Congruences between modular forms given by the divided β family in homotopy theory

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We characterize the 2-line of the p-local Adams-Novikov spectral sequence in terms of modular forms satisfying a certain explicit congruence condition for primes $p \geq 5$. We give a similar characterization of the 1-line, reinterpreting some earlier work of A. Baker and G. Laures. These results are then used to deduce that, for ℓ a prime which generates $\mathbb{Z}_p^\times$, the spectrum $Q(\ell)$ detects the α and β families in the stable stems.

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1 Introduction

The Adams-Novikov spectral sequence

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \Rightarrow (\pi^S_{t-s})_{(p)}$$

is one of the main tools for organizing periodic phenomena in the p-local stable homotopy groups of spheres. Assuming that p is an odd prime, the 1-line is generated by elements

$$\alpha_{i/j} \in \mathrm{Ext}^{1,2(p-1)i-1}_{BP_*BP}(BP_*, BP_*)$$

of order p^j, for $i \geq 1$ and j satisfying

$$j = \nu_p(i) + 1.$$

The elements $\alpha_{i/j}$ are all permanent cycles, and detect the generators of the image of the J-homomorphism. The image of J admits a global description in terms of denominators of Bernoulli numbers: there is a correspondence

$$\alpha_{i/j} \leftrightarrow B_t$$

between the generator $\alpha_{i/j}$ and the t^{th} Bernoulli number for $t = (p-1)i$. The order p^j of the element $\alpha_{i/j}$ is equal to the p-factor of the denominator of the quotient

$$\frac{B_t}{t}.$$

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Thus the 1-line of the Adams-Novikov spectral sequence is governed by the p-adic valuations of the denominators of the Bernoulli numbers. The purpose of this paper is to provide a similar description for the 2-line of the Adams-Novikov spectral sequence, in terms of certain congruences of modular forms.

Let

$$M_k(\Gamma_0(N))$$

denote the space of weight k modular forms for $\Gamma_0(N)$ defined over $\mathbb{Z}$. For a ring R, let

$$M_k(\Gamma_0(N))_R = M_k(\Gamma_0(N)) \otimes R \tag{1.1}$$

be the corresponding space of modular forms defined over R. If $N = 1$, we shall simplify the notation:

$$(M_k)_R := M_k(\Gamma_0(1))_R.$$

We shall sometimes work with modular forms which are simply meromorphic at ∞, which we shall denote

$$M_k(\Gamma_0(N))^0_R = M_k(\Gamma_0(N))_R[\Delta^{-1}]$$

where $\Delta \in M_{12}$ is the discriminant.

Remark 1.2 Implicit in our definition of the notation $M_k(\Gamma_0(N))_R$ given by (1.1) is a non-trivial base change theorem. One typically requires N to be invertible in R, and then one regards the modular forms for a ring R as the sections of a certain line bundle of the base-change of the moduli stack of elliptic curves to R. In most instances considered in this paper, these two notions agree, see [Kat73, 1.7, 1.8].

The q-expansion gives rise to an embedding

$$\begin{aligned} M_k(\Gamma_0(N)) &\hookrightarrow \mathbb{Z}[[q]] \\ f &\mapsto f(q) \end{aligned}$$

and consequently embeddings

$$\begin{aligned} M_k(\Gamma_0(N))_R &\hookrightarrow R[[q]], \\ M_k(\Gamma_0(N))^0_R &\hookrightarrow R((q)). \end{aligned}$$

Therefore, a modular form over R is determined by its weight and its q-expansion.

For any $f \in (M_k)_R$, and any prime ℓ, the power series

$$(V_\ell f)(q) := f(q^\ell)$$

is the q-expansion of a modular form

$$V_\ell f \in M_k(\Gamma_0(\ell))_R.$$

Suppose that p is a prime greater than 3. Miller, Ravenel, and Wilson showed that the 2-line of the Adams-Novikov spectral sequence is generated by elements

$$\beta_{i/j,k} \in \mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$$

for i, j, and k satisfying certain elaborate conditions (see Theorem 11.2). Our main theorem is stated below.

Theorem 1.3 For each additive generator

$$\beta_{i/j,k} \in \mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$$

there is an associated modular form

$$f_{i/j,k} \in M_t$$

(where $t = i(p^2-1)$) satisfying:

(1) The q-expansion $f_{i/j,k}(q)$ is not congruent to 0 mod p.

(2) We have $\mathrm{ord}_q f_{i/j,k}(q) > \frac{t-j(p-1)}{12}$ or $\mathrm{ord}_q f_{i/j,k}(q) = \frac{t-j(p-1)-2}{12}$.

(3) There does not exist a form

$$g \in M_{t'}, \qquad \text{for } t' < t,$$

satisfying

$$f_{i/j,k}(q) \equiv g(q) \mod p^k.$$

(4) For every prime $\ell \neq p$, there exists a form

$$g_\ell \in M_{t-j(p-1)}(\Gamma_0(\ell))$$

satisfying

$$f_{i/j,k}(q^\ell) - f_{i/j,k}(q) \equiv g_\ell(q) \mod p^k.$$

The congruence conditions met by the forms $f_{i/j,k}$ are sharp; we have the following converse theorem.

Theorem 1.4 Suppose there exists a modular form

$$f \in M_t$$

satisfying Conditions 1.3(1)-(4), where $t \equiv 0 \mod (p-1)p^{k-1}$. Then $t = i(p^2-1)$ for some i, and if i is not a power of p, there is a corresponding generator

$$\beta_{i/j,k} \in \mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*).$$

Finally, the congruence condition given in Theorem 1.3 exhibits a certain rigidity, as explained in the following theorem.

Theorem 1.5 Suppose that ℓ_0 is a prime which generates $\mathbb{Z}_p^\times$. Then, if f is a modular form of weight $t \equiv 0 \mod (p-1)p^{k-1}$ satisfying Conditions 1.3(1)-(3), and Condition 1.3(4) for $\ell = \ell_0$, then f satisfies Condition 1.3(4) for *all* primes $\ell \neq p$.

Remark 1.6 In [Lau99], G. Laures introduced the f-invariant, a higher analog of the Adams e-invariant, which gives an injection of the groups $\mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$ into Katz's ring of divided congruences, tensored with $\mathbb{Q}/\mathbb{Z}$. Laures' f-invariant therefore associates to an element $\beta_{i/j,k}$, a *non-congruence* between modular forms. It is natural to ask what relation this non-congruence has to the non-congruences given by condition (3) of Theorem 1.3. Laures and the author have discovered that the f-invariant of $\beta_{i/j,k}$ is precisely represented by the modular forms $f_{i/j,k}$ [BLa]. Thus, Theorem 1.4 gives a precise description of the image of the f-invariant. The image of the elements β_i under the f-invariant has been calculated in a different way by J. Hornbostel and N. Naumann [HN07].

The proofs of Theorems 1.3-1.5 use the spectrum $Q(\ell)$ introduced by the author in [Beh07a], [Beh06]. Analyzing the chromatic spectral sequence

$$E_1^{s,t} = \pi_t M_s Q(\ell) \Rightarrow \pi_{t-s} Q(\ell),$$

we observe that a density result [BL06] relates part of the 2-line of the chromatic spectral sequence of $Q(\ell)$ to the 2-line of the chromatic spectral sequence for the sphere. We also analyze the 0 and 1-lines of the chromatic spectral sequence of $Q(\ell)$, and find:

(1) The 0-line $\pi_t M_0 Q(\ell)$ is concentrated in $t = 0, -1, -2$ (Corollary 8.4).

(2) The 1-line $\pi_t M_1 Q(\ell)$ is generated in degrees congruent to $0 \mod 4$ by the images of the elements $\alpha_{i/j} \in \pi_* M_1 S$ (Corollary 9.10).

In fact, the additive generators of $\pi_{4t} Q(\ell)$ are given by the Eisenstein series $E_{2t} \in (M_{2t})_{\mathbb{Q}}$ (9.8), and the orders of the groups $\pi_{4t} M_1 Q(\ell)$ are directly linked to the p-adic valuation of the denominators of the Bernoulli numbers $\frac{B_{2t}}{4t}$ through the appearance of the Bernoulli numbers in the q-expansions of the Eisenstein series. The relationship was originally made precise by G. Laures [Lau96], and rederived by A. Baker in [Bak99], where Hecke operations are used to conclude that Eisenstein series generate the 1-line of the TMF-Adams Novikov spectral sequence. Our analysis is closely related to these.

As a consequence of our study of the chromatic spectral sequence for $Q(\ell)$ we are able to prove the following theorem.

Theorem (Theorem 12.1) The images of the elements $\alpha_{i/j}$ and the elements $\beta_{i/j,k}$ under the homomorphism

$$\pi_* S_{E(2)} \to \pi_* Q(\ell)$$

are non-trivial.

This theorem shows that the homotopy of the spectrum $Q(\ell)$ is closely tied to Greek letter phenomena. It also gives credibility to the author's hope that the following questions have affirmative answers.

(1) Are the homotopy Greek letter elements $\beta^h_{i/j,k}$ (see [Beh07b]) detected by the spectra $Q(\ell)$ at the primes 2 and 3?

(2) Do the spectra $Q_U(K^{p,\ell})$ (constructed using Shimura varieties of type $U(1,n-1)$ in [BLb]) detect the v_n-periodic Greek letter elements?

Organization of the paper.

In Section 2 we summarize the chromatic spectral sequence. We also recall Morava's change of rings theorem, which relates the terms of the chromatic spectral sequence to the cohomology of the Morava stabilizer groups $\mathbb{S}_n$.

In Section 3 we explain how to associate a p-complete TMF-spectrum to every compact open subgroup of the adele group $GL_2(\mathbb{A}^{p,\infty})$, following standard conventions used in the theory of modular forms. Certain E_∞-operations between these spectra are given by elements of $GL_2(\mathbb{A}^{p,\infty})$.

In Section 4 we explain how to use the GL_2 action of Section 3 to define spectra $Q(S)$ for a set of primes S. These spectra agree with the spectra $Q(\ell)$ defined by the author in [Beh07a], [Beh06] when $S = \{\ell\}$. The approach of this paper, however, mimics that of [BLb]. We explain how the results of [Beh07a], [BL06] arise in this framework. In particular, we identify the $K(2)$-localization of $Q(S)$ as the homotopy fixed points of a dense subgroup Γ_S of the Morava stabilizer group $\mathbb{S}_2$.

In Section 5 we explain how the building resolution of $Q(\ell)$ given in [Beh07a] can be recovered using the methods of [BLb]. We use this resolution to define a finite cochain complex $C^\bullet(\ell)$ of modular forms whose cohomology gives $\pi_* Q(\ell)$.

In Section 6 we express the first differential in $C^\bullet(\ell)$ in terms of the Verschiebung of modular forms.

In Section 7 we describe the chromatic spectral sequence of $Q(\ell)$. In particular, we show that its E_1-term consists of three lines: $M_0Q(\ell)$, $M_1Q(\ell)$, and $M_2Q(\ell)$. We explain how to modify the chain complex $C^\bullet(\ell)$ to compute these terms. We also relate $M_2Q(\ell)$ to the cohomology of the group Γ_ℓ.

In Section 8 we show that $\pi_t M_0Q(\ell)$ is concentrated in $t = 0, -1, -2$. We also deduce that the rest of $\pi_t Q(\ell)$ is p-torsion, and give bounds for the torsion.

