ON THE HOMOTOPY OF $Q(3)$ AND $Q(5)$ AT THE PRIME 2

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ABSTRACT. We study modular approximations $Q(\ell)$, $\ell = 3, 5$, of the $K(2)$-local sphere at the prime 2 that arise from ℓ-power degree isogenies of elliptic curves. We develop Hopf algebroid level tools for working with $Q(5)$ and record Hill, Hopkins, and Ravenel's computation of the homotopy groups of $\mathrm{TMF}_0(5)$. Using these tools and formulas of Mahowald and Rezk for $Q(3)$ we determine the image of Shimomura's 2-primary divided β-family in the Adams-Novikov spectral sequences for $Q(3)$ and $Q(5)$. Finally, we use low-dimensional computations of the homotopy of $Q(3)$ and $Q(5)$ to explore the rôle of these spectra as approximations to $S_{K(2)}$.

CONTENTS

In [3], motivated by [7], the p-local spectrum $Q(\ell)$ ($p \nmid \ell$) is defined as the totalization of an explicit semi-cosimplicial E_∞-ring spectrum of the form

$$Q(\ell)^\bullet = \Big(\mathrm{TMF} \Rightarrow \mathrm{TMF}_0(\ell) \times \mathrm{TMF} \Rrightarrow \mathrm{TMF}_0(\ell) \Big).$$

The spectrum $Q(\ell)$ serves as a kind of approximation to the $K(2)$-local sphere. In [4], it is proven that there is an equivalence

$$Q(\ell)_{K(2)} \simeq (E_2^{h\Gamma_\ell})^{h\mathrm{Gal}}$$

where Γ_ℓ is a certain subgroup of the Morava stabilizer group $\mathbb{S}_2$ coming from isogenies of elliptic curves. The subgroup Γ_ℓ is dense if p is odd and ℓ generates

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a dense subgroup of $\mathbb{Z}_p^\times$ [6]. Based on this, it is conjectured that there are fiber sequences

$$D_{K(2)}Q(\ell) \to S_{K(2)} \xrightarrow{u} Q(\ell) \tag{0.0.1}$$

for such choices of ℓ (and the case of $\ell = 2$ and $p = 3$ is handled by explicit computation in [3], and is closely related to [7]). Density also is used in [5] to show that for such ℓ, $Q(\ell)$ detects the exact divided β family pattern for $p \geq 5$.

However, in the case of $p = 2$, $\mathbb{Z}_2^\times$ is not topologically cyclic, and the closure of Γ_ℓ in $\mathbb{S}_2$ is the inverse image of the closure of the subgroup $\ell^{\mathbb{Z}} < \mathbb{Z}_2^\times$ under the reduced norm

$$N : \mathbb{S}_2 \to \mathbb{Z}_2^\times.$$

It is not altogether clear in this case what the analog of the conjecture (0.0.1) should be, though one possibility is suggested in [6]. Although the 2-primary "duality resolution" of Goerss, Henn, Mahowald, and Rezk (see [8]) seems to take the form of a fiber sequence like (0.0.1), we will observe that the mod $(2, v_1)$-behavior of $Q(3)$ actually precludes $Q(3)$ from being half of the duality resolution (see Remark 4.5.3). The non-density of Γ_3, together with the appearance of both $\mathrm{TMF}_0(3)$ and $\mathrm{TMF}_0(5)$ factors in $\mathrm{TMF} \wedge \mathrm{TMF}$ also suggests that, from a TMF-resolutions perspective, $Q(3)$ alone may not be seeing enough homotopy, and that a combined approach of $Q(3)$ and $Q(5)$ may be required at the prime 2.

The goal of this paper is to explore such an approach by extending the work of Mahowald and Rezk [14] on $Q(3)$, and initiating a similar study of $Q(5)$.

The first testing ground for the effectiveness of $Q(3)$ or $Q(5)$ at detecting v_2-periodic homotopy at the prime 2 is Shimomura's 2-primary divided beta family [16]. To the authors' surprise, $Q(3)$ was found to exactly detect Shimomura's divided beta patterns on the 2-lines of the E_2 term of its Adams-Novikov spectral sequence, as we shall explain in Section 4. Hence $Q(3)$ is all that is needed to detect the shape of the divided beta family. The authors were equally surprised to find no such phenomenon for $Q(5)$ - the beta family for $Q(5)$ has greater v_1-divisibility than that for the sphere. On the other hand, the $K(2)$-localization of $Q(5)$ is built out of homotopy fixed point spectra of groups with larger 2-torsion than $Q(3)$. This raises the possibility that while $Q(5)$ may be less effective when it comes to beta elements, it could detect exotic torsion in higher cohomological degrees that is invisible to $Q(3)$. This possibility is explored through some low dimensional computations.

We now summarize the contents of this paper. In Section 1 we review and expand the theory of $\Gamma_0(5)$-structures on elliptic curves. A $\Gamma_0(5)$-structure is an elliptic curve equipped with a cyclic subgroup of order 5. We recall an explicit description of the scheme representing $\Gamma_1(5)$-structures (elliptic curves with a point of order 5) in terms of *Tate normal form* curves and use this description to present several Hopf algebroids that stackify to the moduli space of $\Gamma_0(5)$-structures. We then use these Hopf algebroids and the geometry of elliptic curves to determine the maps defining $Q(5)^\bullet$.

In Section 2 we compute the homotopy fixed point spectral sequence

$$H^*(\mathbb{F}_5^\times; \pi_* \mathrm{TMF}_1(5)) \implies \pi_* \mathrm{TMF}_0(5).$$

The ring $\pi_* \mathrm{TMF}_1(5)$ and the action of $\mathbb{F}_5^\times$ on it are determined by Tate normal form, allowing us to produce a detailed group cohomology computation. We then compute the differentials and hidden extensions in the spectral sequence by a number of methods: TMF-module structure, transfer-resetriction arguments, and comparision with the homotopy orbit spectral sequence. Our use of the homotopy orbit spectral sequence to determine hidden extensions is somewhat novel and may find use in other contexts.

Since $Q(\ell)$ is the totalization of a cosimplicial spectrum, we can compute the E_2-term of its Adams-Novikov spectral sequence as the cohomology of a double complex. The differentials in the double complex are either internal cobar differentials for the Weierstrass or $\Gamma_0(5)$ Hopf algebroids or external differentials determined by the cosimplicial structure of $Q(\ell)^\bullet$. In Section 3 we review formulas for the external differentials in the $\ell = 3$ and $\ell = 5$ cases. The $Q(3)$ formulas are due to Mahowald and Rezk [14] while those for $Q(5)$ are derived from Section 1.

In Section 4 we compute several chromatic spectral sequences related to $Q(3)$ and $Q(5)$. Definitions are stated in Section 4.1 and the technique we use is carefully laid out in Section 4.4. Stated precisely, we compute $H^{0,*}(M_0^2 C^*_{tot}(Q(3)))$ and $H^{0,*}(M_1^1 C^*_{tot}(Q(5)))$, both of which are related to the divided β family in the $Q(\ell)$ spectra. We compare these groups to Shimomura's 2-primary divided β family for the sphere spectrum (i.e. the groups $\mathrm{Ext}^{0,*}(M_0^2 BP_*)$, reviewed in Theorem 4.2.1). In Theorem 4.2.2 we find that $\mathrm{Ext}^{0,*}(M_0^2 BP_*)$ is isomorphic to $H^{0,*}(M_0^2 C^*_{tot} Q(3))$, so $Q(3)$ precisely detects the divided β family. In contradistinction, Theorem 4.2.4 and Corollary 4.9.4 show that the divided β family for $Q(5)$ has extra v_1-divisibility.

Finally, in Section 5 we compute $\pi_n Q(3)$ and $\pi_n Q(5)$ for $0 \leq n < 48$. These computations give evidence for some homotopy which $Q(5)$ detects which is not detected by $Q(3)$.

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1. Elliptic curves with level 5 structures

We consider the moduli problems of $\Gamma_1(5)$- and $\Gamma_0(5)$-structures on elliptic curves. An elliptic curve with a $\Gamma_1(5)$-structure over a commutative $\mathbb{Z}[1/5]$-algebra R is a pair (C, P) where C is an elliptic curves over R, and $P \in C$ is a point of exact order 5. An elliptic curve with a $\Gamma_0(5)$-structure is a pair (C, H) with C an elliptic curve over R and $H < C$ a subgroup of order 5. Let $\mathcal{M}_i(5)$ denote the moduli stack (over $\mathrm{Spec}(\mathbb{Z}[1/5])$) of $\Gamma_i(5)$-structures.

Let $\mathcal{M}_i^1(5)$ denote the the moduli stack of tuples (C, P, v) (respectively (C, H, v)) where v is a tangent vector at $0 \in C$. Note that in the case where $i = 1$, we can use translation by P to equivalently specify this structure as a tuple (C, P, v') where v' is a tangent vector at P.

Let $\mathcal{M}^1$ denote the moduli stack of elliptic curves with tangent vector. Then the maps in the cosimplicial E_∞ ring $Q(5)^\bullet$ arise by evaluating the TMF-sheaf $\mathcal{O}^{top}$ on maps $\mathcal{M}_0^1(5) \to \mathcal{M}^1$ and $\mathcal{M}_0^1(5) \to \mathcal{M}_0^1(5)$. Recall that the Weierstrass Hopf algebroid (A, Γ) stackifies to $\mathcal{M}^1$; we review the structure of (A, Γ) in 1.2. In this section we produce a Hopf algebroid (B^1, Λ^1) representing $\mathcal{M}_0^1(5)$ and produce Hopf algebroid formulas for the maps in the semi-simplicial stack associated to $Q(\ell)^\bullet$.

1.1. **Representing $\mathcal{M}_1(5)$.** In this section we will give explicit presentations of $\mathcal{M}_1(5)$ and $\mathcal{M}_1^1(5)$. Consider the rings

$$\begin{aligned} B &:= \mathbb{Z}[1/5, b, \Delta^{-1}] \\ B^1 &:= \mathbb{Z}[1/5, a_1, a_2, a_3, \Delta^{-1}]/(a_2^3 + a_3^2 - a_1 a_2 a_3) \end{aligned}$$

where Δ is given respectively by:

$$\begin{aligned} \Delta(b) &= b^5(b^2 - 11b - 1), \\ \Delta(a_1, a_2, a_3) &= -8a_1^2 a_3^2 a_2^2 + 20 a_1 a_3^3 a_2 - a_1^4 a_3^2 a_2 - 11 a_3^4 + a_1^3 a_3^3. \end{aligned}$$

We have the following theorem.

Theorem 1.1.1. The stacks $\mathcal{M}_1(5)$ and $\mathcal{M}_1^1(5)$ affine schemes, given by

$$\begin{aligned} \mathcal{M}_1(5) &= \mathrm{Spec}(B), \\ \mathcal{M}_1^1(5) &= \mathrm{Spec}(B^1). \end{aligned}$$

Proof. We first use the techniques of [10, §4.4] (which is a recapitulation of a method from [13]) to produce an explicit model for $\mathcal{M}_1^1(5)$ as an affine scheme. The procedure is exhibited graphically in Figure 1.1.

