exp(\frac{1}{2}\pi A)$. If $V\subseteq W$ then we have maps $O/U(V)\to O/U(W)$ given by sending $[X]$ to $[X\oplus I_{W-V}]$, and $E(V)\to E(W)$ given by sending A to $A\oplus j$ where j is viewed as a conjugate linear skew-symmetric transformation of $W-V$. Upon taking colimits over finite dimensional quaternionic subspaces of $\mathcal{U}$, we obtain $p:E\to O/U$, which we wish to show is a quasifibration over SO/U, with fiber U/Sp.

For $W\subset\mathcal{U}$, let $QS(W)$ denote the space of quaternionic structures on W. These are the conjugate linear isometries J of W such that $J^2=-I$.

Proposition 4.5. *Let $W\subset\mathcal{U}$ be a finite dimensional quaternionic subspace. Then $U/Sp(W)\cong QS(W)$.*

Proof. $U(W)$ acts transitively on $QS(W)$ with stabilizer $Sp(W)$. $\square$

With the intent of understanding the coset representatives of $O/U(W)$, we give the following two lemmas.

Lemma 4.6. *Suppose that $Y = \exp(A)$, where $A \in \mathfrak{o}(W)$ is conjugate linear. Then $Yi = iY^{-1}$.*

Proof.

$$-iYi = -i\exp(A)i = \exp(-iAi) = \exp(-A) = Y^{-1}. \qquad \square$$

Lemma 4.7. *Suppose that $Y, Z \in O(W)$ satisfy $-iYi = Y^{-1}$ and $-iZi = Z^{-1}$. Then there is an $X \in U(W)$ such that $Y = XZ$ if and only if $Y^2 = Z^2$.*

Proof. Suppose that there is an $X \in U(W)$ such that $Y = XZ$. Observe that

$$Z^{-1}X^{-1}i = Y^{-1}i = iY = iXZ = XZ^{-1}i$$

and therefore $XZX = Z$. But then $Y^2 = XZXZ = Z^2$.

Conversely, suppose that $Y^2 = Z^2$. Then $Y = (Y^{-1}Z)Z$, so we need only show that $Y^{-1}Z \in U(W)$. But $YZ^{-1} = Y^{-1}Z$, so $Y^{-1}Zi = iYZ^{-1} = iY^{-1}Z$. $\square$

We shall say that $X \in SO(W)$ is a *special representative* of the equivalence class $[X] \in SO/U(W)$ if $X = \exp(A)$ for some conjugate linear $A \in \mathfrak{o}(W)$. Observe that by the previous two lemmas, any two special representatives are in the same equivalence class if and only if they have identical squares.

Lemma 4.8. *Every $[X] \in SO/U(W)$ has a special representative.*

Proof. $SO/U(W)$ is geodesically complete, and the geodesics γ of $SO/U(W)$ all take the form $\gamma(t) = [Y\exp(tB)]$ for $Y \in SO(W)$ and $B \in \mathfrak{u}(W)^{\perp}$ (see, for example, [5, VI.2.15]). $\square$

Lemma 4.9. *Suppose that $W \subset \mathcal{U}$ is a finite dimensional quaternionic space. Let X be a special representative of the class $[X] \in SO/U(W)$. Then $p_W^{-1}([X]) = U/Sp(\ker(X^2 - I))$.*

Proof. We claim that if $A \in p_W^{-1}([X])$, then A defines a quaternionic structure on $\ker(X^2 - I)$, that is, $A(\ker(X^2 - I)) \subseteq \ker(X^2 - I)$, and $A^2 = -I$. If $A \in E(W)$ we may regard A as an element of $\mathfrak{u}(W \otimes_{\mathbb{R}} \mathbb{C})$, and write a spectral decomposition

$$A = i\pi_{W'} - i\pi_{W''} + \sum_l \mu_l \pi_{W_l},$$

where $\mu_l \in (-i, i)$. Regarding $X \in U(W \otimes_{\mathbb{R}} \mathbb{C})$, its spectral decomposition is

$$X = \pi_{V'} - \pi_{V''} + \sum_l (\lambda_l \pi_{V_l'} - \lambda_l \pi_{V_l''}),$$