In Section 9 we compute $\pi_{4t}M_1Q(\ell)$, and show that its generators can be expressed as Eisenstein series. The orders of these groups are given by the p-adic valuation of the Bernoulli numbers $B_{2t}/4t$.

In Section 10 we recall theorems of Serre and Swinnerton-Dyer, which relate congruences amongst q-expansions of modular forms to multiplication by the Hasse invariant E_{p-1}.

In Section 11 we analyze $\pi_* M_2Q(\ell)$, and prove Theorems 1.3 – 1.5.

In Section 12 we deduce Theorem 12.1 from our extensive knowledge of the chromatic spectral sequence for $Q(\ell)$.

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2 The chromatic spectral sequence

Let X be a spectrum. The chromatic tower for X is the tower of Bousfield localizations with respect to the Johnson-Wilson spectra $E(n)$.

$$\begin{array}{ccccccc} M_0X & & M_1X & & M_2X & & \\ \| & & \downarrow & & \downarrow & & \\ X_{E(0)} & \longleftarrow & X_{E(1)} & \longleftarrow & X_{E(2)} & \longleftarrow & \cdots \end{array}$$

Here the fibers M_nX are the nth monochromatic layers. They admit a presentation as

$$M_nX \simeq \operatorname*{hocolim}_I M(I)^0 \wedge X_{E(n)} \tag{2.1}$$

where $M(I)^0 = M(i_0, \ldots, i_{n-1})^0$ is the generalized Moore spectrum with top cell in dimension 0 and BP-homology,

$$BP_*M(i_0, \ldots, i_{n-1}) = \Sigma^{-\|I\|-n} BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$$

with

$$\|I\| := \sum_j 2i_j(p^j - 1)$$

and I ranges over a suitable cofinal collection of multi-indices. (This colimit presentation of the monochromatic layers was conjectured in [Rav84, 5.9]. The conjecture was resolved by the Hopkins-Smith periodicity theorem [HS98], which implies the existence of such a system of generalized Moore spectra.)

Applying homotopy to the chromatic tower yields the chromatic spectral sequence

$$E_1^{n,k} = \pi_k M_n X \Rightarrow \pi_{k-n} X_{(p)}$$

which is conditionally convergent if X is harmonic.

Morava's change of rings theorem [Mor85] states that the Adams-Novikov spectral sequence for M_nX takes the form

$$H_c^s(\mathbb{S}_n, (M_nE_n)_t(X))^{\mathrm{Gal}(\mathbb{F}_p)} \Rightarrow \pi_{t-s}(M_nX). \tag{2.2}$$

For $p \gg n$ this spectral sequence is known to collapse. A simple instance of this (for $X = S$) is given by the following lemma.

Lemma 2.3 For $X = S$ and $2p - 2 \geq \max\{n^2, 2n + 2\}$, the spectral sequence (2.2) collapses: the groups

$$H_c^s(\mathbb{S}_n, \pi_t M_n E_n)^{\mathrm{Gal}(\mathbb{F}_p)}$$

are zero unless $t \equiv 0 \mod 2(p-1)$.

Proof The action of an element a of the finite subgroup $\mathbb{F}_p^\times \subset \mathbb{S}_n$ on

$$\pi_{2t} M_n E_n \cong (\pi_{2t} E_n)/(p^\infty, v_1^\infty, \ldots, v_{n-1}^\infty).$$

is given by multiplication by $\widetilde{a}^t$, where $\widetilde{a}$ is the image of a under the Teichmüller embedding $\mathbb{F}_p^\times \subset \mathbb{Z}_p^\times$. Since $\mathbb{F}_p^\times$ is cyclic of order $p-1$, it follows that $\mathbb{F}_{p-1}$ acts trivially if and only if $t \equiv 0 \mod p-1$. Because the subgroup $\mathbb{F}_p^\times \subset \mathbb{S}_n$ is central and Galois invariant, it follows that there is an induced action of $\mathbb{F}_p^\times$ on

$$H_c^s(\mathbb{S}_n, \pi_{2t} M_n E_n)^{\mathrm{Gal}(\mathbb{F}_p)}$$

However, the induced action on cohomology must be trivial, because the action is obtained by restriction from the action of $\mathbb{S}_n$. Thus, the cohomology groups must be trivial, except when $t \equiv 0 \mod p-1$. The result follows from the fact that if $n < p-1$, the group $\mathbb{S}_n$ has cohomological dimension n^2 [Mor85]. □

The sparsity of the spectral sequence (2.2), together with the fact that $E_2^{s,t} = 0$ for $s \leq n^2$ implies the following corollary

Corollary 2.4 For $2(p-1) \geq \max\{n^2, 2(n+1)\}$, we have

$$\pi_t M_n S \cong H_c^s(\mathbb{S}_n, \pi_{t+s} M_n E_n)^{\mathrm{Gal}(\mathbb{F}_p)}$$

where $t = 2k(p-1) - s$ and $0 \leq s < 2(p-1)$.

3 Adelic formulation of TMF

Let $\mathbb{A}$ denote the rational adeles. For a set of finite places S of $\mathbb{Q}$, define

$$\widehat{\mathbb{Z}}^S := \prod_{\ell \notin S} \mathbb{Z}_\ell,$$
$$\mathbb{A}^{S,\infty} := \widehat{\mathbb{Z}}^S \otimes \mathbb{Q}.$$

Fix a rank 2 module:

$$V^p := \mathbb{A}^{p,\infty} \oplus \mathbb{A}^{p,\infty},$$

and let $L^p = \widehat{\mathbb{Z}}^p \oplus \widehat{\mathbb{Z}}^p$ be the canonical lattice contained in V^p.

For an elliptic curve C over an algebraically closed field k of characteristic unequal to ℓ, let

$$T_\ell(C) := \varprojlim_k C(k)[\ell^k]$$

denote the ℓ-adic Tate module of C. The Tate module $T_\ell(C)$ is a free $\mathbb{Z}_\ell$-module of rank 2 [Sil86]. If the characteristic of k is zero or p, the ℓ-adic Tate modules assemble to give the $\mathbb{A}^{p,\infty}$-module

$$V^p(C) := T^p(C) \otimes \mathbb{Q},$$

where

$$T^p(C) := \prod_{\ell \neq p} T_\ell(C).$$

There is a canonical short exact sequence

$$T^p(C) \to V^p(C) \xrightarrow{u} C[tor^p]$$

where $C[tor^p]$ is the subgroup of the group of k-points of C consisting of those points which are torsion of order prime to p.

A *rational uniformization* is an isomorphism

$$\eta : V^p \xrightarrow{\cong} V^p(C).$$

The group $GL_2(\mathbb{A}^{p,\infty})$ acts on the set of rational uniformizations by precomposition.

As explained in [BLb, Sec. 3.2], a rational uniformization η of $V^p(C)$ gives rise to a prime-to-p quasi-isogeny

(3.1) $$\phi_\eta : C \to C_\eta$$

(up to isomorphism of C_η). If the uniformization has the property that

(3.2) $$T^p(C) \subseteq \eta(L^p),$$

the quasi-isogeny is an isogeny. In this case, the (isomorphism class of the) isogeny ϕ_η is characterized by its kernel H_η, which is given by:

$$\ker(\phi) = H_\eta := \text{image}(L^p \hookrightarrow V^p \xrightarrow{\eta} V^p(C) \xrightarrow{u} C[tor^p]).$$

(The case of more general η, not satisfying (3.2), is easily generalized from this, producing quasi-isogenies ϕ_η which need not be isogenies.)

For a subgroup

$$K^p \subset GL_2(\mathbb{A}^{p,\infty}),$$

we let $[\eta]_{K^p}$ denote the K^p-orbit of rational uniformizations generated by η. The we shall refer to the orbit $[\eta]_{K^p}$ as an *K^p-level structure*. If we define

$$K_0^p := GL_2(\widehat{\mathbb{Z}}^p) \subset GL_2(\mathbb{A}^{p,\infty}),$$

then, given a rational uniformization

$$\eta : V^p \xrightarrow{\cong} V^p(C),$$

the isomorphism class of the quasi-isogeny ϕ_η depends *only* on the K_0^p level structure $[\eta]_{K_0^p}$.

If C is an elliptic scheme over a connected base S, we can pick a geometric point s of S and talk about level structures of the fiber C_s, *provided they are* $\pi_1(S,s)$-*invariant*. Given a $\pi_1(S,s)$-invariant K_0^p-level structure $[\eta]_{K_0^p}$ represented by a rational uniformization

$$\eta : V^p \xrightarrow{\eta} V^p(C_s),$$

(satisfying (3.2)) there is an associated subgroup

$$H_{\eta,s} < C_s[tor^p].$$

The $\pi_1(S,s)$-invariance of $[\eta]_{K_0^p}$ implies that $H_{\eta,s}$ extends to a local system over S, giving a subgroup

$$H_\eta < C,$$

and a corresponding isogeny

$$\phi_\eta : C \to C/H_\eta =: C_\eta.$$

Extending this to η not satisfying (3.2) associates to a $\pi_1(S,s)$-invariant K_0^p-level structure $[\eta]_{K_0^p}$ of C_s an quasi-isogeny $\phi_\eta : C \to C_\eta$ of elliptic schemes over S.

Associated to a compact open subgroup

$$K^p \subset GL_2(\mathbb{A}^{p,\infty})$$

is a Deligne-Mumford stack $\mathcal{M}(K^p)$ over $\mathbb{Z}_{(p)}$ of *elliptic curves with K^p-level structure* (see, for instance, [Hid04, 7.1.2], specialized to the group GL_2). For a connected scheme S over $\mathbb{Z}_{(p)}$ with a specified geometric point s, the S-points of $\mathcal{M}(K^p)$ are the groupoids whose objects are tuples

$$(C, [\eta]_{K^p})$$

where:

$$C = \text{elliptic scheme over } S,$$
$$[\eta]_{K^p} = \pi_1(S,s) \text{ invariant } K^p\text{-level structure on } C_s.$$

The morphisms of the groupoid of S-points of $\mathcal{M}(K^p)$

$$\alpha : (C, [\eta]_{K^p}) \to (C', [\eta']_{K^p})$$

are the prime-to-p quasi-isogenies

$$\alpha : C \to C'$$

for which

$$[\alpha_* \circ \eta]_{K^p} = [\eta']_{K^p}.$$

Remark 3.3

(1) If the compact open subgroup is given by

$$K_0^p := GL_2(\widehat{\mathbb{Z}}^p) \subset GL_2(\mathbb{A}^{p,\infty})$$

then there is an isomorphism

$$\mathcal{M}(K_0^p) \xrightarrow{\cong} \mathcal{M}_{ell} \otimes \mathbb{Z}_{(p)}$$
$$(C, [\eta]_{K_0^p}) \mapsto C_\eta$$

where $\mathcal{M}_{ell}/\mathbb{Z}$ is the usual (uncompactified) moduli stack of elliptic curves (see, for instance, [Kud03]).