Suppose (C, P) is a $\Gamma_1(5)$-structure over a commutative ring R in Weierstrass form with $P = (\alpha, \beta)$. For $r, s, t \in R$ and $\lambda \in R^\times$ let $\varphi_{r,s,t,\lambda}$ denote the coordinate change

$$\begin{aligned} x &\mapsto \lambda^{-2} x + r \\ y &\mapsto \lambda^{-3} y + \lambda^{-2} s x + t. \end{aligned}$$

Move P to $(0,0)$ via the coordinate change $\varphi_{-\alpha,0,-\beta,1} : (C, P) \to (C_{\underline{a}'}, (0,0))$ where $C_{\underline{a}'}$ has Weierstrass form

$$y^2 + a_1' xy + a_3' xy = x^3 + a_2' x^2 + a_4' x.$$

(Note that $a_6' = 0$ because $(0,0)$ is on the curve.) Next eliminate a_4' by applying the transformation $\varphi_{0,-a_4'/a_3',0,1}$. The result is a smooth Weierstrass curve

$$T^1(a_1, a_2, a_3) : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 \tag{1.1.2}$$

with $\Gamma_1(5)$-structure $(0,0)$ which we call the *homogeneous Tate normal form* of (C, P). The discriminant of $T^1(a_1, a_2, a_3)$ is

$$\Delta = -8a_1^2 a_3^2 a_2^2 + 20 a_1 a_3^3 a_2 - a_1^4 a_3^2 a_2 - 11 a_3^4 + a_1^3 a_3^3. \tag{1.1.3}$$

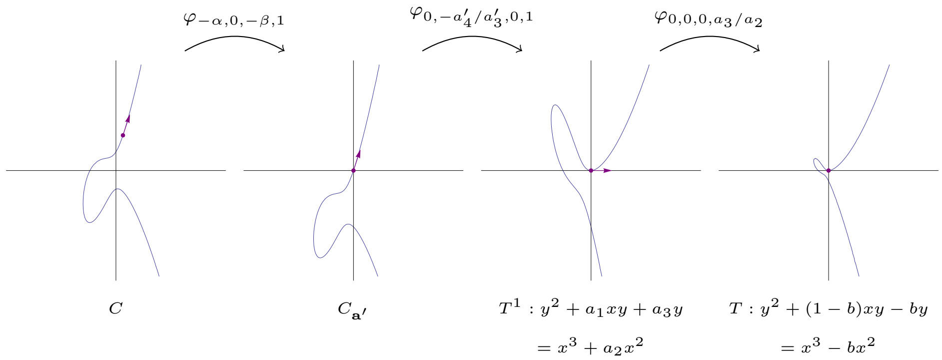


FIGURE 1.1.1. The procedure for putting a $\Gamma_1(5)$-structure in (homogeneous and non-homogeneous) Tate normal form

Since $(0,0)$ has order 5 in $T^1(a_1, a_2, a_3)$ we must have

$$[3](0,0) = [-2](0,0) \tag{1.1.4}$$

where $[n]$ denotes the $\mathbb{Z}$-module structure of the elliptic curve group law. Using the standard formulas for the addition law for an elliptic curve, we see that (1.1.4) is equivalent to

$$a_2^3 + a_3^2 = a_1a_2a_3. \tag{1.1.5}$$

Let $f^1(a_1, a_2, a_3) := a_2^3 + a_3^2 - a_1a_2a_3$ and let

$$B^1 := \mathbb{Z}[a_1, a_2, a_3, \Delta^{-1}]/(f^1).$$

Then

$$\mathcal{M}_1^1(5) = \mathrm{Spec}(B^1).$$

We now consider $\Gamma_1(5)$-structures without distinguished tangent vectors and produce a (non-homogeneous) Tate normal form which is the univeral elliptic curve for $\mathcal{M}_1(5)$. Begin with a $\Gamma_1(5)$-structure (C, P) and change coordinates to put it in homogeneous Tate normal form $T^1(a_1, a_2, a_3)$. Now apply the coordinate transformation $\varphi_{0,0,0,a_3/a_2}$. (This transformation is permissible because $(0,0)$ has order greater than 3.) After applying the transformation, the coefficients of y and x^2 are equal. Let

$$T(b, c) : y^2 + (1-c)xy - by = x^3 - bx^2. \tag{1.1.6}$$

denote the resulting smooth Weierstrass curve.

Since $(0,0)$ has order 5 we know (1.1.4) holds; it follows that

$$b = c \tag{1.1.7}$$

in (1.1.6). Abusing notation, let

$$T(b) : y^2 + (1-b)xy - by = x^3 - bx^2; \tag{1.1.8}$$

we call this the (*non-homogeneous*) *Tate normal form* of (C, P). The discriminant of $T(b)$ is

$$\Delta = b^5(b^2 - 11b - 1). \tag{1.1.9}$$

Let

$$B := \mathbb{Z}[1/5, b, \Delta^{-1}].$$

The preceding two paragraphs show that

$$\mathcal{M}_1(5) = \mathrm{Spec}(B).$$

□

Corollary 1.1.10. The moduli space $\mathcal{M}_1^1(5)$ is represented by

$$\mathrm{Spec}(\mathbb{Z}[1/5, a_1, u^{\pm}, \Delta^{-1}])$$

where

$$\Delta = -11u^{12} + 64a_1u^{11} - 154a_1^2u^{10} + 195a_1^3u^9 - 135a_1^4u^8 + 46a_1^5u^7 - 4a_1^6u^6 - a_1^7u^5.$$

Proof. The relation $a_2^3 + a_3^2 = a_1a_2a_3$ allows us to put $T^1(a_1, a_2, a_3)$ in the form

$$y^2 + a_1xy + u^2(a_1 - u)y = x^3 + u(a_1 - u)x^2$$

with u invertible. The discriminant is then determined by hand. □

The simple structure of $\mathcal{M}_1(5)$ has an immediate topological corollary that we record here.

Corollary 1.1.11. The $K(2)$-localization of $\mathrm{TMF}_1(5)$ is a height 2 Lubin-Tate spectrum for the formal group law $\widehat{T(b)}$ defined over $\mathbb{F}_2$:

$$\mathrm{TMF}_1(5)_{K(2)} \simeq E_2(\mathbb{F}_2, \widehat{T(b)}).$$

Proof. The $K(2)$-localization of $\mathrm{TMF}_1(5)$ is controlled by the $\mathbb{F}_2$-supersingular locus of $\mathcal{M}_1(5)$, $\mathcal{M}_1(5)^{ss}_{\mathbb{F}_2}$. The 2-series of the formal group law for $T = T(b)$ takes the form

$$[2]_{\widehat{T}}(z) = 2z + (b-1)z^2 + 2bz^3 + (b^2 - 2b)z^4 + \cdots .$$

Hence $\widehat{T}$ is supersingular over $\mathbb{F}_2$ if and only if $b = 1$. Note that $\Delta(T(1)) = 11$, a unit in $\mathbb{Z}_2$ and $\mathbb{F}_2$. It follows that

$$\mathcal{M}_1(5)^{ss}_{\mathbb{F}_2} = \mathrm{Spec}(\mathbb{F}_2).$$

Let $E_2 = E_2(\mathbb{F}_2, \widehat{T})$ with $\pi_0 E_2 = \mathbb{Z}_2[[u_1]]$. The map

$$\mathrm{Spec}\, \pi_0 E_2 \to \mathcal{M}_1(5)$$

induced by

$$\begin{aligned} B &\to \pi_0 E_2 \\ b &\mapsto u_1 + 1. \end{aligned}$$

induces the $K(2)$-localization of $\mathrm{TMF}_1(5)$. □

1.2. **Representing maps $\mathcal{M}_1^1(5) \to \mathcal{M}^1$.** There are two important maps $\mathcal{M}_1^1(5) \to \mathcal{M}^1$ which we analyze. On the level of points, the first is the forgetful map

$$\begin{aligned} \mathcal{M}_1^1(5) &\xrightarrow{f} \mathcal{M}^1 \\ (C,P) &\mapsto C. \end{aligned}$$

The second is the quotient map

$$\begin{aligned} \mathcal{M}_1^1(5) &\xrightarrow{q} \mathcal{M}^1 \\ (C,P) &\mapsto C/\langle P\rangle. \end{aligned}$$

Let (A,Γ) denote the usual Weierstrass curve Hopf algebroid with

$$A = \mathbb{Z}[a_1,a_2,a_3,a_4,a_6,\Delta^{-1}], \Gamma = A[r,s,t]$$

that stackifies to $\mathcal{M}^1$.

Theorem 1.2.1. The morphisms f and q above are represented by

$$\begin{aligned} A &\xrightarrow{f^*} B^1 \\ a_i &\mapsto \begin{cases} a_i & \text{if } i=1,2,3, \\ 0 & \text{if } i=4,6, \end{cases} \end{aligned}$$

and

$$\begin{aligned} A &\xrightarrow{q^*} B^1, \\ a_i &\mapsto \begin{cases} a_i & \text{if } i=1,2,3, \\ 5a_1^2a_2 - 10a_1a_3 - 10a_2^2 & \text{if } i=4, \\ a_1^4a_2 - 2a_1^3a_3 - 12a_1^2a_2^2 + 19a_2^3 - a_3^2 & \text{if } i=6. \end{cases} \end{aligned}$$

Computing q requires that we find a Weierstrass curve representation of $C/\langle P\rangle$ in terms of the Weierstrass coeffcients of C. This procedure is well-studied by number theorists under the name *Vélu's formulae* (see [17], [12, §2.4]) and is implemented in the computer algebra system `Magma`. In fact, if ϕ is an isogeny on C in Weierstrass form with kernel H, then Vélu's formulae compute Weierstrass coefficients for the target of ϕ in terms of the Weierstrass coefficients of C and the defining equations subgroup scheme H. We briefly review the formulae here for reference.

Suppose $H < C$ is a finite subgroup with ideal sheaf generated by a monic polynomial $\psi(x)$ where C is a Weierstrass curve of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

For simplicity, assume that the isogeny $\phi : C \to C/H$ has odd degree. (The even degree case can be handled as a separate case, but we will not need it in this paper.) Write

$$\psi(x) = x^n - s_1x^{n-1} + \cdots + (-1)^n s_n.$$

Then C/H has Weierstrass equation

$$y_H^2 + a_1x_Hy_H + a_3y_H = x_H^3 + a_2x_H^2 + (a_4 - 5t)x_H + (a_6 - b_2t - 7w)$$

where

$$\begin{aligned} t &= 6(s_1^2 - 2s_2) + b_2s_1 + nb_4, \\ w &= 10(s_1^3 - 3s_1s_2 + 3s_3) + 2(b_2(s_1^2 - 2s_2) + 3b_4s_1 + nb_6, \end{aligned}$$

and

$$\begin{aligned} b_2 &= a_1^2 + 4a_2, \\ b_4 &= a_1 a_3 + 2a_4, \\ b_6 &= a_3^2 + 4a_6. \end{aligned}$$

Vélu's formulae also give explicit equations for the isogeny $\phi : (x, y) \mapsto (x_H, y_H)$, but they are cumbersome to write down and we will not need them here.

Proof of Theorem 1.2.1. The representation of f is obvious: $T^1(a_1, a_2, a_3)$ is already in Weierstrass form with $a_4, a_6 = 0$.

Consider the case of $C = T(a_1, a_2, a_3)$ with $H = \langle P \rangle$ an order 5 subgroup. Using the elliptic curve addition law we see that H is the subgroup scheme of C cut out by the polynomial $x(x + a_2)$. Putting this data into Vélu's formulae, we find that C/H has Weierstrass form

$$\begin{aligned} y^2 + a_1 xy + a_3 y = x^3 &+ a_2 x^2 \\ &+ (5a_1^2 a_2 - 10a_1 a_3 - 10a_2^2)x \\ &+ (a_1^4 a_2 - 2a_1^3 a_3 - 12a_1^2 a_2^2 + 19a_2^3 - a_3^2) \end{aligned} \tag{1.2.2}$$

from which our formula for q follows. □

1.3. **Hopf algebroids for $\mathcal{M}_0^1(5)$.** Recall that a $\Gamma_0(5)$-structure (C, H) consists of an elliptic curve C along with a subgroup $H < C$ of order 5. Unlike the moduli problem of $\Gamma_1(5)$-structures, $\mathcal{M}_0(5)$ is not representable by a scheme. Still, it is the case that $\mathcal{M}_1(5)$ admits a $C_4 = \mathbb{F}_5^\times$-action such that $\mathcal{M}_0(5)$ is the geometric quotient $\mathcal{M}_1(5)//\mathbb{F}_5^\times$. For $g \in \mathbb{F}_5^\times$, g takes (C, P) to $(C, [\tilde{g}]P)$ for $\tilde{g}$ any lift of g in $\mathbb{Z}$. Similarly, we can write $\mathcal{M}_0^1(5) = \mathcal{M}_1^1(5)//\mathbb{F}_5^\times$.