where $|\lambda_l| = 1$ and $\mathrm{Im}(\lambda_j) < 0$. If $p_W(A) = [i\exp(\frac{1}{2}\pi A)] = [X]$, then we have $-\exp(\pi A) = X^2$, so $V' \oplus V'' = W' \oplus W'' = \ker(X^2 - I) \otimes_{\mathbb{R}} \mathbb{C}$. So $A \in QS(\ker(X^2 - I))$. Conversely, suppose that J is a quaternionic structure on $\ker(X^2 - I)$. Then, regarding J as an element of $\mathfrak{u}(\ker(X^2 - I) \otimes_{\mathbb{R}} \mathbb{C})$ we obtain a spectral decomposition $J = i\pi_{W'} - i\pi_{W''}$. Let

$$A = i\pi_{W'} - i\pi_{W''} + \sum_l \mu_l \pi_{V_l' \oplus V_l''},$$

where $\mu_l \in (-i, i)$ are the unique solutions in the given range to the equation $-\mathrm{e}^{\pi\mu_l} = \lambda_l^2$. Then $(i\exp(\frac{1}{2}\pi A))^2 = X^2$, so $A \in p_W^{-1}([X])$. □

For $V \subseteq W \subset \mathcal{U}$, let

$$\overline{U/Sp}_{V,W} = \varinjlim_{W' \geqslant W} U/Sp\big(V \oplus \big(W - W'\big)\big).$$

Then for a special representative $X \in SO(W)$, $p^{-1}([X])$ may be canonically identified with $\overline{U/Sp}_{\ker(X^2-I),W}$. Of course, $\overline{U/Sp}_{V,W} \cong U/Sp$.

Define a filtration of SO/U by

$$F_n SO/U = \big\{[X] \mid X \text{ is a special representative, } \dim_{\mathbb{C}} \ker\big(X^2 - I\big)^{\perp} \leqslant 2n\big\}.$$

We are implicitly using the fact that any two special representatives of the same coset have identical squares in making this definition. Then the same argument used for the previous spaces works for our present situation, to prove that $p^{-1}(F_n SO/U - F_{n-1} SO/U) \to F_n SO/U - F_{n-1} SO/U$ is a Serre fibration. The key point is that $U_m/Sp_n \times U_{m-2n} \to U_m/U_{2n} \times U_{m-2n}$ is a fibration. Therefore $F_n U/Sp - F_{n-1} U/Sp$ is distinguished.

Just as in the previous section, one may define a neighborhood N_n of $F_{n-1} SO/U$ in $F_n SO/U$ by

$$N_n = \big\{[X] \mid X \text{ is a special representative, } \dim \mathrm{Eig}_{\mathrm{e}^{\pi \mathrm{i}[1/2,3/2]}} X^2 < 2n\big\}.$$

Let f and H_t be defined as in the previous section. These yield the deformations required by the Dold–Thom theorem. One verifies that the induced maps on fibers are weak equivalences by the same methods in the previous section, by the following consequence of Lemma 2.3.

Lemma 4.10. *Suppose that $V \subseteq V' \subseteq W$ where V and V' are even dimensional complex spaces and W is a finite dimensional quaternionic subspace of $\mathcal{U}$. Fix a quaternionic structure J on $V' - V$. Then the map $\overline{U/Sp}_{V,W} \to \overline{U/Sp}_{V',W}$ given by sending A to $A \oplus J$ is a homotopy equivalence.*

4.3. $\Omega U/Sp \simeq BSp \times \mathbb{Z}$

Let $\mathcal{U} \cong \mathbb{H}^\infty$ be a countably infinite dimensional quaternionic inner product space. For finite dimensional $W \subset \mathcal{U}$, $U(W \oplus W)$ is the collection of complex linear isometries of

$W \oplus W$, and $Sp(W \oplus W)$ is the subgroup of quaternion linear isometries of $W \oplus W$. Then $U/Sp = \varinjlim_W U/Sp(W \oplus W)$. Define

$$E(W) = \{A \mid jA = Aj,\ \sigma(A) \subseteq I\} \subseteq H(W \oplus W),$$

where $H(W \oplus W)$ is the collection of all complex linear transformations of $W \oplus W$ which are hermitian. Observe that $\mathfrak{sp}(W \oplus W)^{\perp} = \{A \in \mathfrak{u}(W \oplus W)\colon\ Aj = -jA\}$. Define a map $p_W : E(W) \to U/Sp(W \oplus W)$ by $p_W(A) = [\exp(\pi \mathrm{i} A)]$. Then, analogous to the previous section, we have the following two lemmas which allow us to understand a system of coset representatives of U/Sp. The proofs are nearly identical to those of Lemmas 4.6 and 4.7, respectively.