(2) If the compact open subgroup is given by

$$K_0^p(\ell) := GL_2(\widehat{\mathbb{Z}}^{p,\ell})K_0(\ell) \subset GL_2(\mathbb{A}^{p,\infty}),$$

where $K_0(\ell) \subset GL_2(\mathbb{Z}_\ell)$ is the subgroup of matrices given by

$$K_0(\ell) = \left\{ A \in GL_2(\mathbb{Z}_\ell) \ : \ A \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \mod \ell \right\},$$

then there is an isomorphism [Kud03]

$$\begin{aligned} \mathcal{M}(K_0^p(\ell)) &\xrightarrow{\cong} \mathcal{M}(\Gamma_0(\ell)) \otimes_{\mathbb{Z}[1/\ell]} \mathbb{Z}_{(p)}, \\ (C, [\eta]_{K_0^p(\ell)}) &\mapsto (C_\eta, N_\eta) \end{aligned}$$

where $\mathcal{M}(\Gamma_0(\ell))$ is the moduli stack of elliptic curves with a $\Gamma_0(\ell)$-structure, and N_η is the $\Gamma_0(\ell)$-structure (subgroup of order ℓ) of C associated to the image of the composite

$$\ell^{-1}\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \to V^p \xrightarrow{\eta} V^p(C) \xrightarrow{(\phi_\eta)_*} V^p(C_\eta).$$

(3) If $K_1^p < K_2^p$ is a pair of compact open subgroups, then there is an induced étale cover of moduli stacks:

$$\begin{aligned} \mathcal{M}(K_1^p) &\to \mathcal{M}(K_2^p), \\ (C, [\eta]_{K_1^p}) &\mapsto (C, [\eta]_{K_2^p}). \end{aligned}$$

If K_1^p is normal in K_2^p, the cover is a torsor for K_2^p/K_1^p.

(4) An element $g \in GL_2(\mathbb{A}^{p,\infty})$ gives rise to an isomorphism of stacks

$$\begin{aligned} g^* : \mathcal{M}(gK^pg^{-1}) &\to \mathcal{M}(K^p), \\ (C, [\eta]_{gK^pg^{-1}}) &\mapsto (C, [\eta \circ g]_{K^p}). \end{aligned}$$

Clearly we have

(3.4) $$g^* = (gk)^* \quad \text{for } k \in K^p.$$

The moduli of p-divisible groups corresponding to the moduli space $\mathcal{M}(K^p)$ satisfies Lurie's generalization of the Hopkins-Miller theorem [BLb, Sec. 8.1], and hence the p-completion $\mathcal{M}(K^p)^\wedge_p$ carries a presheaf of p-complete E_∞-ring spectra $\mathcal{E}_{K^p}$ on the site $(\mathcal{M}(K^p)^\wedge_p)_{et}$ such that:

(1) The presheaf $\mathcal{E}_{K^p}$ satisfies homotopy hyperdescent (i.e. it is Jardine fibrant).

(2) For an affine étale open

$$\mathrm{Spf}(R) \xrightarrow{(C,[\eta]_{K^p})} \mathcal{M}(K^p)^\wedge_p$$

the corresponding spectrum of sections

$$E = \mathcal{E}_{K^p}(\mathrm{Spf}(R))$$

is a weakly even periodic elliptic spectrum associated to the elliptic curve C (i.e. $\pi_0(E) = R$, and the formal group associated to E is isomorphic to the formal group of C).

Define $\mathrm{TMF}(K^p)$ to be the global sections

$$\mathrm{TMF}(K^p) := \mathcal{E}_{K^p}(\mathcal{M}(K^p)^\wedge_p).$$

In particular, we have

$$\mathrm{TMF}(K^p_0) = \mathrm{TMF}_p,$$
$$\mathrm{TMF}(K^p_0(\ell)) = \mathrm{TMF}_0(\ell)_p.$$

By the functoriality of Lurie's theorem [BLb, 8.1.4], the action of $GL_2(\mathbb{A}^{p,\infty})$ described in Remark 3.3 gives rise to maps of E_∞-ring spectra

(3.5) $$g_* : \mathrm{TMF}(K^p) \to \mathrm{TMF}(gK^pg^{-1}).$$

4 The spectra $Q(S)$

The collection of compact open subgroups K^p of $GL_2(\mathbb{A}^{p,\infty})$ under inclusion forms a filtered category, and we may take the colimit

(4.1) $$\mathcal{V} := \operatorname*{colim}_{K^p} \mathrm{TMF}(K^p).$$

The action of $GL_2(\mathbb{A}^{p,\infty})$ described above gives $\mathcal{V}$ the structure of a *smooth* $GL_2(\mathbb{A}^{p,\infty})$-*spectrum* [BLb, 10.3]. We may recover each of the spectra $\mathrm{TMF}(K^p)$ from $\mathcal{V}$ by taking homotopy fixed points [BLb, 10.6.5]:

$$\mathrm{TMF}(K^p) \simeq \mathcal{V}^{hK^p}.$$

For a set of primes S not containing p, we have an open subgroup

$$(K^{p,S}_0)_+ := GL_2(\mathbb{A}_S)K^{p,S}_0 \subset GL_2(\mathbb{A}^{p,\infty})$$

where $\mathbb{A}_S = \prod'_{\ell \in S} \mathbb{Q}_\ell$ is the ring of S-adeles and

$$K_0^{p,S} = \prod_{\ell \notin \{p\} \cup S} GL(\mathbb{Z}_\ell).$$

We define a spectrum

$$Q(S) = \mathcal{V}^{h(K_0^{p,S})_+}.$$

The $K(2)$-localization of the spectrum $Q(S)$ is closely related to the $K(2)$-local sphere, as we now explain. Let C_0 be a fixed supersingular curve over $\overline{\mathbb{F}}_p$ (any two are isogenous). Assume (for convenience) that C_0 is defined over $\mathbb{F}_p$ (such a curve exists for every prime p [Wat69]). The quasi-endomorphism ring

$$D := \mathrm{End}^0(C_0)$$

is a quaternion algebra over $\mathbb{Q}$ ramified at p and ∞. The subring of actual endomorphisms

$$\mathcal{O}_D := \mathrm{End}(C_0) \subset D$$

is a maximal order. For our set of primes S, define a group

$$\Gamma_S := (\mathcal{O}_D[S^{-1}])^\times.$$

The group Γ_S is the group of quasi-isogenies of C_0 whose degree lies in

$$\mathbb{Z}[S^{-1}]^\times \subset \mathbb{Q}^\times.$$

The group Γ_S embeds in the (profinite) Morava stabilizer group through its action on the height 2 formal group of C_0:

$$\Gamma_S \hookrightarrow \mathrm{Aut}(\widehat{C}_0) \cong \mathbb{S}_2.$$

Theorem 4.2 (Behrens-Lawson [BL06]) If p is odd, and S contains a generator of $\mathbb{Z}_p^\times$, then the subgroup

$$\Gamma_S \hookrightarrow \mathbb{S}_2$$

is dense.

The universal deformation $\widetilde{C}_0$ of the supersingular curve C_0, by Serre-Tate theory , gives Morava E-theory E_2 the structure of an elliptic spectrum, where

$$\pi_* E_2 = W(\overline{\mathbb{F}}_p)[[u_1]][u^{\pm 1}].$$

Since C_0 is assumed to admit a definition over $\mathbb{F}_p$, there is an action of the Galois group $\mathrm{Gal}(\mathbb{F}_p)$ on the spectrum E_2. Picking a fixed rational uniformization

$$\eta_0 : V^p \xrightarrow{\cong} V^p(C_0)$$

gives, for every K^p, a canonical map of E_∞-ring spectra

$$\mathrm{TMF}(K^p) \xrightarrow{(\widetilde{C}_0,[\eta_0]_{K^p})} E_2$$

classifying the pair $(\widetilde{C}_0, [\eta_0]_{K^p})$, thus a map

$$\mathcal{V} \xrightarrow{(\widetilde{C}_0,\eta_0)} E_2. \tag{4.3}$$

Using the Tate embedding

$$\mathrm{End}^0(C_0) \hookrightarrow \mathrm{End}(V^p(C_0)),$$

the rational uniformization η_0 induces an inclusion

$$\begin{aligned} \gamma : \Gamma_S &\hookrightarrow (K_0^{p,S})_+ \subseteq GL_2(\mathbb{A}^{p,\infty}), \\ \alpha &\mapsto \eta_0^{-1}\alpha\eta_0. \end{aligned}$$

Lemma 4.4

(1) For $\alpha \in \Gamma_S$, the following diagram commutes

$$\begin{array}{ccc} \mathrm{TMF}(K^p) & \xrightarrow{(\widetilde{C}_0,\eta_0)} & E_2 \\ {\scriptstyle \gamma(\alpha)_*}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ \mathrm{TMF}(\gamma(\alpha)K^p\gamma(\alpha)^{-1}) & \xrightarrow[(\widetilde{C},\eta_0)]{} & E_2 \end{array}$$

where $\gamma(\alpha)_*$ is the morphism induced by the action of $GL_2(\mathbb{A}^{p,\infty})$ on TMF, and α_* is the morphism induced by the action of the Morava stabilizer group on E_2 through the inclusion $\Gamma_S \subset \mathbb{S}_2$.

(2) The map

$$\mathrm{TMF}(K^p) \xrightarrow{(\widetilde{C}_0,\eta_0)} E_2$$

is invariant under the action of $\mathrm{Gal}(\mathbb{F}_p)$ on E_2.

Proof Let $\mathrm{Def}(C_0)$ denote the formal moduli of deformations of C_0. For a complete local ring $(R, \mathfrak{m})$, the R-points of $\mathrm{Def}(C_0)$ is the category of tuples

$$(C, \iota, \beta),$$

where

$$\begin{aligned} &C = \text{elliptic curve over } R, \text{ with reduction } \overline{C} \bmod \mathfrak{m}, \\ &\iota : \overline{\mathbb{F}}_p \to R/\mathfrak{m}, \\ &\beta : \iota^* C_0 \xrightarrow{\cong} \overline{C}. \end{aligned}$$

The element $\alpha \in \Gamma_S$ acts on the R-points of $\mathrm{Def}(C_0)$ by

$$\alpha^* : (C, \iota, \beta) \mapsto (C, \iota, \beta \circ \alpha).$$

By Serre-Tate theory, this space is equivalent to the formal moduli $\mathrm{Def}(\widehat{C}_0)$ of deformations of the associated height 2 formal group $\widehat{C}_0$, and the action of Γ_S on $\mathrm{Def}(C_0)$ is compatible with the action of $\mathbb{S}_2$ on $\mathrm{Def}(\widehat{C}_0)$. Part (1) of the lemma follows from the commutativity of the following diagram, which is easily checked on R-points.

$$\begin{array}{ccc} \mathcal{M}(K^p) & \xleftarrow{(C_{\mathrm{univ}}, [(\beta_{\mathrm{univ}})_* \eta_0])} & \mathrm{Def}(C_0) \\ {\scriptstyle \gamma(\alpha)^*}\uparrow & & \uparrow{\scriptstyle \alpha^*} \\ \mathcal{M}(\gamma(\alpha) K^p \gamma(\alpha)^{-1}) & \xleftarrow[(C_{\mathrm{univ}}, [(\beta_{\mathrm{univ}})_* \eta_0])]{} & \mathrm{Def}(C_0) \end{array}$$

Part (2) is checked in a similar manner. □

Lemma 4.4 implies that the morphism (**4.3**) descends to give a morphism

$$Q(S) = \mathcal{V}^{h(K^{p,S})_+} \to \left(E_2^{h\Gamma_S}\right)^{h\,\mathrm{Fr}_p} =: E(\Gamma_S). \tag{**4.5**}$$

Here, if X is a spectrum with an action of the Frobenius $\mathrm{Fr}_p \in \mathrm{Gal}(\mathbb{F}_p)$, the spectrum $X^{h\,\mathrm{Fr}_p}$ is defined to be the homotopy fiber

$$X^{h\,\mathrm{Fr}_p} \to X \xrightarrow{\mathrm{Fr}_p - 1} X.$$

The following theorem is proved in [Beh07a] in the case where S consists of one prime. The proof of the more general case is identical to the proof of Corollary 14.5.6 of [BLb].