While it is typically easier to use this quotient stack presentation of $\mathcal{M}_0(5)$ and $\mathcal{M}_0^1(5)$ (and this will be the perspective we will be taking in the computations later in this paper), we will note that there is also a presentation of these moduli stacks by '(r, s, t)' Hopf algebroids. Let B^1 be as before and define

$$\Lambda^1 := B^1[r, s, t]/\sim$$

where $\sim$ consists of the relations

$$\begin{aligned} 3r^2 &= 2st + a_1 rs + a_3 s + a_1 t - 2a_2 r, \\ t^2 &= r^3 + a_2 r^2 - a_1 rt - a_3 t, \\ s^6 &= -3a_1 s^5 + 9rs^4 + 3a_2 s^4 - 3a_1^2 s^4 + 4ts^3 \\ &\quad + 20a_1 rs^3 + 6a_1 a_2 s^3 + 2a_3 s^3 - a_1^3 s^3 + 6a_1 ts^2 \\ &\quad - 27r^2 s^2 - 18a_2 rs^2 + 12a_1^2 rs^2 - 3a_2^2 s^2 + 3a_1^2 a_2 s^2 \\ &\quad + 3a_1 a_3 s^2 - 12rts - 4a_2 ts + 2a_1^2 ts - 33a_1 r^2 s \\ &\quad - 20a_1 a_2 rs - 6a_3 rs + a_1^3 rs - 3a_1 a_2^2 s - 2a_3 a_2 s \\ &\quad + a_1^2 a_3 s + 4t^2 - 2a_1 rt - 2a_1 a_2 t + 4a_3 t + 27r^3 \\ &\quad + 27a_2 r^2 - 2a_1^2 r^2 + 9a_2^2 r - a_1^2 a_2 r - a_1 a_3 r. \end{aligned}$$

Theorem 1.3.1. The rings (B^1, Λ^1) form a Hopf algebroid stackifying to $\mathcal{M}_0^1(5)$. The structure maps are given by

$$\begin{aligned}
\eta_R(a_1) &= a_1 + 2s \\
\eta_R(a_2) &= a_2 + 3r - s^2 - a_1 s \\
\eta_R(a_3) &= a_3 + 2t + a_1 r \\
\psi(r) &= r \otimes 1 + 1 \otimes r \\
\psi(s) &= s \otimes 1 + 1 \otimes s \\
\psi(t) &= t \otimes 1 + s \otimes r + 1 \otimes t.
\end{aligned}$$

Proof. The reader will note that the structure maps are identical to those for the standard Weierstrass Hopf algebroid (A, Γ). The relations $\sim$ are precisely those required so that $\varphi_{r,s,t,1}$ transforms $T^1(a_1, a_2, a_3)$ such that $a_2^3 + a_3^2 = a_1 a_2 a_3$ into another homogeneous Tate normal curve. □

There are forgetful and quotient maps on $\mathcal{M}_0^1(5)$ that on points take

$$\begin{aligned}
\mathcal{M}_0^1(5) &\xrightarrow{f} \mathcal{M}^1 \\
(C, H) &\mapsto C
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}_0^1(5) &\xrightarrow{q} \mathcal{M}^1 \\
(C, H) &\mapsto C/H.
\end{aligned}$$

(We elide the tangent vectors for concision.)

Corollary 1.3.2. The maps f and q on $\mathcal{M}_0^1(5)$ are represented by

$$\begin{aligned}
(A, \Gamma) &\xrightarrow{f^*} (B^1, \Lambda^1) \\
a_i &\mapsto \begin{cases} a_i & \text{if } i = 1, 2, 3, \\ 0 & \text{if } i = 4, 6 \end{cases} \\
r, s, t &\mapsto r, s, t
\end{aligned}$$

and

$$\begin{aligned}
(A, \Gamma) &\xrightarrow{q^*} (B^1, \Lambda^1) \\
a_i &\mapsto \begin{cases} a_i & \text{if } i = 1, 2, 3, \\ 5a_1^2 a_2 - 10 a_1 a_3 - 10 a_2^2 & \text{if } i = 4, \\ a_1^4 a_2 - 2a_1^3 a_3 - 12 a_1^2 a_2^2 + 19 a_2^3 - a_3^2 & \text{if } i = 6. \end{cases} \\
r, s, t &\mapsto r, s, t
\end{aligned}$$

Proof. This is a consequence of Theorems 1.2.1 and 1.3.1. □

1.4. **The Atkin-Lehner dual.** We will now compute the Atkin-Lehner dual

$$t : \mathcal{M}_0^1(5) \to \mathcal{M}_0^1(5).$$

Each $\Gamma_0(5)$-structure (C, H) can also be represented as a pair (C, ϕ) where $\phi : C \to C'$ has kernel H. On points, the Atkin-Lehner dual takes

$$\mathcal{M}_0^1(5) \xrightarrow{t} \mathcal{M}_0^1(5)$$
$$(C, \phi) \mapsto (C, \widehat{\phi})$$

where $\widehat{\phi}$ is the dual isogeny to ϕ.

We can lift t to stacks closely related to $\mathcal{M}_1^1(5)$. Recall ([11, §2.8]) that for each $\Gamma_0(5)$-structure (C, ϕ) there is an associated scheme-theoretic Weil pairing

$$\langle -, - \rangle_\phi : \ker \phi \times \ker \widehat{\phi} \to \mu_5.$$

Choose a primitive fifth root of unity ζ. For a $\Gamma_1(5)$-structure (C, P) let (C, ϕ_P) denote the associated $\Gamma_0(5)$-structure where $\phi_P : C \to C'$ is an isogeny with kernel $\langle P \rangle$. If we work in $\mathcal{M}_1^1(5)_\zeta$, i.e. $\mathcal{M}_1^1(5)$ considered as a $\mathbb{Z}[\frac{1}{5}, \zeta]$-scheme, then there is a unique $Q \in \ker \widehat{\phi_P}$ such that $\langle P, Q \rangle_\phi = \zeta$. We define

$$t_\zeta : \mathcal{M}_1^1(5)_\zeta \to \mathcal{M}_1^1(5)_\zeta$$

in the obvious way so that $t_\zeta(C, P) = (C', Q)$.

The maps t and t_ζ fit in the commutative diagram

$$\begin{array}{ccc} \mathcal{M}_1^1(5)_\zeta & \xrightarrow{t_\zeta} & \mathcal{M}_1^1(5)_\zeta \\ \downarrow & & \downarrow \\ \mathcal{M}_0^1(5)_\zeta & \xrightarrow[t]{} & \mathcal{M}_0^1(5)_\zeta \end{array}$$

where the vertical maps take (C, P) to (C, ϕ_P).

We can gain some computational control over t via the following method. First, recall from Corollary 1.1.10 that for each homogeneous Tate normal curve $T^1(a_1, a_2, a_3)$ there is a unit u such that $a_2 = u(a_1 - u)$ and $a_3 = u^2(a_1 - u)$. Abusing notation, denote the same curve by $T^1(a_1, u)$, and let H denote the canonical cyclic subgroup of order 5 generated by $(0, 0)$. The defining polynomial for H is $x(x + u(a_1 - u))$. Denote the isogeny with kernel H by ϕ. Note that the range of ϕ is the curve C/H given by Vélu's formulae in (1.2.2).

Using the computer algebra system `Magma`, we can determine that the kernel of $\widehat{\phi}$ is the subgroup scheme determined by

$$f := x^2 + (a_1^2 - a_1 u + u^2)x + \frac{1}{5}(a_1^4 - 7a_1^3 u - 11a_1^2 u^2 + 47a_1 u^3 - 29u^4).$$

Then over the ring $R := \mathbb{Z}[\frac{1}{5}, \zeta][a_1, u^{\pm}]$ the polynomial f splits and we find that

$$(\ker \widehat{\phi})(R) = \{\infty, (x_0, y_{00}), (x_0, y_{01}), (x_1, y_{10}), (x_1, y_{11})\}$$

where

$$
\begin{aligned}
x_0 &= \frac{1}{5}(\zeta^3+\zeta^2-2)a_1^2 + \frac{1}{5}(9\zeta^3+9\zeta^2+7)a_1u + \frac{1}{5}(-11\zeta^3-11\zeta^2-8)u^2,\\
x_1 &= \frac{1}{5}(-\zeta^3-\zeta^2-3)a_1^2 + \frac{1}{5}(-9\zeta^3-9\zeta^2-2)a_1u + \frac{1}{5}(11\zeta^3+11\zeta^2+3)u^2,\\
y_{00} &= \frac{1}{5}(\zeta^2+2\zeta+2)a_1^3 + \frac{1}{5}(\zeta^3+7\zeta^2+17\zeta+5)a_1^2u\\
&\quad + \frac{1}{5}(9\zeta^3-29\zeta^2-31\zeta-14)a_1u^2 + \frac{1}{5}(-11\zeta^3+22\zeta^2+11\zeta+8)u^3,\\
y_{01} &= \frac{1}{5}(-\zeta^3-2\zeta^2-2\zeta)a_1^3 + \frac{1}{5}(-10\zeta^3-16\zeta^2-17\zeta-12)a_1^2u\\
&\quad + \frac{1}{5}(2\zeta^3+40\zeta^2+31\zeta+17)a_1u^2 + \frac{1}{5}(11\zeta^3-22\zeta^2-11\zeta-3)u^3,\\
y_{10} &= \frac{1}{5}(2\zeta^3+\zeta+2)a_1^3 + \frac{1}{5}(16\zeta^3-\zeta^2+6\zeta+4)a_1^2u\\
&\quad + \frac{1}{5}(-40\zeta^3-9\zeta^2-38\zeta-23)a_1u^2 + \frac{1}{5}(22\zeta^3+11\zeta^2+33\zeta+19)u^3,\\
y_{11} &= \frac{1}{5}(-\zeta^3+\zeta^2-\zeta+1)a_1^3 + \frac{1}{5}(-7\zeta^3+10\zeta^2-6\zeta-2)a_1^2u\\
&\quad + \frac{1}{5}(29\zeta^3-2\zeta^2+38\zeta+15)a_1u^2 + \frac{1}{5}(-22\zeta^3-11\zeta^2-33\zeta-14)u^3.
\end{aligned}
$$

Choose (x_0, y_{00}) as a preferred generator of $\widehat{H}$. Let $\zeta' = \langle (0,0), (x_0, y_{00})\rangle_\phi$. Then applying the method of Theorem 1.1.1 we can put $(C/H, (x_0, y_{00}))$ in homogeneous Tate normal form. What we find is a curve $T^1(t^*_{\zeta'}(a_1), t^*_{\zeta'}(u))$ with

$$
\begin{aligned}
t^*_{\zeta'}(a_1) &= \frac{1}{5}(-8\zeta^3-6\zeta^2-14\zeta-7)a_1 + \frac{1}{5}(14\zeta^3-2\zeta^2+12\zeta+6)u,\\
t^*_{\zeta'}(u) &= \frac{1}{5}(-\zeta^3-7\zeta^2-8\zeta-4)a_1 + \frac{1}{5}(8\zeta^3+6\zeta^2+14\zeta+7)u.
\end{aligned} \tag{1.4.1}
$$

Remark 1.4.2. We could produce similar formulas for any of the (x_i, y_{ij}) and these would correspond to different choices of ζ' for the Atkin-Lehner dual on $\Gamma_1(5)$-structures. The applications below will be invariant of this choice.

Equation (1.4.1) permits a description of the Atkin-Lehner dual on the ring of $\Gamma_0(5)$-modular forms. For a congruence subgroup Γ, let $MF(\Gamma)$ denote the ring of Γ-modular forms. Let $MF(\Gamma_1(5))_\zeta$ denote the ring of $\Gamma_1(5)$-modular forms over the ring $\mathbb{Z}[\frac{1}{5}, \zeta]$; it is isomorphic to $\mathbb{Z}[\frac{1}{5}, \zeta][a_1, u^{\pm}, \Delta^{-1}]$. Then

$$MF(\Gamma_0(5)) = (MF(\Gamma_1(5))_\zeta^{Gal})^{\mathbb{F}_5^\times}$$

where Gal dentoes the copy of $\mathbb{F}_5^\times$ acting on coefficients.

Theorem 1.4.3. The map $t^* : MF(\Gamma_0(5)) \to MF(\Gamma_0(5))$ induced by the Atkin-Lehner dual is the restriction of the unique map on $MF(\Gamma_1(5)_\zeta)$ determined by (1.4.1).

2. The homotopy groups of $\mathrm{TMF}_0(5)$

By étale descent along the cover

$$\mathcal{M}_1(5) \to \mathcal{M}_1(5)//\mathbb{F}_5^\times = \mathcal{M}_0(5).$$

we have $\mathrm{TMF}_0(5) \simeq \mathrm{TMF}_1(5)^{h\mathbb{F}_5^\times}$. Hill, Hopkins, and Ravenel [9] computed $\pi_* \mathrm{TMF}_0(5)$ using the homotopy point spectral sequence

$$E_2^{s,t} = H^s(\mathbb{F}_5^\times; \pi_t \mathrm{TMF}_1(5)) \implies \pi_{t-s} \mathrm{TMF}_0(5).$$

As this computation is not yet available in the literature, we reproduce it in this section.

2.1. **Computation of the E_2-term.** Consider the representation of $\mathcal{M}_1(5)$ implicit in Corollary 1.1.10. In the context of spectral sequence computations, we will let $x = u$ and let $y = a_1 - u$. Let σ denote the reduction of 2 in $\mathbb{F}_5^\times$ a generator. Then the action of $\mathbb{F}_5^\times$ on $\pi_* \mathrm{TMF}_1(5) = \mathbb{Z}[1/5, x, y, \Delta^{-1}]$ is determined by

$$\begin{aligned} \sigma \cdot x &= y \\ \sigma \cdot y &= -x. \end{aligned} \tag{2.1.1}$$

(This is computed by putting the $\Gamma_1(5)$-structure given by x, y, and the point $[2] \cdot (0,0)$ in Tate normal form.) The discriminant is given by

$$\Delta = x^5 y^5 (y^2 + 11xy - x^2),$$

so x and y are invertible in $\pi_* \mathrm{TMF}_1(5)$ as well.