Lemma 4.11. *Let $W \subset \mathcal{U}$ be finite dimensional. If $A \in H(W \oplus W)$ has the property that $Aj = jA$, then $X = \exp(\mathrm{i}A)$ has the property that $Xj = jX^{-1}$.*

Lemma 4.12. *Suppose that $Y, Z \in U(W \oplus W)$ have the property that $-jYj = Y^{-1}$ and $-jZj = Z^{-1}$. Then there exists an $X \in Sp(W \oplus W)$ such that $Y = XZ$ if and only if $Y^2 = Z^2$.*

We shall call $X \in U(W \oplus W)$ such that $X = \exp(\pi \mathrm{i} A)$ for some $A \in E(W)$ a *special representative* for the class $[X] \in U/Sp(W \oplus W)$. Note that the previous two lemmas ensure that two special representatives are in the same equivalence class if and only if they have the same squares. An argument similar to that of Lemma 4.8 ensures that every coset of $U/Sp(W \oplus W)$ has a special representative. Define, for a quaternionic space Y,

$$BSp(Y) = \coprod_n \{V \mid V \text{ is a quaternionic supspace of } Y,\ \dim V_{\mathbb{H}} = n\}.$$

For $V \subseteq W \subset \mathcal{U}$, $BSp(V \oplus V) \to BSp(W \oplus W)$ is given by sending Y to $Y \oplus (W - V) \oplus 0$, so that $BSp \times \mathbb{Z} = \varinjlim_W BSp(W \oplus W)$. The fiber of p_W can now be identified.

Lemma 4.13. *Let $W \subset \mathcal{U}$ be finite dimensional. If X is a special representative for $[X] \in U/Sp(W)$, then $p_W^{-1}([X]) \cong BSp(\ker(X^2 - I))$.*

Proof. Suppose $A \in E(W)$. Write the spectral decomposition of A as

$$A = \pi_{W_0} + \sum_l \mu_l \pi_{W_l},$$

where $\mu_l \in (0,1)$ and W_0 and W_l are complex subspaces of $W \oplus W$. These are actually quaternionic subspaces because if $Av = \mu v$, then $Ajv = jAv = j\mu v = \mu jv$, since μ must be real. Similarly, write the spectral decomposition of the special representative X as

$$X = \pi_{V'} - \pi_{V''} + \sum_l (\lambda_l \pi_{V_l'} - \lambda_l \pi_{V_l''}),$$

where $\mathrm{Im}(\lambda_l) > 0$ and $|\lambda_l| = 1$. Now, $p_W(A) = [X]$ if and only if $W_0 \subseteq V' \oplus V'' = \ker(X^2 - I)$, $W_l = V_l' \oplus V_l''$, and $\mu_l \in (0,1)$ is the unique solution of $\mathrm{e}^{2\pi \mathrm{i} \mu_l} = \lambda_l^2$. It is then clear that the map $p_W^{-1}([X]) \to BSp(\ker(X^2 - I))$ given by sending A to $\ker(A - I)$ is a homeomorphism. $\square$

For $V \subseteq W$, define $E(V) \to E(W)$ by sending A to $A \oplus \pi_{(W-V)\oplus 0}$, and $U/Sp(V \oplus V) \to U/Sp(W \oplus W)$ by sending $[X]$ to $[X \oplus I_{(W-V)\oplus(W-V)}]$. Taking colimits over W we obtain $p : E \to U/Sp$, which we shall see is a quasifibration. Define

$$\overline{BSp}_{V,W} = \varinjlim_{W' \geqslant W} BSp\big(V \oplus \big(W' - W\big) \oplus \big(W' - W\big)\big),$$

where $V \subseteq W \oplus W \subset \mathcal{U} \oplus \mathcal{U}$. Upon stabilization, the above lemma yields that for a special representative $X \in Sp(W)$, $p^{-1}([X]) = \overline{BSp}_{\ker(X^2 - I), W}$. Define a filtration of U/Sp by

$$F_n U/Sp = \big\{[X] \mid X \text{ is a special representative, } \dim_{\mathbb{C}} \ker\big(X^2 - I\big)^{\perp} \leqslant 2n\big\}.$$