Theorem 4.6 The map (**4.5**) induces an equivalence

$$Q(S)_{K(2)} \to E(\Gamma_S).$$

5 The building resolution

If $S = \{\ell\}$ is a set containing one prime, the spectrum $Q(\ell)$ defined in Section 4 is equivalent to the spectrum constructed in [Beh07a]. In [Beh07a], the spectrum $Q(\ell)$ was defined to be the totalization of a certain semi-cosimplicial E_∞-ring spectrum. This description is recovered as follows.

The group $GL_2(\mathbb{Q}_\ell)$ acts on its building $\mathcal{B} = \mathcal{B}(GL_2(\mathbb{Q}_\ell))$ with compact open stabilizers. Explicitly, the building $\mathcal{B}$ is equivariantly homeomorphic to the geometric realization of a semi-simplicial $GL_2(\mathbb{Q}_\ell)$-set $\mathcal{B}_\bullet$ of the form

$$\mathcal{B}_\bullet = \left(GL_2(\mathbb{Q}_\ell)/GL_2(\mathbb{Z}_\ell) \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \begin{array}{c} GL_2(\mathbb{Q}_\ell)/K_0(\ell) \\ \times \\ GL_2(\mathbb{Q}_\ell)/GL_2(\mathbb{Z}_\ell) \end{array} \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} GL_2(\mathbb{Q}_\ell)/K_0(\ell). \right) \tag{5.1}$$

The action of $GL_2(\mathbb{Q}_\ell)$ on the building $\mathcal{B}$ extends to an action of $(K_0^{p,\ell})_+$, simply by letting the local factors away from ℓ act trivially. Regarded as a semi-simplicial $(K_0^{p,\ell})_+$-set, we have

$$\mathcal{B}_\bullet = \left((K_0^{p,\ell})_+/K_0^p \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \begin{array}{c} (K_0^{p,\ell})_+/K_0^p(\ell) \\ \times \\ (K_0^{p,\ell})_+/K_0^p \end{array} \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} (K_0^{p,\ell})_+/K_0^p(\ell) \right). \tag{5.2}$$

The canonical $(K_0^{p,\ell})_+$-equivariant morphism

$$\mathcal{V} \xrightarrow{c} \underline{\mathrm{Map}}(\mathcal{B}, \mathcal{V})^{sm} \tag{5.3}$$

(given by the inclusion of the constant functions) is an equivalence [BLb, Lem. 13.2.3]. Here, $\underline{\mathrm{Map}}(-,-)^{sm}$ is defined to be the colimit of the U-fixed point spectra, as U ranges over the open subgroups of $GL_2(\mathbb{A}^{p,\infty})$. The argument in [BLb] relies on the fact that the building $\mathcal{B}$ is not only contractible, but possesses a canonical contracting homotopy with excellent equivariance properties.

The semi-simplicial decomposition of $\mathcal{B}$ induces an equivariant equivalence

$$\underline{\mathrm{Map}}(\mathcal{B}, \mathcal{V})^{sm} \simeq \mathrm{Tot}\, \underline{\mathrm{Map}}(\mathcal{B}_\bullet, \mathcal{V})^{sm}$$

and therefore an equivalence on fixed point spectra:

$$\begin{aligned} Q(\ell) &= \mathcal{V}^{h(K_0^{p,\ell})_+} \\ &\simeq \mathrm{Tot}\left(\underline{\mathrm{Map}}(\mathcal{B}_\bullet, \mathcal{V})^{sm}\right)^{h(K_0^{p,\ell})_+}. \end{aligned}$$

Using Shapiro's lemma in the context of smooth $(K_0^{p,\ell})_+$-spectra gives an equivalence

$$(\underline{\mathrm{Map}}((K_0^{p,\ell})_+/K^p, \mathcal{V})^{sm})^{h(K_0^{p,\ell})_+} \simeq \mathcal{V}^{hK^p}.$$

Since we have

$$\begin{aligned} \mathcal{V}^{hK_0^p} &\simeq \mathrm{TMF}(K_0^p) = \mathrm{TMF}_p \\ \mathcal{V}^{hK_0^p(\ell)} &\simeq \mathrm{TMF}(K_0^p(\ell)) = \mathrm{TMF}_0(\ell)_p \end{aligned}$$

Thus there is an induced semi-cosimplicial decomposition

$$Q(\ell) \simeq \mathrm{Tot}\, Q(\ell)^\bullet \tag{5.4}$$

where

$$Q(\ell)^\bullet = \begin{pmatrix} & \mathrm{TMF}_0(\ell)_p & \to \\ \mathrm{TMF}_p \ \begin{matrix}\to\\ \to\end{matrix} & \times & \to\ TMF_0(\ell)_p \\ & \mathrm{TMF}_p & \to \end{pmatrix}. \tag{5.5}$$

For $p \geq 5$, the homotopy groups of TMF_p and $\mathrm{TMF}_0(\ell)_p$ are concentrated in even degrees, and there are isomorphisms

$$\pi_{2k}\,\mathrm{TMF}_p \cong (M_k)^0_{\mathbb{Z}_p},$$
$$\pi_{2k}\,\mathrm{TMF}_0(\ell)_p \cong M_k(\Gamma_0(\ell))^0_{\mathbb{Z}_p}.$$

Applying homotopy to the semi-cosimplicial spectrum $Q(\ell)^\bullet$ (5.5) gives a semi-cosimplicial abelian group

$$C(\ell)^\bullet_{2k} := \begin{pmatrix} & M_k(\Gamma_0(\ell))^0_{\mathbb{Z}_p} & \to \\ (M_k)^0_{\mathbb{Z}_p} \ \begin{matrix}\to\\ \to\end{matrix} & \times & \to\ M_k(\Gamma_0(\ell))^0_{\mathbb{Z}_p} \\ & (M_k)^0_{\mathbb{Z}_p} & \to \end{pmatrix}. \tag{5.6}$$

The Bousfield-Kan spectral sequence for $Q(\ell)^\bullet$ takes the form

$$E_1^{s,t} = C(\ell)^s_t \Rightarrow \pi_{t-s}Q(\ell). \tag{5.7}$$

Proposition 5.8 For $p \geq 5$, the spectral sequence (5.7) collapses at E_2 to give an isomorphism

$$\pi_n Q(\ell) \cong H^0(C(\ell)^\bullet_n) \oplus H^1(C(\ell)^\bullet_{n+1}) \oplus H^2(C(\ell)^\bullet_{n+2}).$$

Proof The rings of modular forms M_* and $M_*(\Gamma_0(\ell))$ are concentrated in even weights. This easily follows in the case of $\Gamma_0(\ell)$ from the fact that the inversion $[-1] : C \to C$ gives an automorphism of any $\Gamma_0(\ell)$-structure. Thus there is no room for differentials, or hidden extensions, in the spectral sequence (5.7). □

In fact, since we have argued that $H^s(C(\ell)^\bullet_t)$ is non-zero unless $t \equiv 0 \mod 4$, we have

Corollary 5.9 For $p \geq 5$, there are isomorphisms

$$\pi_t Q(\ell) \cong H^s(C(\ell)^\bullet)_{t+s}$$

where $t = 4k - s$ and $0 \leq s < 4$.

6 Effect of coface maps on modular forms

Suppose that $p \geq 5$. In this section we will deduce the effect of the two initial cosimplicial coface maps of $C(\ell)^\bullet_{2k}$ on the level of q-expansions. To aid in this, we recall from [Beh07a] that the semi-cosimplicial resolution of $Q(\ell)$ may be constructed by applying the Goerss-Hopkins-Miller presheaf to a semi-simplicial object in the site $(\mathcal{M}_{ell})_{et}$:

$$\mathcal{M}_\bullet := \left((\mathcal{M}_{ell})_p \;\begin{array}{c}\leftarrow\\ \leftarrow\end{array}\; \begin{array}{c}\mathcal{M}(\Gamma_0(\ell))_p\\ \amalg \\ (\mathcal{M}_{ell})_p\end{array} \;\begin{array}{c}\leftarrow\\ \leftarrow\\ \leftarrow\end{array}\; \mathcal{M}(\Gamma_0(\ell))_p \right). \tag{6.1}$$

The coface maps $d_i : \mathcal{M}_1 \to \mathcal{M}_0$ are given on R-points by

$$\begin{aligned} d_i : (\mathcal{M}_{ell})_p &\to (\mathcal{M}_{ell})_p \\ d_0 : C &\mapsto C/C[\ell] \\ d_1 : C &\mapsto C \\ d_i : (\mathcal{M}(\Gamma_0(\ell)))_p &\to (\mathcal{M}_{ell})_p \\ d_0 : (C,H) &\mapsto C/H \\ d_1 : (C,H) &\mapsto C \end{aligned}$$

Proposition 6.2 Consider the morphisms

$$d_0, d_1 : (M_k)^0_{\mathbb{Z}_p} \to M_k(\Gamma_0(\ell))^0_{\mathbb{Z}_p} \times (M_k)^0_{\mathbb{Z}_p}$$

induced by the initial coface maps of the cosimplicial abelian group $C(\ell)^\bullet_{2k}$. On the level of q-expansions, the maps are given by

$$\begin{aligned} d_0(f(q)) &:= (\ell^k f(q^\ell), \ell^k f(q)), \\ d_1(f(q)) &:= (f(q), f(q)). \end{aligned}$$

Proof It is clear from the description of the map d_1 that its effect on q-expansions is as given. Choosing an embedding $\mathbb{Z}_p \hookrightarrow \mathbb{C}$, by the q-expansion principle, it suffices to verify these identities hold when we base-change to $\mathbb{C}$ and consider the Tate curve:

$$C_q := \mathbb{C}^\times / q^{\mathbb{Z}}.$$

The group of ℓth roots of unity $\mu_\ell \subset \mathbb{C}^\times$ induces a $\Gamma_0(\ell)$-structure on the Tate curve C_q:

$$\mu_\ell \subset \mathbb{C}^\times / q^{\mathbb{Z}} = C_q.$$

This level structure is the kernel of the isogeny

$$\begin{aligned}\phi_{\mu_\ell} : C_q = \mathbb{C}^\times/q^{\mathbb{Z}} &\to \mathbb{C}^\times/q^{\ell\mathbb{Z}} = C_{q^\ell},\\ z &\mapsto z^\ell.\end{aligned}$$

The invariant differential dz/z on C_q transforms under this isogeny by

$$\phi^*(dz/z) = \ell dz/z.$$

It follows that d_0 on the $M_k(\Gamma_0(\ell))^0_{\mathbb{Z}_p}$-component is given by

$$\begin{aligned}d_0 : (M_k)^0_{\mathbb{Z}_p} &\to M_k(\Gamma_0(\ell))^0_{\mathbb{Z}_p},\\ f(q) &\mapsto \ell^k f(q^\ell),\end{aligned}$$

as desired. The ℓth power map

$$\begin{aligned}[\ell] : C_q &\to C_q\\ z &\mapsto z^\ell\end{aligned}$$

transforms the invariant differential by

$$[\ell]^*(dz/z) = \ell dz/z.$$

If follows that the $(M_k)^0_{\mathbb{Z}_p}$ component of d_0 is given by

$$\begin{aligned}d_0 : (M_k)^0_{\mathbb{Z}_p} &\to (M_k)^0_{\mathbb{Z}_p},\\ f(q) &\mapsto \ell^k f(q).\end{aligned}$$

□

7 The chromatic spectral sequence for $Q(\ell)$

The following lemma implies that the chromatic resolution of $Q(\ell)$ is finite.