Proposition 2.1.2. The E_2-term of the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$ is given by

$$H^*(\mathbb{F}_5^\times; \pi_* \mathrm{TMF}_1(5)) = \mathbb{Z}[1/5][b_2, b_4, \delta, \eta, \nu, \gamma, \xi, \Delta^{-1}]/\sim$$

where $\Delta = \delta^2(b_4 + 11\delta)$ and $\sim$ consists of the relations

$$\begin{aligned} b_4^2 &= b_2^2\delta - 4\delta^2, & \eta\nu &= 0, \\ 2\eta &= 0, & b_2\nu &= 0, \\ 2\nu &= 0, & b_2\xi &= \delta\eta^2, \\ 2\gamma &= 0, & \nu\gamma &= 0, \\ 4\xi &= 0, & b_4\xi &= b_2^2\xi + 2\delta\xi + \delta\eta\gamma, \\ \nu^2 &= 2\xi, & b_4\nu &= 0, \\ \gamma^2 &= (b_2^2 + \delta)\eta^2, & b_4\gamma &= (b_4 + \delta)b_2\eta, \\ & & \gamma b_2 &= \eta(b_2^2 + b_4). \end{aligned}$$

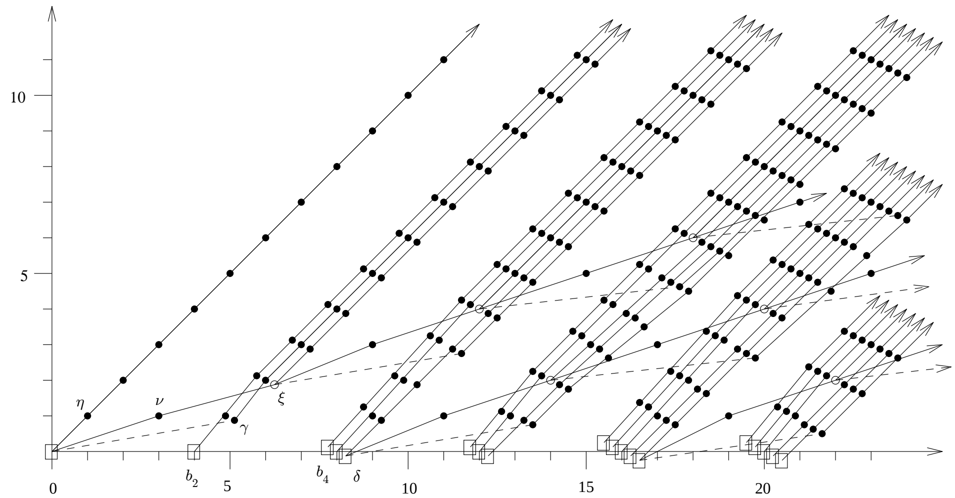


FIGURE 2.1.1. The E_2-term of the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$

The generators lie in bidegrees $(t-s, s)$:

$$\begin{aligned} |b_2| &= (4,0), \\ |b_4| = |\delta| &= (8,0), \\ |\eta| &= (1,1), \\ |\nu| &= (3,1), \\ |\gamma| &= (5,1), \\ |\xi| &= (6,2). \end{aligned}$$

Figure 2.1.1 shows a picture of the the subring of the E_2-term of the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$ with δ not inverted. Here and elsewhere in this paper, we use boxes $\square$ to represent $\mathbb{Z}$'s (or $\mathbb{Z}[1/5]$'s in this case), filled circles $\bullet$ to represent $\mathbb{Z}/2$'s, and open circles $\circ$ to represent $\mathbb{Z}/4$'s.

Proof. This is a rote but fairly involved calculation following from (2.1.1). Let $R = \mathbb{Z}[1/5, \Delta^{\pm}]$. The first step is to determine the structure of $\pi_* \mathrm{TMF}_1(5)$ as a $R[\mathbb{F}_5^\times]$-module.

We begin by setting some notation for $R[\mathbb{F}_5^\times]$-modules. With a slight abuse of notation, let R denote the $R[\mathbb{F}_5^\times]$-module with trivial action, let $R(-1)$ denote R with the sign action $\sigma \cdot n = -z$, let $\tau = R^2$ with the twist action $\sigma \cdot (m,n) = (n,m)$, and let $\psi = R^2$ with the cycle action $\sigma \cdot (m,n) = (n,-m)$. Then we have

$$\begin{aligned} \pi_{8n} \mathrm{TMF}_1(5) &= \tau\{x^{4n}, x^{4n-1}y, \ldots, x^{2n+1}y^{2n-1}\} \oplus R\{x^{2n}y^{2n}\}, \\ \pi_{8n+4} \mathrm{TMF}_1(5) &= \tau\{x^{4n+2}, x^{4n+1}y, \ldots, x^{2n+2}y^{2n}\} \oplus R(-1)\{x^{2n+1}y^{2n+1}\}, \\ \pi_{4n+2} \mathrm{TMF}_1(5) &= \psi\{x^{2n+1}, x^{2n}y, \ldots, x^{n+1}y^n\}. \end{aligned}$$

We can now see that

$$b_2 := x^2 + y^2,$$
$$b_4 := x^3y - xy^3,$$
$$\delta := x^2y^2$$

generate $(\pi_* \mathrm{TMF}_1(5))^{\mathbb{F}_5^\times} = H^0(\mathbb{F}_5^\times; \pi_* \mathrm{TMF}_1(5))$ as an R-module, with relation

$$b_4^2 = b_2^2\delta - 4\delta^2.$$

(Warning: The b_2 and b_4 here are not related to the b_2 and b_4 mentioned in relation to Vélu's formulae.) Note that δ is almost a cube root of Δ: we have

$$\Delta = \delta^2(b_4 + 11\delta).$$

To compute the higher cohomology of $\pi_* \mathrm{TMF}_1(5)$ we begin by noting that

$$H^*(\mathbb{F}_5^\times; R) = R[\beta]/4\beta,$$
$$H^*(\mathbb{F}_5^\times; R(-1)) = R[\beta]/2\beta[1],$$
$$H^*(\mathbb{F}_5^\times; \tau) = R[\beta]/2\beta,$$
$$H^*(\mathbb{F}_5^\times; \psi) = R[\beta]/2\beta[1]$$

where β has cohomological degree 2, [1] denotes a cohomological degree shift by 1, and each cohomology ring has the obvious $H^*(\mathbb{F}_5^\times; R)$-module structure. We define

$$\eta \in H^{1,0}(\mathbb{F}_5^\times; \mathbb{Z}\{x, y\}),$$
$$\nu \in H^1(\mathbb{F}_5^\times; \mathbb{Z}\{xy\}),$$
$$\gamma \in H^1(\mathbb{F}_5^\times; \mathbb{Z}\{x^3, y^3\})$$

to be the unique non-trivial elements in their respective cohomology groups, and define

$$\xi := \beta x^2y^2 = \beta\delta.$$

From this description we can verify that the ring in the proposition has the correct additive structure and it remains to show that the multiplicative relations hold.

Note that the relations in $\sim$ give the correct additive structure of $H^*(\mathbb{F}_5^\times; \pi_* \mathrm{TMF}_1(5))$ and hence there are no other relations. While some of these relations follow from simple dimensional considerations or the additive structure, many of them can be checked explicitly on the cochain level. To this end we have the following 1-cochain representatives, whose values on $\sigma^i \in \mathbb{F}_5^\times$ are displayed below.

g	1	σ	σ^2	σ^3
$\eta(g)$	0	x	$x+y$	y
$\nu(g)$	0	xy	0	xy
$\gamma(g)$	0	x^3	x^3+y^3	y^3

Each of these 1-cochains $\phi(g)$ satisfies the 1-cocycle condition

$$(\delta\phi)(g_1, g_2) = g_1\phi(g_2) - \phi(g_1g_2) + \phi(g_2) = 0.$$

We also record a 2-cocycle $\beta(g_1, g_2)$ which represents β; its values on (g_1, g_2) are recorded in the following table.

$g_2 \backslash g_1$	1	σ	σ^2	σ^3
1	0	0	0	0
σ	0	0	0	1
σ^2	0	0	1	1
σ^3	0	1	1	1

Recall for 1-cocycles $\phi(g)$ and $\psi(g)$, the explicit chain-level formula for the 2-cocycle $\phi \cup \psi$ (see for instance, [1]):

$$(\phi \cup \psi)(g_1, g_2) = (g_1\phi(g_2))\psi(g_1).$$

With these explicit cochain representatives, the desired relations may be checked (when they are not already obvious from dimensional considerations). To illustrate this, we prove the relation $b_4\xi = b_2^2\xi + 2\delta\xi + \delta\eta\gamma$. Since δ is invertible, to prove $b_4\xi = b_2^2\xi + 2\delta\xi + \delta\eta\gamma$ it suffices to establish the relation $b_4\beta = b_2^2\beta + 2\delta\beta + \eta\gamma$, or equivalently (since all of these elements have order 2)

$$\eta\gamma + \beta(b_4 + b_2^2 - 2\delta) = 0.$$

Using our explicit cochain representatives, we compute that $\eta\gamma + \beta(b_4 + b_2^2 - 2\delta)$ is represented by the 2-cocycle $\psi(g_1, g_2)$ whose values are given by the following table.

$g_2 \backslash g_1$	1	σ	σ^2	σ^3
1	0	0	0	0
σ	0	x^3y	$-x^4 - xy^3$	$x^4 + x^3y - xy^3$
σ^2	0	$-x^4 + x^3y$	$-2xy^3$	$x^4 + x^3y$
σ^3	0	$x^3y - xy^3 + y^4$	$x^4 - xy^3$	$x^4 + x^3y + y^4$

This 2-cocycle is seen to be the coboundary of the following 1-cochain ϕ:

g	1	σ	σ^2	σ^3
$\phi(g)$	0	$-xy^3$	$-xy^3$	$x^4 - xy^3$

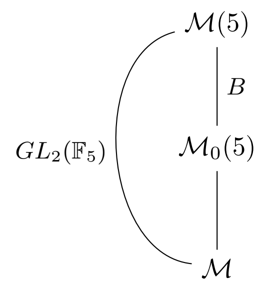
□

2.2. **Computation of the differentials and hidden extensions.** The following sequence of propositions specifies the behavior of the homotopy fixed point spectral sequence

$$H^s(\mathbb{F}_5^\times; \pi_t \, \mathrm{TMF}_1(5)) \Rightarrow \pi_{t-s} \, \mathrm{TMF}_0(5) \tag{2.2.1}$$

culminating in Theorem 2.2.12, a complete description of $\pi_* \, \mathrm{TMF}_0(5)$.

Let $\mathcal{M}(5)$ denote the moduli space of elliptic curves with full level structure, and $\mathrm{TMF}(5)$ the corresponding spectrum of topological modular forms. Utilizing the following portion of [11, Diagram 7.4.3]:



(where B is the Borel subgroup of upper triangular matrices), the spectrum $\mathrm{TMF}(5)$ has an action of $GL_2(\mathbb{F}_5)$, and we have

$$\mathrm{TMF} \simeq \mathrm{TMF}(5)^{hGL_2(\mathbb{F}_5)},$$
$$\mathrm{TMF}_0(5) \simeq \mathrm{TMF}(5)^{hB}.$$

Lemma 2.2.2. The transfer-restriction composition

$$\pi_* \mathrm{TMF} \xrightarrow{\mathrm{Res}} \pi_* \mathrm{TMF}_0(5) \xrightarrow{\mathrm{Tr}} \pi_* \mathrm{TMF}$$

is multiplication by $[GL_2(\mathbb{F}_5) : B] = 6$.

Proof. The theorem is true on the level of homotopy fixed point spectral sequence E_2-terms: the composite

$$H^s(GL_2(\mathbb{F}_5); \pi_t \mathrm{TMF}(5)) \xrightarrow{\mathrm{Res}} H^s(B; \pi_t \mathrm{TMF}(5)) \xrightarrow{\mathrm{Tr}} H^s(GL_2(\mathbb{F}_5); \pi_t \mathrm{TMF}(5))$$

is multiplication by $[GL_2(\mathbb{F}_5) : B] = 6$. Since there are no nontrivial elements of $E_\infty^{s,t}$ with $t - s = 0$ and $s > 0$, it follows that the transfer-restriction on the unit $1_{\mathrm{TMF}} \in \pi_0 \mathrm{TMF}$ is given by

$$\mathrm{Tr}\,\mathrm{Res}(1_{\mathrm{TMF}}) = 6 \cdot 1_{\mathrm{TMF}}.$$

We compute, using the projection formula, that for $a \in \pi_* \mathrm{TMF}$, we have

$$\mathrm{Tr}\,\mathrm{Res}(a) = \mathrm{Tr}\,\mathrm{Res}(a \cdot 1_{\mathrm{TMF}}) = \mathrm{Tr}((\mathrm{Res}\, a) \cdot 1_{\mathrm{TMF}_0(5)}) = a \cdot \mathrm{Tr}(1_{\mathrm{TMF}_0(5)}) = 6 \cdot a.$$

□

We deduce the following corollary.