Then the techniques used in the previous sections go through in this instance to prove that $p^{-1}(F_n U/Sp - F_{n-1} U/Sp) \to F_n U/Sp - F_{n-1} U/Sp$ is a Serre fibration, hence $F_n U/Sp - F_{n-1} U/Sp$ is distinguished. Techniques completely analogous to those used in the previous sections provide the neighborhoods and deformations required by the Dold–Thom theorem. By a proof similar to that of Lemma 3.4, one obtains the following lemma, which verifies that the induced maps on fibers are homotopy equivalences.

Lemma 4.14. *Suppose that we have finite dimensional quaternionic spaces $V \subseteq V' \subseteq W \oplus W$. Let $V'' \subseteq V' - V$. Then the natural map $\overline{BSp}_{V,W} \to \overline{BSp}_{V',W}$ given by sending X to $X \oplus V''$ is a homotopy equivalence.*

4.4. $\Omega Sp \simeq Sp/U$

Let $\mathcal{U} \cong \mathbb{H}^\infty$ be a countably infinite dimensional quaternionic inner product space. For finite dimensional $W \subset \mathcal{U}$, $Sp(W)$ is the space of quaternionic isometries of W. Then $Sp = \varinjlim_W Sp(W)$. Define

$$E(W) = \big\{A \mid \sigma(A) \subseteq [-1, 1],\ Aj = -jA\big\} \subseteq H(W),$$

where $H(W)$ is the space of all complex linear hermitian operators on W. Define $p_W : E(W) \to Sp(W)$ by $p_W(A) = -\exp(\pi \mathrm{i} A)$. We need a convenient model for $Sp/U(W)$.

Lemma 4.15. *Let $W \subset \mathcal{U}$ be a finite dimensional quaternionic subspace. Then there is an isomorphism*

$$Sp/U(W) \cong \{V \mid V \textit{ is a complex subspace of } W,\ W = V \oplus jV\}.$$

Proof. $Sp(W)$ acts transitively on this space, with stabilizer $U(W)$. $\square$

With this in mind we may identify the fiber of p_W.

Lemma 4.16. *Let $W \subset \mathcal{U}$ be a finite dimensional quaternionic subspace. For $X \in Sp(W)$, $p_W^{-1}(X) \cong Sp/U(\ker(X - I))$.*

Proof. For $A \in E(W)$, write the spectral decomposition of A

$$A = \pi_{W'} - \pi_{W''} + \sum_l (\mu_l \pi_{W_l'} - \mu_l \pi_{W_l''}),$$

where $\mu_l \in (0,1)$ and $jW' = W''$ and $jW_l' = W_l''$. The latter conditions are seen to be necessary since if $Av = \mu v$, then $Ajv = -jAv = -j\mu v = -\mu jv$. Similarly, write X as

$$X = \pi_V - \pi_{V_0} + \sum_l \bigl(\lambda_l \pi_{V_l'} + \overline{\lambda_l} \pi_{V_l''}\bigr),$$

where $|\lambda_l| = 1$, $\mathrm{Im}(\lambda_l) < 0$, V and V_0 are quaternionic subspaces of W, and $jV_l' = V_l''$. This condition is required since if $Xv = \lambda v$, then $Xjv = jXv = j\lambda v = \overline{\lambda} jv$. So $p_W(A) = X$ if and only if $W' \oplus W'' = V$, $W_l' = V_l'$, $W_l'' = V_l''$, and $\mu_l \in (0,1)$ are the unique solutions to the equation $-\mathrm{e}^{\pi \mathrm{i} \mu_l} = \lambda_l$. It follows immediately that $p_W^{-1}(X) = Sp/U(\ker(X - I))$. $\square$

Let Y be a quaternionic vector space, and define $Y^{\mathbb{C}} = \{v \mid iv = vi\} \subseteq Y$. For $V \subseteq W$, define maps $E(V) \to E(W)$ by sending A to $A \oplus \pi_{(W-V)^{\mathbb{C}}}$. Taking the colimit over all $W \subset \mathcal{U}$ yields $p: E \to Sp$. The proof that this is a quasifibration is completely analogous to the previous sections. Since $E(W)$ is contractible for all W, E is contractible, and the previous lemma implies that the fiber of p is Sp/U.