Lemma 7.1 The spectrum $Q(\ell)$ is $E(2)$-local.

Proof The spectra TMF_p and $\mathrm{TMF}_0(\ell)_p$ are $E(2)$-local. By (5.4), the spectrum $Q(\ell)$ is $E(2)$-local. □

We deduce that the chromatic resolution for $Q(\ell)$ takes the following form.

$$\begin{array}{ccccc} M_0Q(\ell) & & M_1Q(\ell) & & M_2Q(\ell) \\ \| & & \downarrow & & \downarrow \\ Q(\ell)_{E(0)} & \longleftarrow & Q(\ell)_{E(1)} & \longleftarrow & Q(\ell) \end{array}$$

Applying homotopy, we get a three line spectral sequence

$$E_1^{n,k} = \left\{ \begin{array}{ll} \pi_k M_n Q(\ell), & n \leq 2 \\ 0, & n > 2 \end{array} \right\} \Rightarrow \pi_{k-n} Q(\ell). \tag{7.2}$$

Assuming that $p \geq 5$, applying M_n to the cosimplicial resolution (5.4), we get spectral sequences

$$E_2^{s,t} = H^s(C(\ell)^\bullet[p^{-1}])_t \Rightarrow \pi_{t-s} M_0 Q(\ell) \tag{7.3}$$

$$E_2^{s,t} = H^s(C(\ell)^\bullet/p^\infty[v_1^{-1}])_t \Rightarrow \pi_{t-s} M_1 Q(\ell) \tag{7.4}$$

$$E_2^{s,t} = H^s(C(\ell)^\bullet/(p^\infty, v_1^\infty))_t \Rightarrow \pi_{t-s} M_2 Q(\ell) \tag{7.5}$$

Here, the E_2-terms are the cohomology of the cosimplicial abelian group obtained from applying the functor $\pi_*(M_n -)$ to (5.4). The values of the resulting cosimplicial abelian group are given by the following lemma.

Lemma 7.6 Let $p \geq 5$ and $(N, p) = 1$. Then

$$\begin{aligned} \pi_{2*} M_0 \operatorname{TMF}_0(N)_p &= M_*(\Gamma_0(N))^0_{\mathbb{Q}_p}, \\ \pi_{2*} M_1 \operatorname{TMF}_0(N)_p &= M_*(\Gamma_0(N))^0_{\mathbb{Z}_p}/p^\infty[E_{p-1}^{-1}], \\ \pi_{2*} M_2 \operatorname{TMF}_0(N)_p &= M_*(\Gamma_0(N))^0_{\mathbb{Z}_p}/(p^\infty, E_{p-1}^\infty). \end{aligned}$$

Proof This is a direct application of (2.1). Here, E_{p-1} is the $(p-1)$st Eisenstein series, which reduces to the Hasse invariant v_1 mod p [Kat73, Sec. 2.1]. □

In particular, since $C(\ell)^\bullet_t$ is non-zero only when $t \equiv 0 \mod 4$, the same argument proving Corollary 5.9 gives:

Corollary 7.7 For $p \geq 5$, there are isomorphisms

$$\begin{aligned} \pi_t M_0 Q(\ell) &\cong H^s(C(\ell)^\bullet[p^{-1}])_{t+s} \\ \pi_t M_1 Q(\ell) &\cong H^s(C(\ell)^\bullet/(p^\infty)[v_1^{-1}])_{t+s} \\ \pi_t M_2 Q(\ell) &\cong H^s(C(\ell)^\bullet/(p^\infty, v_1^\infty))_{t+s} \end{aligned}$$

where $t = 4k - s$ and $0 \leq s < 4$.

We end this section by relating $M_2Q(\ell)$ to the subgroup $\Gamma_\ell \subset \mathbb{S}_n$. By Theorem 4.6 there are equivalences

$$M_2Q(\ell) \simeq M_2(Q(\ell)_{K(2)}) \simeq M_2((E_2^{h\Gamma_\ell})^{h\,\mathrm{Fr}_p}).$$

We recall the following result from [Beh07a].

Proposition 7.8 The group Γ_ℓ acts on the building $\mathcal{B}$ for $GL_2(\mathbb{Q}_\ell)$ with finite stabilizers, given by groups of automorphisms of supersingular curves.

We deduce the following.

Lemma 7.9 There is an equivalence

$$M_2((E_2^{h\Gamma_\ell})^{h\,\mathrm{Fr}_p}) \simeq ((M_2E_2)^{h\Gamma_\ell})^{h\,\mathrm{Fr}_p}.$$

Proof Since the spectra $M(I)^0$ are finite, we have

$$M(I)^0 \wedge ((E_2^{h\Gamma_\ell})^{h\,\mathrm{Fr}_p}) \simeq ((M(I)^0 \wedge E_2)^{h\Gamma_\ell})^{h\,\mathrm{Fr}_p}.$$

The result would follow from (2.1) if we could commute the homotopy colimit over I with the homotopy fixed points with respect to Γ_ℓ. However, by Proposition 7.8, the group Γ_ℓ acts on the building $\mathcal{B}$ for $GL_2(\mathbb{Q}_\ell)$ with finite stabilizers. Since $\mathcal{B}$ is contractible and finite dimensional, we conclude that the group Γ_ℓ has finite virtual cohomological dimension. □

We conclude that there is an equivalence

$$M_2Q(\ell) \simeq ((M_2E_2)^{h\Gamma_\ell})^{h\,\mathrm{Fr}_p}$$

and a homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s(\Gamma_\ell, \pi_t M_2E_2)^{\mathrm{Gal}(\mathbb{F}_p)} \Rightarrow \pi_{t-s}M_2Q(\ell). \tag{7.10}$$

Lemma 7.11 For $p \geq 5$, we have:

(1) $H^s(\Gamma_\ell, \pi_t M_2E_2)^{\mathrm{Gal}(\mathbb{F}_p)} = 0$ for $s > 2$.

(2) $H^s(\Gamma_\ell, \pi_t M_2E_2)^{\mathrm{Gal}(\mathbb{F}_p)} = 0$ for $t \not\equiv 0 \mod 4$.

(3) There are isomorphisms

$$\pi_t M_2Q(\ell) \cong H^s(\Gamma_\ell, \pi_{t+s} M_2E_2)^{\mathrm{Gal}(\mathbb{F}_p)}$$

where $t = 4k - s$ and $0 \leq s < 4$.

Proof (1) follows from Proposition 7.8, together with the fact that the coefficients are p-local and the building is contractible and 2-dimensional. (2) follows from the fact that there is a central, Galois invariant element $[-1] \in \Gamma_\ell$ (given by inversion) which acts on $\pi_{2i}M_2E_2$ by $(-1)^i$. (3) follows from (1) and (2), using the spectral sequence (**7.10**). □

Combining Corollary 7.7 with Lemma 7.11, we get

Corollary 7.12 For $p \geq 5$, there are isomorphisms

$$H^s(C(\ell)^\bullet/(p^\infty, v_1^\infty))_t \cong H^s(\Gamma_\ell, \pi_t M_2E_2)^{\mathrm{Gal}(\mathbb{F}_p)}.$$

Remark 7.13 One could give a purely algebraic proof of Corollary 7.12 which makes no reference to topology. In the context of the exposition of this paper it happens to be quicker (but arguably less natural) to use topological constructions.

8 $M_0Q(\ell)$

Let $p \geq 5$ and ℓ be a topological generator of $\mathbb{Z}_p^\times$. In this section we will concern ourselves with locating the non-trivial homotopy of $M_0Q(\ell)$.

Proposition 8.1 The groups

$$H^s(C^\bullet(\ell))_{2t}$$

consist entirely if p^j-torsion if

$$t \equiv 0 \mod (p-1)p^{j-1},$$

and are zero if $t \not\equiv 0 \mod (p-1)$.

Proof Consider the central element

$$[\ell] := \begin{bmatrix} \ell & 0 \\ 0 & \ell \end{bmatrix} \in GL_2(\mathbb{Q}_\ell).$$

Let $\mathcal{V}$ be the smooth $GL_2(\mathbb{A}^{p,\infty})$-spectrum of (**4.1**). We assume that $\mathcal{V}$ is fibrant as a smooth $GL_2(\mathbb{A}^{p,\infty})$-spectrum, so that homotopy fixed points are equivalent to point-set level fixed points

$$\mathcal{V}^{hU} \simeq \mathcal{V}^U$$

for U an open subgroup of $GL_2(\mathbb{A}^{p,\infty})$ [BLb, Cor.10.5.5]. Because $[\ell]$ is central, the action of $[\ell]$ on $\mathcal{V}$ is $GL_2(\mathbb{A}^{p,\infty})$-equivariant. Because $[\ell]$ is contained in the subgroup $(K_0^{p,\ell})_+$, it acts as the identity on

$$Q(\ell) \simeq \mathcal{V}^{(K_0^{p,\ell})_+}.$$

However, the morphism c of (5.3) is compatible with the action of $[\ell]$, where we let $[\ell]$ act on $\underline{\mathrm{Map}}(\mathcal{B},\mathcal{V})^{sm}$ through its action on the target $\mathcal{V}$. We deduce that the endomorphism $[\ell]$ acts on the cosimplicial object $Q(\ell)^\bullet$, where the action is given level-wise on each factor by the endomorphism

$$[\ell] : \mathrm{TMF}(K^p) \to \mathrm{TMF}(K^p)$$

(where K^p is either K_0^p or $K_0^p(\ell)$). The endomorphism $[\ell]$ is the induced action of $[\ell]$ on the fixed point spectrum

$$\mathrm{TMF}(K^p) \simeq \mathcal{V}^{hK^p}.$$

The action of $[\ell]$ on the homotopy groups of $\mathrm{TMF}(K^p)$ is given by

$$\begin{aligned}[\ell] : \pi_{2k}\,\mathrm{TMF}(\Gamma_0(N)) &\to \pi_{2k}\,\mathrm{TMF}(\Gamma_0(N)),\\ f &\mapsto \ell^k f.\end{aligned}$$

This is easily deduced from the fact that the induced quasi-isogeny (3.1)

$$C_\eta \to C_{\eta\circ[\ell]}$$

is isomorphic to the ℓth power map of elliptic curves. It follows that

$$[\ell] : H^s(C(\ell)^\bullet_{2k}) \to H^s(C(\ell)^\bullet_{2k})$$

acts by multiplication by ℓ^k. However, since we have shown that $[\ell]$ acts by the identity on $\pi_* Q(\ell)$, Proposition 5.8 implies that $[\ell]$ acts by the identity on $H^s(C(\ell)^\bullet_{2k})$. We deduce that multiplication by $\ell^k - 1$ is the zero homomorphism on $H^s(C(\ell)^\bullet_{2k})$. Since ℓ was assumed to be a topological generator of $\mathbb{Z}_p^\times$, the proposition follows. □

We immediately deduce:

Corollary 8.2 We have

$$H^s(C^\bullet(\ell)[p^{-1}])_t = 0$$

for $t \neq 0$.