Corollary 2.2.3. Suppose that $z \in \pi_* \mathrm{TMF}$ satisfies $2z \neq 0$, then $\mathrm{Res}(z)$ in $\pi_* \mathrm{TMF}_0(5)$ is nonzero. Morover, if z has Adams-Novikov filtration s_1, and $2z$ has Adams filtration s_2, then the Adams-Novikov filtration s of $\mathrm{Res}(z)$ satisfies $s_1 \leq s \leq s_2$.

We may now begin computing differentials in the homotopy fixed point spectral sequence (2.2.1).

Proposition 2.2.4. In the homotopy fixed point spectral sequence (2.2.1), $E_2 = E_3$ and the d_3-differentials are determined by

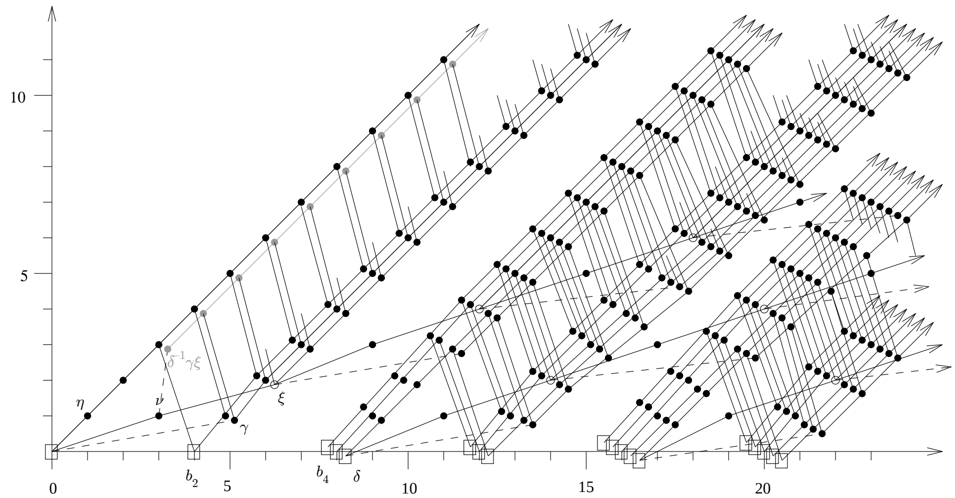


FIGURE 2.2.1. The d_3-differentials in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$.

$$d_3 b_2 = \eta^3,$$
$$d_3\xi = \delta^{-1}\eta\xi^2,$$
$$d_3\gamma = \delta^{-1}\eta\gamma\xi.$$

Figure 2.2.1 shows the d_3 differentials in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$. While most terms involving δ^{-1} are excluded, those depicted are shown in gray.

Proof. First note that $d_3 a_1^2 h_1 = h_1^4$ in the Adams-Novikov spectral sequence for TMF (we use the notation of [2]). Under the restriction map $\mathrm{TMF} \to \mathrm{TMF}_0(5)$, this differential maps to $d_3 b_2\eta = \eta^4$, from which it follows that $d_3 b_2 = \eta^3$.

By Corollary 2.2.3, 2ν must be detected in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$ in Adams-Novikov filtration between 1 and 3. Since $2\nu = 0$ in the E_2-page, it follows that in fact the filtration has to be between 2 and 3, and the only candidates live in filtration 3. Given the differential in the previous paragraph, the only possible candidate to detect 2ν in $\mathrm{TMF}_0(5)$ is $\delta^{-1}\gamma\xi$; moreover, the class representing $2\eta\nu$, i.e. $\delta^{-1}\eta\gamma\xi$, must die in the spectral sequence. This is only possible if $d_3\gamma = \delta^{-1}\eta\gamma\xi$.

Since $\delta^{-1}\gamma\xi$ is a permanent cycle, we have

$$0 = d_3\delta^{-1}\gamma\xi = (d_3\delta^{-1}\xi)\gamma - \delta^{-1}\xi(d_3\gamma).$$

Hence $d_3\xi = \delta^{-1}\eta\xi^2$. □

Figure 2.2.2 shows the resulting E_4-term in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$. Terms involving δ^{-1} are excluded on the 0, 1 and 2-lines, and in lines greater than 2 are shown in gray.

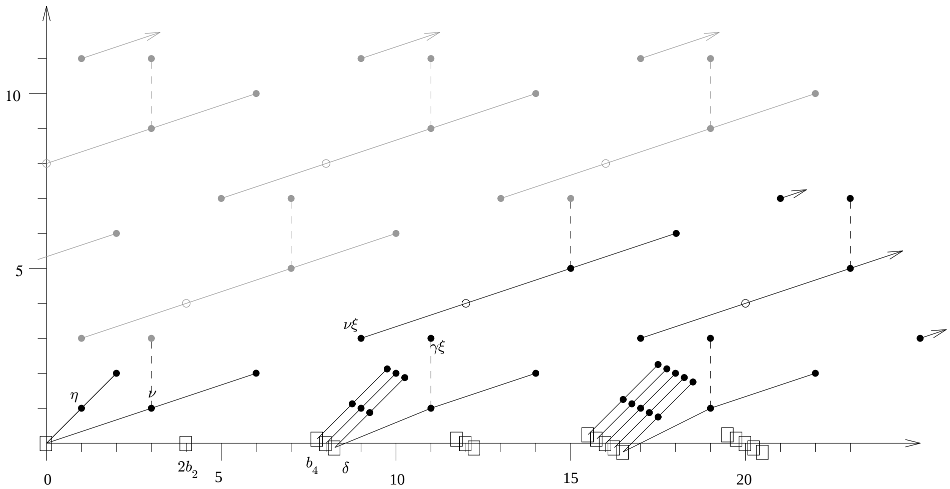


FIGURE 2.2.2. The $E_4 = E_5$ term in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$.

In the following sequence of propositions, we will establish the rest of the differentials in the homotopy fixed point spectral sequence. Figure 2.2.3 displays these differentials. In this figure, the gray patterns represent the (infinite rank) *bo*-patterns.

Proposition 2.2.5. In the homotopy fixed point spectral sequence for $\pi_* \mathrm{TMF}_0(5)$, $E_4 = E_5$ and the d_5-differentials are determined by

$$d_5\delta = \delta^{-1}\nu\xi^2.$$

Proof. The element $\overline{\kappa} \in \pi_{20}S$ is in the Hurewicz image of TMF. In the Adams-Novikov spectral sequence for TMF, $d_5\Delta = \nu\overline{\kappa}$. Since

$$\mathrm{Res}(\Delta) = \delta^2(b_4 + 11\delta),$$
$$\mathrm{Res}(\bar{\kappa}) = \delta\xi^2$$

we deduce that

$$\begin{aligned}\nu\delta\xi^2 &= d_5(\delta^2(b_4 + 11\delta)) \\ &= 2\delta d_5(\delta)(b_4 + 11\delta) + \delta^2 d_5(b_4) + 11\delta^2 d_5(\delta) \\ &= 2\delta b_4 d_5(\delta) + 33\delta^2 d_5(\delta) + \delta^2 d_5(b_4).\end{aligned}$$

Since the only available class for $d_5(\delta)$ to hit is 2-torsion in the E_5-page, we deduce that

$$\delta^2 d_5(\delta) + \delta^2 d_5(b_4) = \nu\delta\xi^2$$

and one of the following different scenarios occurs:

$$\text{Case 1}: \quad d_5(\delta) = \delta^{-1}\nu\xi^2 \text{ and } d_5(b_4) = 0,$$
$$\text{Case 2}: \quad d_5(\delta) = 0 \text{ and } d_5(b_4) = \delta^{-1}\nu\xi^2.$$

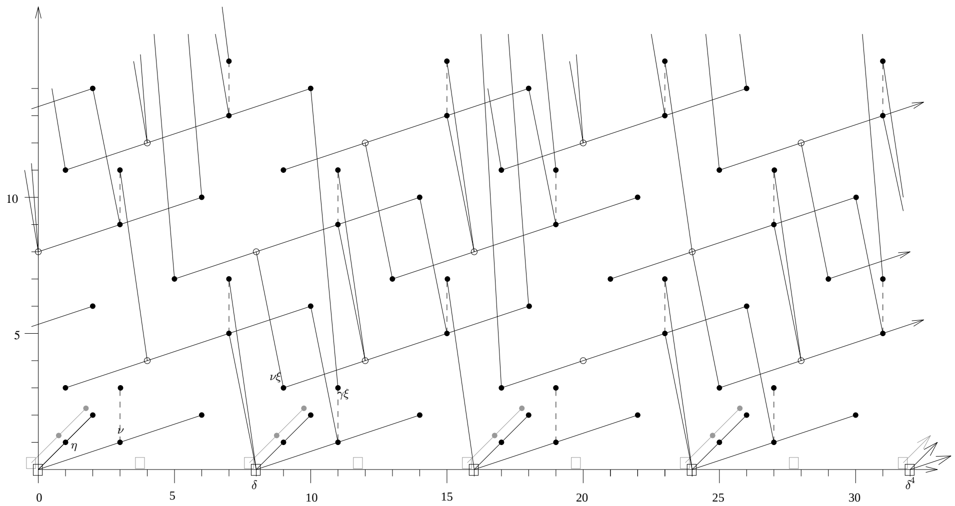


FIGURE 2.2.3. The E_4 term in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$ with d_r-differentials, $r \geq 4$.

In case 2, we can multiply by the permanent cycle $\mathrm{Res}(\bar{\kappa}) = \delta\xi^2$ to deduce

$$d_5(b_4\delta\xi^2) = \nu\xi^4.$$

However, using the relations Proposition 2.1.2, we have

$$b_4\delta\xi^2 = \delta^3\eta^4 + 2\delta^2\xi^2 + \delta^2\xi\gamma\eta.$$

Since $\delta^3\eta^4 = \delta^2\xi\gamma\eta = 0$ in E_3, and since $\delta^2\xi^2$ persists to the E_5-page, it follows that $\nu\xi^4$ is 2-divisible in E_5, which is not true. Thus "Case 2" above is actually impossible, and we must be in "Case 1." We deduce that

$$d_5\delta = \delta^{-1}\nu\xi^2.$$

□

Proposition 2.2.6. In the homotopy fixed point spectral sequence for $\pi_* \mathrm{TMF}_0(5)$, $E_6 = E_7$ and the d_7-differentials are determined by

$$\begin{aligned} d_7(2\delta) = d_7(b_4) &= \delta^{-2}\gamma\xi^3, \\ d_7(2\delta^3) = d_7(\delta^2 b_4) &= \gamma\xi^3, \\ d_7(\delta^2) &= \delta^{-1}\gamma\xi^3, \\ d_7(\delta b_4) &= 0. \end{aligned}$$

Proof. For the first differential, note that $2\nu\overline{\kappa}$ is 0 in $\pi_* \mathrm{TMF}$, from which we deduce that the class represented by $\delta^{-1}\gamma\xi\overline{\kappa}$ is 0 in $\pi_* \mathrm{TMF}_0(5)$ via the restriction map. The element $\gamma\xi^3$ detects this class, so it must be the target of a differential, and the only (not necessarily exclusive) possibilities at this point are:

$$\begin{aligned}&\text{Case 1:} \quad d_7(2\delta^3) = \gamma\xi^3,\\ &\text{Case 2:} \quad d_7(\delta^{2-i}b_4 b_2^{2i}) = \gamma\xi^3, \quad \text{for some } i \geq 0,\\ &\text{Case 3:} \quad d_7(\delta^{2-i}b_2^{2i+2}) = \gamma\xi^3, \quad \text{for some } i \geq 0.\end{aligned}$$

Multiplying by the permanent cycle $\mathrm{Res}(\bar{\kappa}) = \delta\xi^2$, Cases 2 yields

$$d_7(\delta^{5-i}\eta^4 b_2^{2i} + 2\delta^{4-i}\xi^2 b_2^{2i} + \delta^{4-i}\xi\gamma\eta) \neq 0$$