4.5. $\Omega Sp/U \simeq U/O$

Let $\mathcal{U} \cong \mathbb{H}^\infty$ be an infinite dimensional quaternionic space endowed with a real inner product such that multiplication by i and multiplication by j are real isometries. For a finite dimensional right quaternionic subspace $W \subset \mathcal{U}$, regard $Sp(W)$ as the collection of real isometries X of W that are right quaternion linear, in the sense that for all $\alpha \in \mathbb{H}$, $X(v\alpha) = (Xv)\alpha$. The elements of $Sp(W)$ may be regarded as matrices with quaternion coefficients. Then $U(W)$ is the subgroup of $Sp(W)$ consisting of all X which are left complex linear, in the sense that $X(iv) = iX(v)$. Let $W^{\mathbb{R}}$ be the real subspace of W given by $\{v \mid vi = iv \text{ and } vj = jv\}$. The Lie algebra of Sp is given by

$$\mathfrak{sp}(W) = \mathfrak{o}\bigl(W^{\mathbb{R}}\bigr) \oplus iS\bigl(W^{\mathbb{R}}\bigr) \oplus jS\bigl(W^{\mathbb{R}}\bigr) \oplus kS\bigl(W^{\mathbb{R}}\bigr),$$

where $S(X)$ denotes symmetric linear transformations of a space X. The Lie subalgebra corresponding to $\mathfrak{u}(W)$ is $\mathfrak{o}(W^{\mathbb{R}}) \oplus iS(W^{\mathbb{R}})$. We let

$$E(W) = \bigl\{ jA + kB \mid \sigma(A),\ \sigma(B) \subseteq [-1,1] \bigr\} \subseteq jS\bigl(W^{\mathbb{R}}\bigr) \oplus kS\bigl(W^{\mathbb{R}}\bigr).$$

Define $p_W : E(W) \to Sp/U(W)$ by $p_W(A) = [i \exp(\frac{1}{2}\pi A)]$. We identify U/O in the following proposition.

Proposition 4.17. *Let W be a finite dimensional quaternionic inner product space. Then there is an isomorphism*

$$U/O(W) \cong \{V \mid V \text{ is a right complex subspace of } W,\ W = V \oplus iV = V \oplus Vj\}.$$

Proof. $U(W)$ acts transitively on this space, with stabilizer $O(W)$. □

To understand the coset representatives of $U/O(W)$, we give the following two lemmas. Their proofs are completely analogous to the proofs of Lemmas 4.6 and 4.7.

Lemma 4.18. *Let $W \subset \mathcal{U}$ be a right quaternionic subspace of finite dimension. Suppose that $A \in \mathfrak{sp}(W)$ has the property that $Ai = -iA$. Then $X = \exp(A)$ has the property that $Xi = iX^{-1}$.*

Lemma 4.19. *Suppose that $W \subset \mathcal{U}$ is a right quaternionic subspace of finite dimension. If $Y, Z \in Sp(W)$ possess the property that $-iYi = Y^{-1}$ and $-iZi = Z^{-1}$, then there exists an $X \in U(W)$ such that $Y = XZ$ if and only if $Y^2 = Z^2$.*

We shall call an $X \in Sp(W)$ such that there exists an $A \in \mathfrak{sp}(W)$ such that $Ai = -iA$, yielding $X = \exp(A)$ a *special representative* of $[X] \in Sp/U(W)$. The above two lemmas imply that two special representatives are in the same coset if and only if they have identical squares. The argument of Lemma 4.8 shows that any coset has a special representative. With this knowledge we may proceed to identify the fiber of p_W.