We can be more specific in the case of $s = 0$.

Lemma 8.3 We have

$$H^0(C^\bullet(\ell)[p^{-1}])_0 = \mathbb{Q}_p.$$

Proof We must analyze the kernel of the cosimplicial differential

$$d_0 - d_1 : (M_0)^0_{\mathbb{Q}_p} \to M_0(\Gamma_0(\ell))^0_{\mathbb{Q}_p} \oplus (M_0)^0_{\mathbb{Q}_p}.$$

We claim that is is given by the subspace generated by $1 \in (M_0)_{\mathbb{Q}_p}$. Indeed, suppose that $f \in (M_0)^0_{\mathbb{Q}_p}$ satisfies

$$d_0(f) - d_1(f) = 0.$$

By Proposition 6.2, it follows that

$$f(q^\ell) - f(q) = 0.$$

Writing $f(q) = \sum a_n q^n$, we find

$$a_n = \begin{cases} a_{n/\ell}, & n \equiv 0 \mod \ell \\ 0, & n \not\equiv 0 \mod \ell. \end{cases}$$

It follows by induction that $f(q) = a_0$. □

Applying this knowledge to the spectral sequence (7.3), we deduce:

Corollary 8.4 We have

$$\pi_t(M_0Q(\ell)) = 0$$

if $t \notin \{0, -1, -2\}$, and

$$\pi_0(M_0Q(\ell)) = \mathbb{Q}_p.$$

9 $M_1Q(\ell)$: Eisenstein series and the α-family

Let $p \geq 5$ and assume that ℓ is a topological generator of $\mathbb{Z}_p^\times$. In this section we will compute

$$H^0(C^\bullet(\ell)/p^\infty),$$

the 0th cohomology of the cochain complex associated to the cosimplicial abelian group $C^\bullet(\ell)$ tensored with the group $\mathbb{Z}/p^\infty$. These computations will allow us to determine part of the 1-line of the chromatic spectral sequence for $Q(\ell)$.

We have

$$H^0(C^\bullet(\ell)/p^\infty) = \varinjlim_j H^0(C^\bullet(\ell)/p^j),$$

so it suffices to compute the latter. Our explicit determination of the first differential in $C^\bullet(\ell)$ implies that

$$\mathcal{A}_{t/j} := H^0(C^\bullet(\ell)/p^j)_{2t} \tag{9.1}$$

$$= \left\{ f \in (M_t)^0_{\mathbb{Z}/p^j} \ : \ \begin{array}{l} \text{(i) } (\ell^t - 1)f(q) \equiv 0 \mod p^j, \\ \text{(ii) } \ell^t f(q^\ell) - f(q) \equiv 0 \mod p^j \end{array} \right\} \tag{9.2}$$

Lemma 9.3 A modular form $f \in M_t^0$ represents an element of the group $\mathcal{A}_{t/j}$ if and only if

(1) $p^i f \equiv 0 \mod p^j$ for $t = (p-1)p^{i-1}s$, and $(s,p) = 1$,

(2) $f(q) \equiv a \mod p^j$ for $a \in \mathbb{Z}/p^j$.

Proof Since ℓ was assumed to be a topological generator of $\mathbb{Z}_p$,

$$\nu_p(\ell^t - 1) = i$$

for $t = (p-1)p^{i-1}s$, with $(s,p) = 1$. Condition (i) of (9.2) states that

$$(\ell^t - 1)f \equiv 0 \mod p^j.$$

This proves (1).

Because $\ell^t f(q) \equiv f(q) \mod p^j$, we deduce that condition (ii) of (9.2) may be rewritten as

$$f(q^\ell) \equiv f(q) \mod p^j.$$

But, writing

$$f(q) = \sum_n a_n q^n$$

for $a_n \in \mathbb{Z}/p^j$, we see that

$$a_n \equiv \begin{cases} 0, & n \equiv 0 \mod \ell, \\ a_{n/\ell}, & n \not\equiv 0 \mod \ell. \end{cases}$$

Therefore, we inductively deduce that $a_n \equiv 0 \mod p^j$ unless $n = 0$. □

Let $E_k \in M_k$ denote the weight k normalized Eisenstein series (for $k \geq 4$ even), with q-expansion

$$E_k(q) = 1 - \frac{2k}{B_k} \sum_{i=1}^{\infty} \sigma_{k-1}(i) q^i \in \mathbb{Q}[[q]], \tag{9.4}$$

where

$$\sigma_k(i) := \sum_{d|i} d^k.$$

The following lemma follows immediately from the Clausen-von Staudt Theorem on denominators of Bernoulli numbers.

Lemma 9.5 If $p-1$ divides k, the q-expansion of E_k is p-integral. For $k \equiv 0 \mod (p-1)p^{j-1}$ we have

$$E_k(q) \equiv 1 \mod p^j.$$

Lemma 9.6 For each even weight $k \geq 4$ there exists a modular form

$$e_k \in (M_k)^0_{\mathbb{Z}_{(p)}}$$

such that

(1) if $k \equiv 0 \mod p-1$, we have

$$e_k = E_k,$$

(2) the q-expansion of e_k satisfies

$$e_k(q) = 1 + \text{higher terms},$$

(3) if $k_1 \equiv k_2 \mod (p-1)p^{j-1}$, then

$$e_{k_1}(q) \equiv e_{k_2}(q) \mod p^j.$$

Proof Observe that for any even $k \geq 4$, there exist modular forms e_k satisfying condition (2) (one can simply take $e_k = E_4^i E_6^j$ for appropriate i and j). Fix such choices of e_k for even k satisfying $4 \leq k < p-1$ and $k = p+1$. Also set $e_0 = 1$. For even $k \geq p-1$ satisfying $k \neq p+1$ set

$$e_k = e_{k-(p-1)t} E_{(p-1)t}$$

for t chosed such that

$$0 \leq k-(p-1)t < p-1 \quad \text{or} \quad k-(p-1)t = p+1.$$

Then condition (1) is obviously satisfied, and condition (3) is satisfied by Lemma 9.5. □

The following lemma provides a convenient basis for p-integral modular forms which we shall make frequent use of.

Lemma 9.7 The forms

$$\{\Delta^k e_{t-12k} \ : \ k \in \mathbb{Z}, t-12k \geq 4 \text{ and even}\},$$

together with

$$\Delta^k \qquad \text{if } t = 12k,$$

form an integral basis of M_t^0.

Proof Since

$$\Delta(q) = q + \cdots$$

we have

$$\Delta^k(q)e_{t-12k} = q^k + \cdots .$$

This establishes linear independence. We may deduce that these forms span M_t^0 by the explicit calculation

$$M_* = \mathbb{Z}[E_4, E_6, \Delta^{-1}]/(\Delta = \frac{E_4^3 - E_6^2}{1728}).$$

□

Proposition 9.8 The groups $\mathcal{A}_{t/\infty} = \operatorname{colim}_j \mathcal{A}_{t/j}$ are given by

$$\mathcal{A}_{t/\infty} = \mathbb{Z}/p^j\{E_t/p^j\}$$

for $t = (p-1)p^{j-1}s$, where $(s,p) = 1$ and $t \geq 4$, and

$$\mathcal{A}_{0/\infty} = \mathbb{Z}/p^\infty.$$

(Here, the element E_t/p^j is the image of the element $E_t \in \mathcal{A}_{t/j}$.)

Proof This follows immediately from Lemmas 9.3 and 9.7, provided we can show that E_t lies in $\mathcal{A}_{t/j}$. This again follows from criterion (2) of Lemma 9.3: by Lemma 9.5

$$E_t(q) \equiv 1 \mod p^j.$$

□

We obtain the zero-line of spectral sequence (**7.3**) as a corollary.

Corollary 9.9 We have

$$H^0(C^\bullet(\ell)/p^\infty[v_1^{-1}])_{2t} \cong \begin{cases} \mathbb{Z}/p^j, & t = (p-1)p^{j-1}s \text{ and } (s,p) = 1, \\ 0, & t \not\equiv 0 \mod (p-1). \end{cases}$$

Combining this with Corollary 7.7 and Proposition 8.1, we find:

Corollary 9.10 We have

$$\pi_t M_1 Q(\ell) \cong \begin{cases} \mathbb{Z}/p^j, & t = 2(p-1)p^{j-1}s \text{ and } (s,p) = 1, \\ 0, & t \not\equiv 0, -1, -2 \mod 2(p-1). \end{cases}$$

10 Mod p^j congruences

Let $p \geq 5$. The congruence

$$E_{p-1}(q) \equiv 1 \mod p$$

implies the congruence

(10.1)
$$E_{p-1}^{p^{j-1}}(q) \equiv 1 \mod p^j.$$

It follows that multiplication by $E_{p-1}^{p^{j-1}}$ induces an injection

$$\cdot E_{p-1}^{p^{j-1}} : M_t(\Gamma_0(N))_{\mathbb{Z}/p^j} \hookrightarrow M_{t+(p-1)p^{j-1}}(\Gamma_0(N))_{\mathbb{Z}/p^j}.$$

(Here we regard E_{p-1} as a modular form for $\Gamma_0(N)$.) The image of this inclusion is characterized by the following theorem of Serre [Kat73, Cor. 4.4.2].

Theorem 10.2 (Serre) Let f_i be an elements of $M_{k_i}(\Gamma_0(N))_{\mathbb{Z}/p^j}$ for $i = 1, 2$ and $k_1 < k_2$. Then

$$f_1(q) = f_2(q) \in \mathbb{Z}/p^j[[q]]$$

if and only if

(1) $k_1 \equiv k_2 \mod (p-1)p^{j-1}$, and

(2) $f_2 = E_{p-1}^{\frac{k_2-k_1}{p-1}} f_1$.

11 $M_2Q(\ell)$: The β-family congruences

Let $p \geq 5$, and let ℓ be a topological generator of $\mathbb{Z}_p^\times$. In this section we prove Theorem 1.3 and Theorem 1.4. The key observation is the following.