If $i > 0$, this is a contradiction because

$$\delta^{5-i}\eta^4 b_2^{2i} = 2\delta^{4-i}\xi^2 b_2^{2i} = \delta^{4-i}\xi\gamma\eta = 0$$

in the E_7-page for $i > 0$. Therefore Case 2 for $i > 0$ cannot occur. Similarly, multiplying Case 3 by $\bar{\kappa}$ gives

$$d_7(\delta^{5-i}b_2^{2i}\eta^4) \neq 0,$$

again a contradiction. We conclude that either Case 1 or Case 2 with $i = 0$ must hold. Therefore

$$\begin{aligned} d_7(2\delta^3) &= a\gamma\xi^3,\\ d_7(\delta^2 b_4) &= b\gamma\xi^3 \end{aligned}$$

with $a = 1$ or $b = 1$. Multiplying both of the above differentials by $\bar{\kappa}$ yields:

$$\begin{aligned} d_7(2\delta^4\xi^2) &= a\delta\gamma\xi^5,\\ d_7(2\delta^4\xi^2) &= b\delta\gamma\xi^5. \end{aligned}$$

We deduce that $a = b = 1$. Hence we deduce that

$$\begin{aligned} d_7(2\delta^3) &= \gamma\xi^3,\\ d_7(\delta^2 b_4) &= \gamma\xi^3. \end{aligned}$$

We now turn our attention to $d_7(2\delta)$, $d_7(b_4)$, and $d_7(\delta^2)$. The only possible targets for these differentials are $\delta^{-2}\gamma\xi^3$ and $\delta^{-1}\gamma\xi^3$, respectively. Write

$$\begin{aligned} d_7(2\delta) &= c\delta^{-2}\gamma\xi^3,\\ d_7(b_4) &= d\delta^{-2}\gamma\xi^3,\\ d_7(\delta^2) &= e\delta^{-1}\gamma\xi^3. \end{aligned}$$

Then we have

$$\begin{aligned} \gamma\xi^3 &= d_7(\delta^2 b_4)\\ &= d_7(\delta^2)b_4 + \delta^2 d_7(b_4)\\ &= e\delta^{-2}\gamma\xi^3 b_4 + d\gamma\xi^3. \end{aligned}$$

Using the relations we find that $\delta^{-2}\gamma\xi^3 b_4 = 0$, and we therefore deduce that $d = 1$. Similarly, we have

$$\begin{aligned} \gamma\xi^3 &= d_7(2\delta^3)\\ &= d_7(\delta^2)2\delta + \delta^2 d_7(2\delta)\\ &= 2e\delta^{-1}\gamma\xi^3 + c\gamma\xi^3. \end{aligned}$$

Since $2\delta^{-1}\gamma\xi^3 = 0$, we deduce that $c = 1$. We have shown

$$d_7(2\delta) = \delta^{-2}\gamma\xi^3,$$
$$d_7(b_4) = \delta^{-2}\gamma\xi^3.$$

To establish the final d_7 differential on δ^2, note that the restriction map $\mathrm{TMF} \to \mathrm{TMF}_0(5)$ takes $2\nu\Delta$ to $2\nu\Delta$ which is nonzero in $\pi_* \mathrm{TMF}_0(5)$. Since $2\nu\Delta\bar{\kappa} = 0 \in \pi_* \mathrm{TMF}$, we know $2\nu\Delta\bar{\kappa} = 0 \in \pi_* \mathrm{TMF}_0(5)$. The element $\gamma\xi^3\delta^3$ detects this class. It follows that $\delta^{-1}\gamma\xi^3$ must be the target of a differential. By the same argument used earlier, multiplication by $\bar{\kappa}$ shows that the only possible sources of a differential killing $\delta^{-1}\gamma\xi^3$ are δ^2 and δb_4. Write

$$d_7(\delta^2) = e\delta^{-1}\gamma\xi^3,$$
$$d_7(\delta b_4) = f\delta^{-1}\gamma\xi^3.$$

so that e or f equals 1 mod 2. Multiplying both of these differentials by $\bar{\kappa}$ yields

$$d_7(\delta^3\xi^2) = e\gamma\xi^5,$$
$$d_7(2\delta^3\xi^2) = f\gamma\xi^5.$$

Thus we have $e \equiv 1 \bmod 2$, and $f \equiv 0 \bmod 2$, and

$$d_7(\delta^2) = \delta^{-1}\gamma\xi^3,$$
$$d_7(\delta b_4) = 0.$$

□

To handle the next round of differentials we will need the following lemma.

Lemma 2.2.7. The Hurewicz image of the element κ in $\pi_{14}\,\mathrm{TMF}$ is restricts to a non-trivial class in $\pi_{14}\,\mathrm{TMF}_0(5)$, detected by $\nu^2\delta$ in the homotopy fixed point spectral sequence.

Proof. Applying Corollary 2.2.3 to the class $\Delta^4\kappa \in \pi_{110}\,\mathrm{TMF}$ of order 4, we find that $\mathrm{Res}(\Delta^4\kappa)$ is non-trivial, and detected in the homotopy fixed point spectral sequence by a class in filtration between 4 and 14. Given our d_5-differentials, the only candidate is $\nu^2\delta^{13}$. Since E_2 is δ-periodic, and since κ is detected in filtration 2 in TMF, it follows that on the level of E_2 pages κ restricts to $\nu^2\delta$. The lemma follows, since $\nu^2\delta$ is not the target of a differential. □

Proposition 2.2.8. In the homotopy fixed point spectral sequence for $\pi_*\,\mathrm{TMF}_0(5)$, $E_8 = E_9 = E_{10}$ and the d_{11}-differentials are determined by

$$d_{11}(\gamma\xi) = \delta^{-4}\xi^7.$$

Proof. In $\pi_*\,\mathrm{TMF}$ we have $\bar{\kappa}^3\kappa = 0$. The restriction of this element in $\mathrm{TMF}_0(5)$ is detected in the homotopy fixed point spectral sequence by $\delta^4\xi^7$, so the latter must be the target of a differential. The only possibility is $d_{11}(\delta^8\gamma\xi) = \delta^4\xi^7$. Since δ^4 persists to the E_{11}-page, and there are no non-trivial targets for $d_{11}(\delta^4)$, it follows that E_{11} is δ^4-periodic, and the proposition follows. □

Proposition 2.2.9. In the homotopy fixed point spectral sequence for $\pi_* \mathrm{TMF}_0(5)$, $E_{12} = E_{13}$ and the d_{13}-differentials are determined by

$$d_{13}(\delta\nu\xi) = \delta^{-4}\xi^8,$$
$$d_{13}(\delta^3\nu^2) = \delta^{-2}\nu\xi^7.$$

Proof. In $\pi_* \mathrm{TMF}$ we have $\bar{\kappa}^6 = 0$. Since $\mathrm{Res}(\bar{\kappa}^6)$ is detected by $\delta^6\xi^{12}$ in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$, the latter must be the target of a differentials. Since $\bar{\kappa}\delta^6\xi^{12}$ is non-trivial in E_{13}, if $d_r(x) = \delta^6\xi^{12}$ it must be the case that $\bar{\kappa} \cdot x \neq 0$. The only such candidate is

$$d_{13}(\delta^{11}\nu\xi^5) = \delta^6\xi^{12}.$$

Dividing by $\bar{\kappa}^2$, it follows that we have

$$d_{13}(\delta^9\nu\xi) = \delta^4\xi^8.$$

Since δ^4 persists to E_{13} with no possible targets for a non-trivial $d_{13}(\delta^4)$, it follows that

$$d_{13}(\delta\nu\xi) = \delta^{-4}\xi^8.$$

The differential $d_{13}\delta^3\nu^2 = \delta^{-2}\nu\xi^7$ actually follows from the differential above, though perhaps not so obviously, so we will explain in more detail. The element $\xi^3\nu$ persists to the E_{13}-page, and there are no possibilities for it supporting a non-trivial d_{13}-differential. However, by the previous paragraph,

$$\bar{\kappa}^4\xi^3\nu = \delta^4\xi^{11}\nu = d_{13}(\delta^9\xi^4\nu^2) \neq 0 \in E_{13}.$$

Dividing by $\bar{\kappa}^2$, we get

$$d_{13}(\delta^7\nu^2) = \delta^2\xi^7\nu$$

and thus

$$d_{13}(\delta^3\nu^2) = \delta^{-2}\xi^7\nu.$$

□

This concludes the determination of the differentials in the homotopy fixed point spectral sequence, there are no further possibilities. We now turn to determining the hidden extensions in this spectral sequence. To do this, we will recompute $\pi_* \mathrm{TMF}_0(5)$ using a homotopy orbit spectral sequence. This different presentation will turn out to elucidate the multiplicative structure missed by the homotopy fixed point spectral sequence.

The Tate spectral sequence

$$\widehat{H}^s(\mathbb{F}_5^\times; \pi_t \mathrm{TMF}_1(5)) \Rightarrow \pi_{t-s} \mathrm{TMF}_1(5)^{t\mathbb{F}_5^\times}$$

can be easily computed from the homotopy fixed point spectral sequence — one simply has to invert ξ. A picture of the resulting spectral sequence (just from E_4 and beyond) is displayed in Figure 2.2.4.

Note that everything dies in this spectral sequence. Therefore, we have established the following lemma. (There may be other more conceptual ways of proving the following Lemma — for instance, it is well known to hold $K(2)$-locally, and the unlocalized statement might follow from the fact that $\mathcal{M}_1(5) \to \mathcal{M}_0(5)$ is a Galois cover).

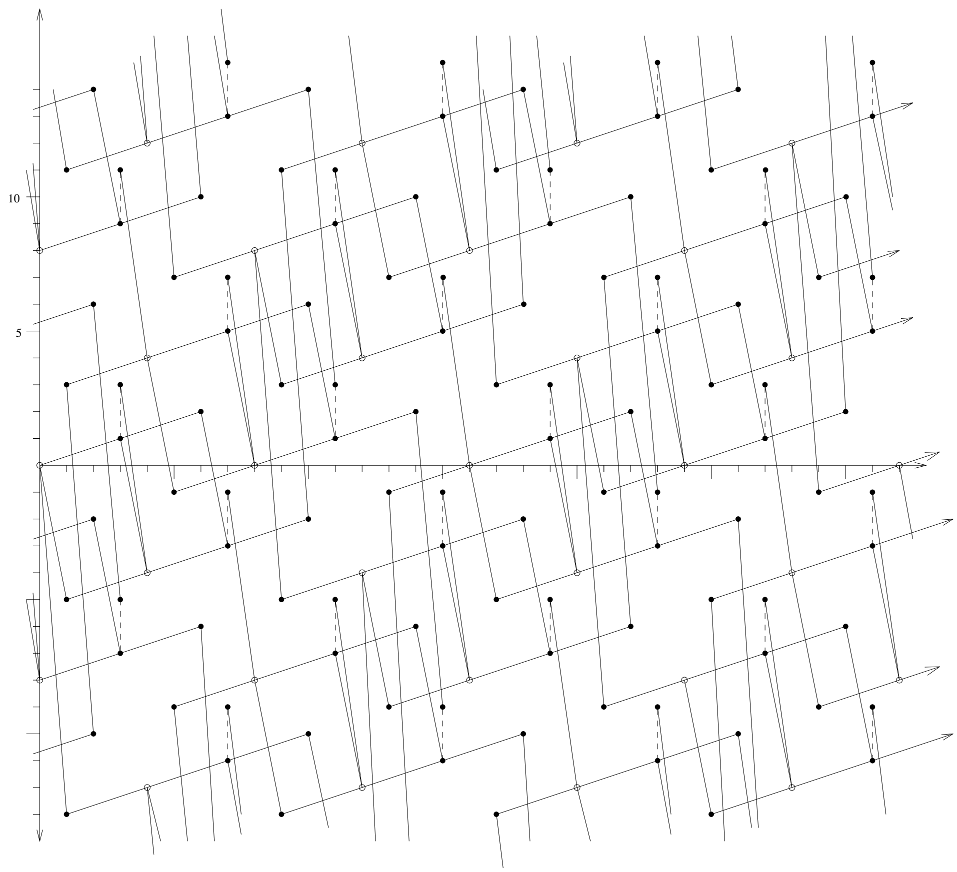


FIGURE 2.2.4. The E_4 term in the Tate spectral sequence for $\mathrm{TMF}_1(5)^{t\mathbb{F}_5^\times}$ with d_r-differentials, $r \geq 4$.