Lemma 4.20. *Let $W \subset \mathcal{U}$ be a finite dimensional right quaternionic subspace. For a special representative X of $[X] \in Sp/U(W)$, we have $p_W^{-1}([X]) \cong U/O(\ker(X^2 - I))$.*

Proof. Suppose $A \in E(W)$. Being careful to write our eigenvalues on the right since A is a right skew-hermitian operator, we may express a spectral decomposition of A as

$$A = \pi_{W'} i - \pi_{W''} i + \sum_l (\pi_{W_l'} i\mu_l - \pi_{W_l''} i\mu_l),$$

where $\mu_l \in (0, 1)$, W', W'', W_l', and W_l'' are right complex spaces, $iW' = W''$, $W'j = W''$, $iW_l' = W_l''$, and $W_l'j = W_l''$. For if $Av = vi\mu$, then $Aiv = -iAv = -ivi\mu = iv(-i\mu)$ and $A(vj) = (Av)j = vi\mu j = vj(-i\mu)$. Similarly, write the spectral decomposition of the special representative X as

$$X = \pi_{V'} - \pi_{V''} + \pi_{V_0'} i - \pi_{V_0''} i + \sum_l \bigl(\pi_{V_l'}\lambda_l + \pi_{V_l''}\overline{\lambda_l} - \pi_{\widetilde{V}_l'}\lambda_l - \pi_{\widetilde{V}_l''}\overline{\lambda}_l\bigr),$$

where $|\lambda_l| = 1$, $\operatorname{Im}(\lambda_l^2) < 0$, $\operatorname{Im}(\lambda_l) > 0$, V' and V'' are quaternionic spaces, $iV_l' = V_l''$, $V_l'j = V_l''$. $i\widetilde{V}_l' = \widetilde{V}_l''$, and $\widetilde{V}_l'j = \widetilde{V}_l''$. For if $Xv = v\lambda$, then $Xiv = iX^{-1}v = iv\overline{\lambda}$, and $Xvj = v\lambda j = vj\overline{\lambda}$. Now, if $-\exp(\pi \mathrm{i}A) = X^2$, we see that $\mu_l \in (0, 1)$ are the unique solutions to $-\mathrm{e}^{\pi \mathrm{i}\mu_l} = \lambda_l^2$, $W' \oplus W'' = V' \oplus V'' = \ker(X^2 - I)$, $W_l' = V_l' \oplus \widetilde{V}_l'$, and $W_l'' = V_l'' \oplus \widetilde{V}_l''$. The result follows immediately. □

For $V \subseteq W$, define $i_{V,W} : E(V) \to E(W)$ by

$$i_{V,W}(A) = A \oplus (\pi_{(k+1)(W-V)^{\mathbb{R}}} - \pi_{(i-j)(W-V)^{\mathbb{R}}}).$$

Taking the colimit over $W \subset \mathcal{U}$, we obtain a map $p : E \to Sp/U$, which, by repeating the techniques of the previous sections, is a quasifibration with fiber U/O.

4.6. $\Omega U/O = BO \times \mathbb{Z}$

Let $\mathcal{U} \cong \mathbb{C}^\infty$. Fix a complex conjugation $c : \mathcal{U} \to \mathcal{U}$. For the purposes of this section, all finite dimensional complex subspaces of $\mathcal{U}$ are assumed to be closed under the conjugation map c. For a complex finite dimensional $W \subset \mathcal{U}$, the *real subspace* of W is defined to be $W^{\mathbb{R}} = \{v \in W\colon\ v = \overline{v}\}$. $U(W \oplus W)$ is the collection of complex isometries of $W \oplus W$, and $O(W \oplus W)$ is the collection of all $X \in U(W \oplus W)$ such that $X = \overline{X}$. Define

$$E(W) = \left\{A \mid \overline{A} = A,\ \sigma(A) \subseteq [0,1]\right\} \subseteq H(W \oplus W).$$

Define $p_W : E(W) \to U/O(W)$ by $p_W(A) = [\exp(\pi \mathrm{i} A)]$. Observe that we have the following two lemmas, whose proofs are analogous to those of Lemmas 4.6 and 4.7.