Lemma 11.1 The inclusion $\Gamma_\ell \hookrightarrow \mathbb{S}_2$ induces an isomorphism

$$H_c^0(\mathbb{S}_2, \pi_t M_2 E_2)^{\mathrm{Gal}(\mathbb{F}_p)} \xrightarrow{\cong} H^0(\Gamma_\ell, \pi_t M_2 E_2)^{\mathrm{Gal}(\mathbb{F}_p)}.$$

Proof By Theorem 4.2, the group Γ_ℓ is dense in $\mathbb{S}_2$. Since $\mathbb{S}_2$ acts continuously on $\pi_t M_2 E_2$, the invariants of $\mathbb{S}_2$ are the same as the invariants of Γ_ℓ. □

Combined with Corollary 7.12, we have an isomorphism

$$H^0(C(\ell)^\bullet/(p^\infty, v_1^\infty))_t \cong H_c^0(\mathbb{S}_2, \pi_t M_2 E_2)^{\mathrm{Gal}(\mathbb{F}_p)}.$$

The right-hand side has been computed by Miller-Ravenel-Wilson [MRW77]:

Theorem 11.2 (Miller-Ravenel-Wilson) The groups $H_c^0(\mathbb{S}_2, \pi_* M_2 E_2)^{\mathrm{Gal}(\mathbb{F}_p)}$ are generated by elements

$$\beta_{i/j,k} \in H_c^0(\mathbb{S}_2, \pi_{2i(p^2-1)-2j(p-1)} M_2 E_2)^{\mathrm{Gal}(\mathbb{F}_p)}$$

which generate cyclic summands of order p^k. Here, for $i = sp^n$ with $(s,p) = 1$, the indices j and k are taken subject to

(1) $p^{k-1} | j$,

(2) $j \leq p^{n-k+1} + p^{n-k} - 1$,

(3) either $j > p^{n-k} + p^{n-k-1} - 1$ or $p^k \not| j$.

We now compute

$$H^0(C^\bullet(\ell)/(p^\infty, v_1^\infty))_*$$

in terms of modular forms.

We have

$$H^0(C^\bullet(\ell)/(p^\infty, v_1^\infty))_{2t} = \varinjlim_k \varinjlim_{\substack{j = sp^{k-1} \\ s \geq 0}} H^0(C^\bullet(\ell)/(p^k, v_1^j))_{2t+2j(p-1)},$$

so it suffices to compute the latter. Proposition 6.2, Lemma 7.6, and Theorem 10.2

imply that, for $j \equiv 0 \mod p^{k-1}$, we have:

$$\begin{aligned}
\mathcal{B}_{t/j,k} &:= H^0(C^\bullet(\ell)/(p^k, v_1^j))_{2t+2j(p-1)} \\
&= \ker \left(\frac{M^0_{t+j(p-1)}}{(p^k, E^j_{p-1})} \xrightarrow{d_0 - d_1} \begin{array}{c} \frac{M^0_{t+j(p-1)}}{(p^k, E^j_{p-1})} \\ \oplus \\ \frac{M_{t+j(p-1)}(\Gamma_0(\ell))^0}{(p^k, E^j_{p-1})} \end{array} \right) \\
&= \ker \left(\frac{(M_{t+j(p-1)})^0_{\mathbb{Z}/p^k}}{(M_t)^0_{\mathbb{Z}/p^k}} \xrightarrow{d_0 - d_1} \begin{array}{c} \frac{(M_{t+j(p-1)})^0_{\mathbb{Z}/p^k}}{(M_t)^0_{\mathbb{Z}/p^k}} \\ \oplus \\ \frac{M_{t+j(p-1)}(\Gamma_0(\ell))^0_{\mathbb{Z}/p^k}}{M_t(\Gamma_0(\ell))^0_{\mathbb{Z}/p^k}} \end{array} \right) \\
&= \left\{ f \in \frac{(M_{t+j(p-1)})^0_{\mathbb{Z}/p^k}}{(M_t)^0_{\mathbb{Z}/p^k}} \ : \ \begin{array}{ll} \text{(i)} & (\ell^{t+j(p-1)} - 1) f(q) = g_1(q), \\ & \text{for } g_1 \in (M_t)^0_{\mathbb{Z}/p^k} \\ \text{(ii)} & \ell^{t+j(p-1)} f(q^\ell) - f(q) = g_2(q), \\ & \text{for } g_2 \in M_t(\Gamma_0(\ell))^0_{\mathbb{Z}/p^k} \end{array} \right\}.
\end{aligned}$$

Here, we are regarding the space of mod p^k modular forms of weight t as being embedded in the space of mod p^k modular forms of weight $t + j(p-1)$ through the inclusion induced by multiplication by E^j_{p-1} using Theorem 10.2.

For a finitely generated abelian p-group A, we shall say that $a \in A$ is an *additive generator of order* p^k if a generates a cyclic subgroup of A of order p^k.

Theorem 11.3 There is a one-to-one correspondence between the additive generators of order p^k in

$$H^0(C(\ell)^\bullet/(p^\infty, v_1^\infty))_{2t}$$

and the modular forms $f \in M^0_{t+j(p-1)}$ for $j \equiv 0 \mod p^{k-1}$ satisfying

(1) We have $t \equiv 0 \mod (p-1)p^{k-1}$.

(2) The q-expansion $f(q)$ is not congruent to 0 mod p.

(3) We have $\operatorname{ord}_q f(q) > \frac{t}{12}$ or $\operatorname{ord}_q f(q) = \frac{t-2}{12}$.

(4) There does not exist a form $f' \in M^0_{t'}$ such that $f'(q) \equiv f(q) \mod p^k$ for $t' < t + j(p-1)$.

$(5)_\ell$ There exists a form

$$g \in M_t(\Gamma_0(\ell))^0$$

satisfying

$$f(q^\ell) - f(q) \equiv g(q) \mod p^k.$$

We will need to make use of the following lemma.

Lemma 11.4 There exist homomorphisms

$$r_m : M^0_{t+m} \to M^0_t$$

such that, if $j \equiv 0 \mod p^{k-1}$, the short exact sequences

$$0 \to (M_t)^0_{\mathbb{Z}/p^k} \xrightarrow{\cdot E^j_{p-1}} (M_{t+j(p-1)})^0_{\mathbb{Z}/p^k} \to \frac{(M_{t+j(p-1)})^0_{\mathbb{Z}/p^k}}{(M_t)^0_{\mathbb{Z}/p^k}} \to 0$$

are split by the mod p^k reduction of $r_{j(p-1)}$.

Proof Using the basis of Lemma 9.7 we define explicit splitting morphisms

$$r_m : M_{t+m} \to M_t$$

whose effect on basis vectors is given by

$$r_m(\Delta^n e_{t+m-12n}) = \begin{cases} \Delta^n e_{t-12n}, & t-12n = 0, \text{ or } t-12n = 2i \text{ for } i \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

We just need to verify that $r_{j(p-1)}$ reduces to give the appropriate splittings. By Condition (3) of Lemma 9.6, and (**10.1**), we have

$$e_t(q)E^j_{p-1}(q) \equiv e_t(q) \equiv e_{t+j(p-1)}(q) \mod p^k.$$

We therefore compute

$$\begin{aligned} r_{j(p-1)}(\Delta^n e_{t-12n}E^j_{p-1}) &\equiv r_{j(p-1)}(\Delta^n e_{t+j(p-1)-12n}) \mod p^k \\ &\equiv \Delta^n e_{t-12n} \mod p^k. \end{aligned}$$

□

The splittings of Lemma 11.4 induce splitting homomorphisms which give short exact sequences

(11.5) $$0 \leftarrow (M_t)^0_{\mathbb{Z}/p^k} \xleftarrow{r_{j,k}} (M_{t+j(p-1)})^0_{\mathbb{Z}/p^k} \xleftarrow{\iota_{j,k}} \frac{(M_{t+j(p-1)})^0_{\mathbb{Z}/p^k}}{(M_t)^0_{\mathbb{Z}/p^k}} \leftarrow 0.$$

These short exact sequences are compatible as k and j vary.

Lemma 11.6 For t even, the image of the homomorphism $\iota_{j,k}$ is given by

$$\{f \in (M_{t+j(p-1)})^0_{\mathbb{Z}/p^k} \ : \ \mathrm{ord}_q f(q) > \frac{t}{12} \text{ or } \mathrm{ord}_q f(q) = \frac{t-2}{12}\}.$$

Proof A basis of $(M_t)^0_{\mathbb{Z}/p^k}$ is given by

$$\{\Delta^n e_{t-12n} \ : \ n \leq \frac{t}{12},\ n \neq \frac{t-2}{12}\}.$$

The image of this basis under $\iota_{j,k}$ is spanned by

$$\{\Delta^n e_{t-12n+j(p-1)} \ : \ n \leq \frac{t}{12},\ n \neq \frac{t-2}{12}\}.$$

Since

$$\Delta^n e_{t-12n+j(p-1)} = q^n + \cdots$$

we deduce the result. □

Proof of Theorem 11.3 Suppose that $b' \in \mathcal{B}_{t/j,k'}$ is an additive generator of order p^k. Let f' be the lift

$$f' := \iota_{j,k'}(b') \in (M_{t+j(p-1)})^0_{\mathbb{Z}/p^{k'}}.$$

Since b' is assumed to be an additive generator of order p^k and $\iota_{j,k'}$ is injective, we deduce that f' is a modular form in $(M_{t+j(p-1)})_{\mathbb{Z}/p^{k'}}$ of exact order p^k. Hence $f' = p^{k'-k}f$ for some modular form $f \in (M_{t+j(p-1)})^0_{\mathbb{Z}/p^k}$. It is simple to check that the image

$$b \in \frac{(M_{t+j(p-1)})^0_{\mathbb{Z}/p^k}}{(M_t)^0_{\mathbb{Z}/p^k}}$$

represents an element of $\mathcal{B}_{t/j,k}$.

It follows that the additive generators of order p^k in

$$H^0(C(\ell)^\bullet/(p^\infty, v_1^\infty))_t = \operatorname*{colim}_{k'} \operatorname*{colim}_{\substack{j=sp^{k'-1} \\ s\geq 1}} \mathcal{B}_{t/j,k'}$$

exactly correspond to the additive generators of order p^k in $\mathcal{B}_{t/j,k}$ which are not in the image of the inclusion

$$\cdot E_{p-1}^{p^{k-1}} : \mathcal{B}_{t/j-p^{k-1},k} \hookrightarrow \mathcal{B}_{t/j,k}.$$

Suppose that b is such an additive generator. Let f be the lift

$$f := \iota_{j,k}(b) \in (M_{t+j(p-1)})^0_{\mathbb{Z}/p^k}.$$

Then by Lemma 11.6 the lift f satisfies

$$\operatorname{ord}_q f(q) > \frac{t}{12} \text{ or } \operatorname{ord}_q f(q) = \frac{t-2}{12}.$$

From the definition of $\mathcal{B}_{t/j,k}$ we have

$$\begin{aligned} &\text{(i)} \quad (\ell^{t+j(p-1)} - 1)f(q) = g_1(q), \\ &\qquad \text{for } g_1 \in (M_t)^0_{\mathbb{Z}/p^k} \\ &\text{(ii)} \quad \ell^{t+j(p-1)} f(q^\ell) - f(q) = g_2(q), \\ &\qquad \text{for } g_2 \in M_t(\Gamma_0(\ell))^0_{\mathbb{Z}/p^k} \end{aligned}$$

Since $j \equiv 0 \mod p^{k-1}$ we deduce that

$$\ell^{t+j(p-1)} \equiv \ell^t.$$

Let $v = \nu_p(\ell^t - 1)$. Condition (i) above implies that

$$f(q) \equiv \frac{g_1(q)}{\ell^t - 1} \mod p^{k-v}.$$

But, if $b'' \in \mathcal{B}_{t/j,k-v}$ is the image of the mod p^{k-v} reduction of b, then

$$f(q) \equiv \iota_{j,k-v}(b'') \mod p^{k-v}$$

and thus, by the exactness of (**11.5**), we have

$$r_{j,k-v}(f) = g_1 = 0.$$

Thus we actually have

$$(\ell^t - 1)f(q) \equiv 0 \mod p^k.$$

Since $f(q)$ has order p^k, we deduce that

$$\ell^t \equiv 1 \mod p^k.$$

Since ℓ is a topological generator of $\mathbb{Z}_p^\times$, we deduce that

$$t \equiv 0 \mod (p-1)p^{k-1}.$$

Thus condition (ii) may be rewritten as

$$f(q^\ell) - f(q) = g_2(q) \text{ for } g_2 \in M_t(\Gamma_0(\ell))^0_{\mathbb{Z}/p^k}.$$

We have therefore verified conditions (1)–(5) of Theorem 11.3.