Lemma 2.2.10. The Tate spectrum $\mathrm{TMF}_1(5)^{t\mathbb{F}_5^\times}$ is trivial, and therefore the norm map

$$N : \mathrm{TMF}_1(5)_{h\mathbb{F}_5^\times} \to \mathrm{TMF}_1(5)^{h\mathbb{F}_5^\times}$$

is an equivalence.

Thus the homotopy groups of $\pi_* \mathrm{TMF}_1(5)_{h\mathbb{F}_5^\times} = \pi_* \mathrm{TMF}_0(5)$ are isomorphic to $\pi_* \mathrm{TMF}_1(5)^{h\mathbb{F}_5^\times}$ as modules over $\pi_* \mathrm{TMF}$. However, these $\pi_* \mathrm{TMF}$-modules are computed in an entirely different way by the homotopy orbit spectral sequence

$$H_s(\mathbb{F}_5^\times; \pi_t \mathrm{TMF}_1(5)) \Rightarrow \pi_{s+t} \mathrm{TMF}_0(5).$$

Nevertheless, the homotopy orbit spectral sequence (with differentials) can be computed by simply truncating the Tate spectral sequence (and manually computing H_0 where appropriate). The resulting homotopy orbit spectral sequence is displayed in Figure 2.2.5. As with our other spectral sequences, we are displaying the E_4-page, with all remaining differentials. The (infinite rank) *bo* patterns are displayed in gray.

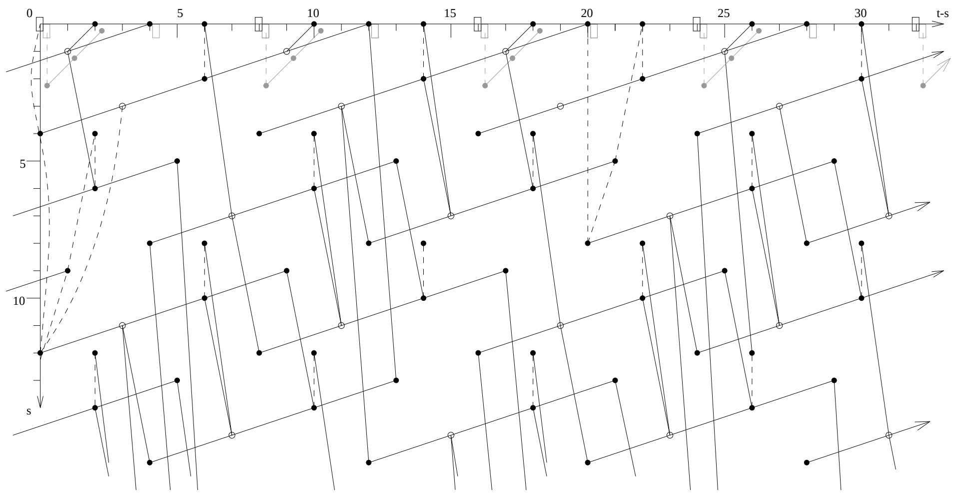


FIGURE 2.2.5. The E_4 term in the homotopy orbit spectral sequence for $\mathrm{TMF}_1(5)_{h\mathbb{F}_5^\times}$ with d_r-differentials, $r \geq 4$.

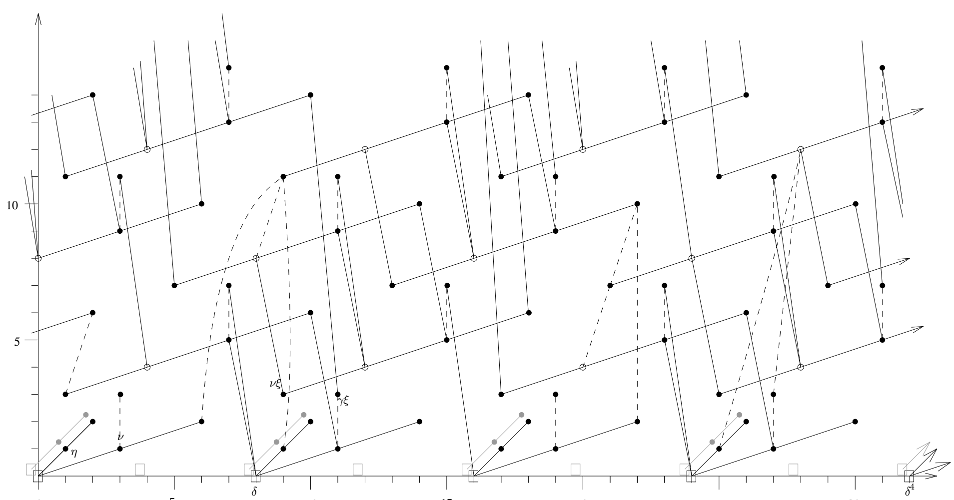


FIGURE 2.2.6. The hidden extensions in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$.

There are many hidden extensions (as $\pi_* \mathrm{TMF}$ modules) in the homotopy orbit spectral sequence (HOSS) which are not hidden in the homotopy fixed point spectral sequence (HFPSS). Since $\pi_0 \mathrm{TMF}_0(5)$ is seen to be torsion free in the HFPSS, there must be additive extensions as indicated in Figure 2.2.5, and $1 \in \pi_0 \mathrm{TMF}_0(5)$ must be detected on the $s = 12$ line. Since the HFPSS shows η, η^2, and ν are nontrivial in $\pi_* \mathrm{TMF}_0(5)$, there must be corresponding hidden extensions in the HOSS. Multiplying these by $\bar{\kappa}$ in the HOSS yields hidden η and η^2 extensions supported by $\bar{\kappa}$.

We will now deduce the hidden extensions in the HFPSS from multiplicative structure in the HOSS. The resulting extensions are displayed in Figure 2.2.6.

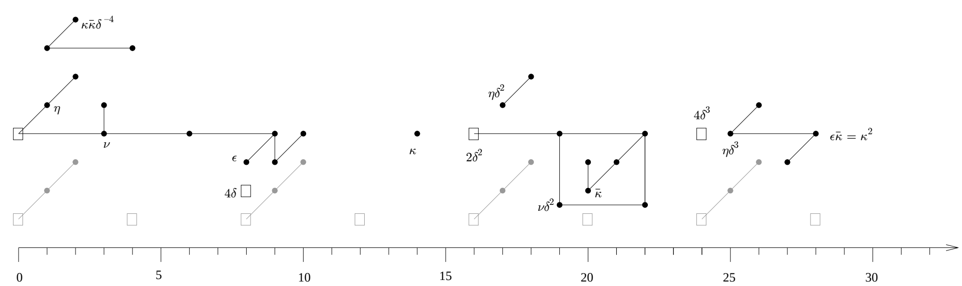
Since $\eta\bar{\kappa}$ and $\eta^2\bar{\kappa}$ are seen to be be non-trivial in $\pi_* \mathrm{TMF}_0(5)$ using hidden extensions in the HOSS, we obtain corresponding new hidden extensions in the HFPSS. With the one exception $\eta \cdot \delta^2\gamma\xi$, all of the other hidden extensions displayed in Figure 2.2.6 follow from non-hidden extensions in the HOSS. The remaining extension is addressed in the following lemma.

Lemma 2.2.11. In the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$, there is a hidden extension

$$\eta \cdot \delta^2\gamma\xi = \delta^{-1}\xi^6.$$

Proof. Observe that since ν^3 is non-trivial in $\pi_* \mathrm{TMF}_0(5)$, and in $\pi_* \mathrm{TMF}$ we have $\nu^3 = \eta\epsilon$, it must follow that ϵ is detected by $\delta^{-2}\xi^4$ in the HFPSS. Thus $\bar{\kappa}\epsilon$ is detected by $\delta^{-1}\xi^6$. However, $\bar{\kappa}\epsilon$ is η-divisible in $\pi_* \mathrm{TMF}$. It follows that it must also be η-divisible in $\pi_* \mathrm{TMF}_0(5)$, and the hidden extension claimed is the only possibility to make this happen. □

Theorem 2.2.12. The homotopy groups $\pi_* \mathrm{TMF}_0(5)$ are given by the following δ^4-periodic pattern.



3. $Q(\ell)$-SPECTRA

We now begin working with the $Q(\ell)$ spectra in earnest. We review the definition of $Q(\ell)$ in 3.1 and in 3.2 recall the double complex that computes the E_2-term of its Adams-Novikov spectral sequence.

In previous sections we have focused on data for $Q(5)$ but in 3.3 we review formulas of Mahowald and Rezk from [14] related to $Q(3)$. Finally in 3.4 we recall the formuals of Section 1 in forms that will be useful in subsequent calculations.

3.1. **Definitions.** In [3], the p-local spectrum $Q(\ell)$ $(p \nmid \ell)$ is defined as the totalization of an explicit semi-cosimplicial E_∞-ring spectrum of the form

$$Q(\ell)^\bullet = \left(\mathrm{TMF} \Rightarrow \mathrm{TMF}_0(\ell) \times \mathrm{TMF} \Rrightarrow \mathrm{TMF}_0(\ell) \right).$$

The coface maps from level 0 to level 1 are given by:

$$d_0 = q^* \times \psi^\ell,$$
$$d_1 = f^* \times 1,$$

and the coface maps from level 1 to level 2 are given by

$$d_0 = t^* \circ \pi_2,$$
$$d_1 = f^* \circ \pi_1,$$
$$d_2 = \pi_2,$$

where π_i are the projections onto the components. These maps are induced by the maps of stacks

$$\begin{aligned} \psi^\ell : \mathcal{M}^1 \to \mathcal{M}^1, &\quad (C, \vec{v}) \mapsto (C, \ell \cdot \vec{v}), \\ f : \mathcal{M}_0^1(\ell) \to \mathcal{M}^1, &\quad (C, H, \vec{v}) \mapsto (C, \vec{v}), \\ q : \mathcal{M}_0^1(\ell) \to \mathcal{M}^1, &\quad (C, H, \vec{v}) \mapsto (C/H, (\phi_H)_* \vec{v}), \\ t : \mathcal{M}_0^1(\ell) \to \mathcal{M}_0^1(\ell), &\quad (C, H, \vec{v}) \mapsto (C/H, \widehat{H}, (\phi_H)_* \vec{v}), \end{aligned}$$

where $\phi_H : (C, H) \to C/H$ is the quotient isogeny. The map $\psi^\ell : MF_k \to MF_k$ is analogous to an Adams operation, and acts by multiplication by ℓ^k. Formulas for f^*, q^*, and t^*, on the level of modular forms are typically computed differently for different choices of ℓ, and are more complicated.

3.2. **The double complex.** As done in the special case of $\ell = 2$ and $p = 3$ in [3], one can form a total cochain complex to compute the E_2-term for the Adams-Novikov spectral sequence for $Q(\ell)$. Let (A, Γ) denote the usual elliptic curve Hopf algebroid, and let $(B^1(\ell), \Lambda^1(\ell))$ denote a Hopf algebroid which stackifies to give $\mathcal{M}_0^1(\ell)$. Let $C^*_\Gamma(A)$, $C^*_{\Lambda^1(\ell)}(B^1)$ denote the corresponding cobar complexes, so the corresponding Adams-Novikov spectral sequences take the form

$$E_2^{s,2t} = H^s(\mathcal{M}, \omega^{\otimes t}) = H^s(C^*_\Gamma(A)_{2t}) \Rightarrow \pi_{2t-s}\,\mathrm{TMF},$$
$$E_2^{s,2t} = H^s(\mathcal{M}_0(\ell), \omega^{\otimes t}) = H^s(C^*_{\Lambda^1(\ell)}(B^1(\ell))_{2t}) \Rightarrow \pi_{2t-s}\,\mathrm{TMF}_0(\ell).$$

Corresponding to the cosimplicial decomposition of $Q(\ell)$ we can form a double complex $C^{*,*}(Q(\ell))$

(3.2.1)
$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \uparrow & & \uparrow & & \uparrow & & \\ C^1_\Gamma(A) & \longrightarrow & C^1_{\Lambda^1(\ell)}(B^1(\ell)) \oplus C^1_\Gamma(A) & \longrightarrow & C^1_{\Lambda^1(\ell)}(B^1(\ell)) & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ C^0_\Gamma(A) & \longrightarrow & C^0_{\Lambda^1(\ell)}(B^1(\ell)) \oplus C^0_\Gamma(A) & \longrightarrow & C^0_{\Lambda^1(\ell)}(B^1(\ell)) & \longrightarrow & \cdots \end{array}$$

Let $C^*_{tot}(Q(\ell))$ denote the corresponding total complex. Then the Adams-Novikov spectral sequence for $Q(\ell)$ takes the form

$$E_2^{s,2t} = H^s(C^*_{tot}(Q(\ell))_{2t}) \Rightarrow \pi_{2t-s} Q(\ell).$$

3.3. **Recollections about** $Q(3)$**.** Mahowald and Rezk [14] performed a study of the explicit formulas for $Q(3)$ similar to our current treatment of $Q(5)$. We summarize some of their results here for the reader's convenience.