Lemma 4.21. *Let W be a finite dimensional complex space. Then if $A \in \mathfrak{u}(W \oplus W)$ has the property that $\overline{A} = -A$, then $X = \exp(A)$ has the property that $X^{-1} = \overline{X}$.*

Lemma 4.22. *Suppose that W is a finite dimensional complex space. Then if $Y, Z \in U(W \oplus W)$ have the property that $Y^{-1} = \overline{Y}$ and $Z^{-1} = \overline{Z}$, then there exists an $X \in O(W \oplus W)$ such that $Y = XZ$ if and only if $Y^2 = Z^2$.*

If $X \in U(W \oplus W)$, and $X = \exp(A)$ for some $A \in \mathfrak{u}(W \oplus W)$ such that $\overline{A} = -A$, then we shall say that X is a *special representative* of $[X] \in U/O(W \oplus W)$. Evidently two special representatives represent the same equivalence class if and only if they have identical squares. The argument of Lemma 4.8 implies that every coset has a special representative. The following lemma identifies the fiber of p_W.

Lemma 4.23. *Let $W \subset \mathcal{U}$ be a finite dimensional complex space. If $X \in U(W \oplus W)$ is a special representative for $[X] \in U/O(W \oplus W)$, then $p_W^{-1}([X]) \cong BO(\ker(X^2 - I)^{\mathbb{R}})$.*

Proof. If $A \in E(W)$, then A admits a spectral decomposition

$$A = \pi_{W_0} + \sum_l \mu_l \pi_{W_l},$$

where $\mu_l \in (0,1)$. We claim that the spaces W_l are closed under the conjugation in W. Indeed, if $Av = \mu v$, then $A\overline{v} = \overline{Av} = \overline{\mu v} = \mu \overline{v}$. The special representative X has a spectral decomposition

$$X = \pi_{V_0'} - \pi_{V_0''} + \sum_l (\lambda_l \pi_{V_l'} - \lambda_l \pi_{V_l''}),$$

where $\mathrm{Im}(\lambda_l) > 0$. We claim that V_l', V_l'' are closed under conjugation. Indeed, if $Xv = \lambda v$ then $X\overline{v} = \overline{X^{-1}v} = \overline{\overline{\lambda}v} = \lambda \overline{v}$. So if $\exp(2\pi \mathrm{i} A) = X^2$, then the eigenvalues $\mu_l \in (0,1)$ must be the unique solutions to the equation $\mathrm{e}^{2\pi \mathrm{i} \mu_l} = \lambda_l^2$. Also $W_l = V_l' \oplus V_l''$ for all $l \neq 0$ and $W_0 \subseteq V_0' \oplus V_0''$ is simply a subspace closed under conjugation. Define $\phi : p_W^{-1}([X]) \to$

$BO(\ker(X^2-I)^{\mathbb{R}})$ by $\phi(A)=\ker(A-I)^{\mathbb{R}}$. This map clearly has a continuous inverse ψ, namely, for a real subspace, $V\subseteq\ker(X^2-I)^{\mathbb{R}}$, let $W_0=V\oplus iV$. Then define

$$\psi(A)=\pi_{W_0}+\sum_l \mu_l\pi_{W_l}. \qquad \square$$

For $V\subseteq W\subset\mathcal{U}$, complex finite dimensional subspaces closed under conjugation, define $U/O(V\oplus V)\to U/O(W\oplus W)$ by sending X to $X\oplus I_{(W-V)\oplus(W-V)}$. Define $E(V)\to E(W)$ by sending A to $A\oplus\pi_{(W-V)\oplus 0}$. Taking the colimit over W, we obtain a map $p:E\to U/O$, which, by arguments completely analogous to those given in the previous sections, is a quasifibration. For $V\subseteq W$, let $BO(V^{\mathbb{R}}\oplus V^{\mathbb{R}})\to BO(W^{\mathbb{R}}\oplus W^{\mathbb{R}})$ be defined by sending Y to $Y\oplus(W-V)^{\mathbb{R}}\oplus 0$, and define $BO\times\mathbb{Z}=\varinjlim_W BO(W^{\mathbb{R}}\oplus W^{\mathbb{R}})$. Upon stabilization the previous lemma yields that $p^{-1}([X])\simeq BO\times\mathbb{Z}$, which completes the proof of the real Bott periodicity theorem.

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