For the converse direction, suppose $f \in (M_{t+j(p-1)})_{\mathbb{Z}/p^k}$ satisfies conditions (1)–(5) of Theorem 11.3. Then by Lemma 11.6, f is in the image of $\iota_{j,k}$. Consider the image

$$b = [f] \in \frac{(M_{t+j(p-1)})^0_{\mathbb{Z}/p^k}}{(M_t)^0_{\mathbb{Z}/p^k}}.$$

of f in the quotient. Observe that by (2), the element b has order p^k. We just need to verify that it is an element of $\mathcal{B}_{t/j,k}$, which amounts to seeing that f satisfies conditions (i) and (ii) above. But condition (1) implies that

$$\ell^t \equiv \ell^{t+j(p-1)} \equiv 1 \mod p^k.$$

This immediately implies that f satisfies condition (i). Condition (ii) then follows from condition (5). □

Observe that if S is a set of primes which contains ℓ and does not contain p, then we have

$$\Gamma_\ell \subseteq \Gamma_S \subset \mathbb{S}_2.$$

Since Γ_ℓ is dense in $\mathbb{S}_2$, the subgroup Γ_S is dense in $\mathbb{S}_2$. We therefore deduce the following lemma.

Lemma 11.7 For a set of primes S not containing p and containing ℓ, there is an isomorphism

$$H^0(\Gamma_S, \pi_t M_2 E_2)^{\mathrm{Gal}(\mathbb{F}_p)} \xrightarrow{\cong} H^0(\Gamma_\ell, \pi_t M_2 E_2)^{\mathrm{Gal}(\mathbb{F}_p)}.$$

In particular, letting ℓ' be a prime in S, we have a zig-zag

$$H^0(\Gamma_\ell, \pi_t M_2 E_2)^{\mathrm{Gal}(\mathbb{F}_p)} \xleftarrow{\cong} H^0(\Gamma_S, \pi_t M_2 E_2)^{\mathrm{Gal}(\mathbb{F}_p)} \hookrightarrow H^0(\Gamma_{\ell'}, \pi_t M_2 E_2)^{\mathrm{Gal}(\mathbb{F}_p)}.$$

If ℓ' also generates $\mathbb{Z}_p^\times$, then the inclusion is an isomorphism. Corollary 7.12 allows us to deduce:

Corollary 11.8 For any prime $\ell' \neq \ell$ There is an inclusion

$$H^0(C(\ell)^\bullet/(p^\infty, v_1^\infty))_t \hookrightarrow H^0(C(\ell')^\bullet/(p^\infty, v_1^\infty))_t.$$

If f satisfies Conditions (1)-(4) and $(5)_\ell$ of Theorem 11.3, then it satisfies condition $(5)_{\ell'}$.

We finish this section by observing that the results of this section combine to give proofs of some of the theorems stated in Section 1

Proofs of Theorems 1.3, 1.4, 1.5 Corollary 11.8 implies Theorem 1.5. The element

$$\beta_{i/j,k} \in H^0(\mathbb{S}_2, \pi_{2i(p^2-1)-2j(p-1)} M_2 E_2)^{\mathrm{Gal}(\mathbb{F}_p)}$$

detects a corresponding Greek letter element

$$\beta_{i/j,k} \in \mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$$

in the chromatic spectral sequence if $i > 0$ and $i \neq p^n$ [MRW77] (if $i = p^n$, then j must be greater than or equal to p^n). Thus Theorems 1.3 and 1.4 follow from Theorem 11.3. Note that the modular forms $f = f_{i/j,k}$ of Theorems 1.3 and 1.4 are taken to be holomorphic at the cusps, whereas in Theorem 11.3, they are merely assumed to be meromorphic at the cusps. This discrepancy is resolved by noting that if i, j, k are chosen such that $\beta_{i/j,k}$ exists in $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$, then

$$t = i(p^2 - 1) - j(p - 1) \geq 0.$$

Therefore, condition (2) of Theorem 11.3 guarantees that the modular forms in question are holomorphic at the cusps. □

12 Greek letter elements in the Hurewicz image of $Q(\ell)$

Since the cosimplicial spectrum $Q(\ell)^\bullet$ is a cosimplicial object in the category of E_∞-ring spectra, the equivalence $Q(\ell) \simeq \mathrm{Tot}\, Q(\ell)^\bullet$ (**5.4**) allows us to regard $Q(\ell)$ as an E_∞-ring spectrum. In particular, it possesses a unit map

$$S \to Q(\ell)$$

which, by Lemma 7.1, localizes to give a map

$$S_{E(2)} \to Q(\ell).$$

In this section we prove:

Theorem 12.1 The images of the elements $\alpha_{i/j}$ and the elements $\beta_{i/j,k}$ under the homomorphism

$$\pi_* S_{E(2)} \to \pi_* Q(\ell)$$

are non-trivial.

We first will need a lemma.

Lemma 12.2 The map

$$\pi_t M_1 S \to \pi_t M_1 Q(\ell)$$

is an isomorphism for $t \equiv 0 \mod 4$.

Proof Let C be an ordinary elliptic curve over $\overline{\mathbb{F}}_p$, so that there is an isomorphism of formal groups

$$C^\wedge \cong \widehat{\mathbb{G}}_m.$$

Let U be the formal neighborhood of the associated point of $\mathcal{M}(K_0^p)$, which carries a universal deformation $\widetilde{C}/U$ of C. Let

$$E = \mathcal{E}_{K_0^p}(U)$$

be the sections of the sheaf $\mathcal{E}$ over U. By Serre-Tate theory, and the deformation theory of p-divisible groups [BLb, 7.1], we deduce that

$$U \cong \mathrm{Spf}(W(\bar{\mathbb{F}}_p)[[x]])$$

and therefore that E is an even periodic ring spectrum with $\pi_0(E) \cong W(\bar{\mathbb{F}}_p)[[x]]$, with associated formal group given by $\widetilde{C}^\wedge$. The cofiber

$$E \xrightarrow{\cdot x} E \to E/x$$

is an even periodic ring spectrum [EKMM97]. The restriction $\widetilde{C}_{\mathrm{can}}$ of the deformation $\widetilde{C}$ to $\pi_0(E/x) \cong W(\bar{\mathbb{F}}_p)$ is the canonical deformation of C (the deformation whose p-divisible group splits). The formal group $\widetilde{C}^\wedge_{\mathrm{can}}$ is therefore a universal deformation of $\mathbb{G}_m/\bar{\mathbb{F}}_p$, and we conclude that there is an isomorphism

$$\widetilde{C}^\wedge_{\mathrm{can}} \cong \widehat{\mathbb{G}}_m$$

between the formal group for E/x and the multiplicative formal group. In particular, this implies that there is an equivalence of ring spectra

$$K_p \otimes_{\mathbb{Z}_p} W(\bar{\mathbb{F}}_p) \cong E/x,$$

where K_p is the p-adic K-theory spectrum. Now, the unit map $S \to K_p$ induces an inclusion

$$\pi_{2t} M_1 S \hookrightarrow \pi_{2t} M_1 K_p$$

(it gives the Adams e-invariant). Therefore the unit map for E/x induces an inclusion

$$\pi_{2t} M_1 S \hookrightarrow \pi_{2t} M_1 (E/x).$$

However, the unit for E/x is homotopic to the composite

$$S \to Q(\ell) \simeq \mathrm{Tot}\, Q(\ell)^\bullet \to Q(\ell)^0 = \mathrm{TMF}_p \to E \to E/x$$

because all of the maps in the composite are maps of ring spectra. We deduce that the maps

$$\pi_t M_1 S \to \pi_t M_1 Q(\ell)$$

are injective for $t \equiv 0 \mod 2$. By Corollary 9.10, these (finite) groups are abstractly isomorphic for $t \equiv 0 \mod 4$ and $t \neq 0$. The result for $t \neq 0$ therefore is proven. The cases of $t = 0$ follows immediately from the fact that the map $\pi_0 S \to \pi_0 Q(\ell) \cong \mathbb{Z}_p$ is a map of rings. □

Proof of Theorem 12.1 Consider the map of chromatic spectral sequences:

$$\begin{array}{ccc} \pi_k M_n S_{E(2)} & \Longrightarrow & \pi_{k-n} S_{E(2)} \\ \downarrow & & \downarrow \\ \pi_k M_n Q(\ell) & \Longrightarrow & \pi_{k-n} Q(\ell) \end{array}$$

The elements $\alpha_{i/j} \in \pi_{2i(p-1)}(M_1 S)$ are known to be permanent cycles for $i > 0$, and therefore map to permanent cycles in the chromatic spectral sequence for $Q(\ell)$. By Lemma 12.2, the images of $\alpha_{i/j}$ in $\pi_t M_1 S$ are nontrivial, and generate these groups for $t \equiv 0 \mod 4$. Since, by Corollary 8.4, $\pi_t M_0 Q(\ell)$ is zero for $t \neq 0, -1, -2$, there are no non-trivial differentials

$$d_1 : \pi_t M_0 Q(\ell) \to \pi_t M_1 Q(\ell)$$

for $t > 0$. We deduce that the images of the elements $\alpha_{i/j}$ in the chromatic spectral sequence for $Q(\ell)$ are non-trivial permanent cycles, and hence witness the non-triviality of the images of the elements $\alpha_{i/j}$ in $\pi_* Q(\ell)$. As a side-effect, we have also determined that the groups $\pi_t M_1 Q(\ell)$ are generated by permanent cycles for $t \equiv 0 \mod 4$. That, combined with the fact that $\pi_t M_0 Q(\ell)$ is zero for t positive, allows us to deduce that there are no non-trivial differentials killing elements of $\pi_t M_2 Q(\ell)$ for $t \equiv 0 \mod 4$. To complete the proof of the theorem, it suffices to show that the images of the elements $\beta_{i/j,k}$ are non-trivial under the homomorphism

$$\pi_t M_2 S \to \pi_t M_2 Q(\ell) \tag{12.3}$$

where $t = 2i(p^2 - 1) - 2j(p - 1)$. But, for such t, the map (12.3) is given by the composite of isomorphisms

$$\begin{aligned} \pi_t M_2 S &\cong H^0_c(\mathbb{S}_2, \pi_t M_2 E_2)^{\mathrm{Gal}(\mathbb{F}_p)} \\ &\cong H^0(\Gamma_\ell, \pi_t M_2 E_2)^{\mathrm{Gal}(\mathbb{F}_p)} \\ &\cong \pi_t M_2 Q(\ell) \end{aligned}$$

given by Corollary 2.4, Lemma 11.1, and Lemma 7.11. □

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