The moduli space $\mathcal{M}_1^1(3)$ is represented by the affine scheme $\mathrm{Spec}(B^1(3))$ with

$$B^1(3) = \mathbb{Z}[1/3, a_1, a_3, \Delta^{-1}]$$

with

$$\Delta = a_3^3(a_1^3 - 27a_3).$$

The corresponding universal $\Gamma_1(3)$ structure is carried by the Weierstrass curve

$$y^2 + a_1xy + a_3y = x^3$$

with point $P = (0,0)$ of order 3. The $\mathbb{G}_m$-action on $\mathcal{M}_1^1(3)$ induces a grading on $B^1(3)$, for which a_i has weight i. It follows that

$$\pi_* \mathrm{TMF}_1(3) = \mathbb{Z}[1/3, a_1, a_3, \Delta^{-1}]$$

with topological degrees $|a_i| = 2i$. The spectrum $\mathrm{TMF}_1(3)$ admits a complex orientation with $v_1 = a_1$ and $v_2 = a_3$.

The group $\mathbb{F}_3^\times = \{\pm 1\}$ acts on $\mathcal{M}_1^1(5)$ by sending an R-point (C, P) (where P is a point of exact order 3 on C) to the R-point $(C, [-1](P))$. This induced action of $\mathbb{F}_3^\times$ on the ring $B^1(3)$ is given by

$$[-1](a_1) = -a_1,$$
$$[-1](a_3) = -a_3.$$

We have

$$\mathcal{M}_0^1(3) = \mathcal{M}_1^1(3)//\mathbb{F}_3^\times$$

and hence an equivalence

$$\mathrm{TMF}_0(3) \simeq \mathrm{TMF}_1(3)^{h\mathbb{F}_3^\times}.$$

The resulting homotopy fixed point spectral sequence takes the form

$$H^s(\mathbb{F}_3^\times; \pi_t \mathrm{TMF}_1(3)) \Rightarrow \pi_{t-s} \mathrm{TMF}_0(3).$$

In particular, the ring of modular forms (meromorphic at the cusps) for $\Gamma_0(3)$ is the subring

$$MF(\Gamma_0(3)) = H^0(\mathbb{F}_3^\times; MF(\Gamma_1(3)) = \mathbb{Z}[1/3, a_1^2, a_1a_3, a_3^2, \Delta^{-1}] \subset B^1(3).$$

Mahowald and Rezk also compute the effects of the maps

$$q^*, f^* : A \to B^1(3),$$
$$t^* : B^1(3) \to B^1(3)$$

as

$$\begin{aligned} f^*(a_1) &= a_1, & q^*(a_1) &= a_1, \\ f^*(a_2) &= 0, & q^*(a_2) &= 0, \\ f^*(a_3) &= a_3, & q^*(a_3) &= 3a_3, \\ f^*(a_4) &= 0, & q^*(a_4) &= -6a_1a_3, \\ f^*(a_6) &= 0, & q^*(a_6) &= -(9a_3^2 + a_1^3a_3), \end{aligned}$$

$$\begin{aligned} t^*(a_1^2) &= -3a_1^2, \\ t^*(a_1a_3) &= \frac{1}{3}a_1^4 - 9a_1a_3, \\ t^*(a_3^2) &= -\frac{1}{27}a_1^6 + 2a_1^3a_3 - 27a_3^2. \end{aligned}$$

3.4. **The formulas for $Q(5)$.** The moduli space $\mathcal{M}_1^1(5)$ is represented by the affine scheme $\mathrm{Spec}(B^1(5))$ with

$$B^1(5) = \mathbb{Z}[1/5, a_1, u, \Delta^{-1}]$$

with

$$\Delta = -11u^{12} + 64a_1u^{11} - 154a_1^2u^{10} + 195a_1^3u^9 - 135a_1^4u^8 + 46a_1^5u^7 - 4a_1^6u^6 - a_1^7u^5.$$

The corresponding universal $\Gamma_1(5)$ structure is carried by the Weierstrass curve

$$y^2 + a_1xy + (a_1u^2 - u^3)y = x^3 + (a_1u - u^2)x^2$$

with point $P = (0,0)$ of order 5. The $\mathbb{G}_m$-action on $\mathcal{M}_1^1(5)$ induces a grading on $B^1(5)$, for which a_1 and u both have weight 1. It follows that

$$\pi_* \mathrm{TMF}_1(5) = \mathbb{Z}[1/5, a_1, u, \Delta^{-1}]$$

with topological degrees $|a_1| = |u| = 2$. The spectrum $\mathrm{TMF}_1(5)$ admits a complex orientation with $v_1 = a_1$ and $v_2 \equiv u^3 \mod (2, v_1)$.

The group $\mathbb{F}_5^\times \cong C_4$ acts on $\mathcal{M}_1^1(5)$: for $5 \nmid n$, the mod 5 reduction $[n] \in \mathbb{F}_5^\times$ acts by sending an R-point (C, P) (where P is a point of exact order 5 on C) to the R-point $(C, [n](P))$. This induced action of the generator $[2]$ of $\mathbb{F}_5^\times$ on the ring $B^1(5)$ is given by

$$\begin{aligned} [2](a_1) &= a_1 - 2u, \\ [2](u) &= a_1 - u. \end{aligned}$$

These have the more convenient expression

$$\begin{aligned} [2](u) &= b_1, \\ [2](b_1) &= -u \end{aligned}$$

where $b_1 := a_1 - u$. We have

$$\mathcal{M}_0^1(5) = \mathcal{M}_1^1(5)//\mathbb{F}_5^\times$$

and hence an equivalence

$$\mathrm{TMF}_0(5) \simeq \mathrm{TMF}_1(5)^{h\mathbb{F}_5^\times}.$$

The resulting homotopy fixed point spectral sequence takes the form

$$H^s(\mathbb{F}_5^\times; \pi_t \mathrm{TMF}_1(5)) \Rightarrow \pi_{t-s} \mathrm{TMF}_0(5).$$

In particular, the ring of modular forms (meromorphic at the cusps) for $\Gamma_0(5)$ is the subring

$$MF(\Gamma_0(5)) = H^0(\mathbb{F}_5^\times; MF(\Gamma_1(5)) = \frac{\mathbb{Z}[1/5, b_2, b_4, \delta][\Delta^{-1}]}{(b_4^2 = b_2^2\delta - 4\delta^2)} \subset B^1(5)$$

where

$$\begin{aligned} b_2 &:= u^2 + b_1^2, \\ b_4 &:= u^3 b_1 - u b_1^3, \\ \delta &:= u^2 b_1^2. \end{aligned}$$

Note that δ is almost a cube root of Δ; we have

$$\Delta = 11\delta^3 + \delta^2 b_4.$$

The effect of the maps

$$\begin{aligned} q^*, f^* &: A \to B^1(5), \\ t^* &: B^1(5) \to B^1(5) \end{aligned}$$

is

$$\begin{aligned} f^*(a_1) &= a_1, & q^*(a_1) &= a_1, \\ f^*(a_2) &= a_1 u - u^2, & q^*(a_2) &= -u^2 + a_1 u, \\ f^*(a_3) &= a_1 u^2 - u^3, & q^*(a_3) &= -u^3 + a_1 u^2, \\ f^*(a_4) &= 0, & q^*(a_4) &= -10u^4 + 30 a_1 u^3 - 25 a_1^2 u^2 + 5 a_1^3 u, \\ f^*(a_6) &= 0, & q^*(a_6) &= -20u^6 + 59 a_1 u^5 - 70 a_1^2 u^4 + 45 a_1^3 u^3 - 15 a_1^4 u^2 + a_1^5 u, \end{aligned}$$

$$\begin{aligned} t^*(a_1) &= \frac{1}{5}(-8\zeta^3 - 6\zeta^2 - 14\zeta - 7)a_1 + \frac{1}{5}(14\zeta^3 - 2\zeta^2 + 12\zeta + 6)u, \\ t^*(u) &= \frac{1}{5}(-\zeta^3 - 7\zeta^2 - 8\zeta - 4)a_1 + \frac{1}{5}(8\zeta^3 + 6\zeta^2 + 14\zeta + 7)u. \end{aligned}$$

In the formulas for t^*, we use ζ to denote a 5th root of unity. This results in the following formula for f^*, q^*, t^* on rings of modular forms:

$$\begin{aligned} f^*(c_4) &= b_2^2 - 12 b_4 + 12\delta, & q^*(c_4) &= b_2^2 + 228 b_4 + 492\delta, \\ f^*(c_6) &= -b_2^3 + 18 b_2 b_4 - 72 b_2 \delta, & q^*(c_6) &= -b_2^3 + 522 b_2 b_4 + 10,008 b_2 \delta, \end{aligned}$$

$$\begin{aligned} t^*(b_2) &= -5 b_2, \\ t^*(b_4) &= \frac{1}{5}(11 b_2^2 - 117 b_4 - 88\delta), \\ t^*(\delta) &= \frac{1}{5}(b_2^2 - 22 b_4 + 117\delta). \end{aligned}$$

4. Detection of the β-family by $Q(3)$ and $Q(5)$

The Miller-Ravenel-Wilson divided β-family [15] is an important algebraic approximation of the $K(2)$-local sphere at the prime 2. It was computed for the prime 2 by Shimomura in [16]. Here we use the standard chain of Bockstein spectral sequences and the formulas of 3.3 and 3.4 to compute algebraic chromatic data in the $Q(3)$ and $Q(5)$ spectra. These are compared to Shimomura's calculations, resulting in Theorems 4.2.2 and 4.2.4. The surprising observation is that $Q(3)$ precisely detects the divided β-family, while the analgous family in $Q(5)$ has extra v_1-divisibility.

4.1. **The chromatic spectral sequence.** Following [15], given a BP_*-module N, we will let

$$M_i^{n-i}N := N/(p, \cdots, v_{i-1}, v_i^\infty, \cdots, v_{n-1}^\infty)[v_n^{-1}].$$

If N is a BP_*BP-comodule, then so is $M_i^{n-i}N$. Letting $\mathrm{Ext}^{*,*}(N)$ denote the groups

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, N),$$

there is a chromatic spectral sequence

$$E_1^{n,s,t} = \mathrm{Ext}^{s,t}(M_0^n N) \Rightarrow \mathrm{Ext}^{s+n,t}(N).$$

The groups $\mathrm{Ext}^{0,*}(M_0^n BP_*)$ detect the nth Greek letter elements in $\mathrm{Ext}^{*,*}(BP_*)$.

The E_1-term of this spectral sequence may be computed by first computing the groups $\mathrm{Ext}^{*,*}(M_n^0)$ and then using the v_i-Bockstein spectral sequences (BSS) of the form

$$\mathrm{Ext}^{*,*}(M_{i+1}^{n-i-1}N) \otimes \mathbb{F}_p[v_i]/(v_i^\infty) \Rightarrow \mathrm{Ext}^{*,*}(M_i^{n-i}N).$$

4.2. **Statement of results.** For the remainder of this section we work exclusively at the prime 2. Shimomura used these spectral sequences to make the following computation.

Theorem 4.2.1 ([16])**.** The groups $\mathrm{Ext}^0(M_0^2 BP_*)$ are spanned by the elements:

$$\frac{1}{2^k v_1^j}, \qquad j \geq 1 \text{ and } k \leq k(j);$$

$$\frac{v_2^{m2^n}}{2^k v_1^j}, \qquad 2 \nmid m,\ k \leq k(j), \text{ and}$$

$$j \leq \begin{cases} a(1), & k = 3, n = 2, \\ a(n-k+1), & \text{otherwise}, \end{cases}$$

where

$$k(j) := \begin{cases} 1, & j \not\equiv 0 \mod 2, \\ \nu_2(j) + 2, & j \equiv 0 \mod 2, \end{cases}$$

and

$$a(i) := \begin{cases} 1, & i = 0, \\ 2, & i = 1, \\ 3 \cdot 2^{i-1}, & i \geq 2. \end{cases}$$

The 'names' $v_2^i/2^k v_1^j$ are not the exact names of BP_*BP-primitives in $M_0^2 BP_*$, but rather the names of the elements detecting them in the sequence of BSS's:

$$\mathrm{Ext}^{*,*}(M_2^0 BP_*) \otimes \frac{\mathbb{F}_2[v_0, v_1]}{(v_0^\infty, v_1^\infty)} \Rightarrow \mathrm{Ext}^{*,*}(M_1^1 BP_*) \otimes \frac{\mathbb{F}_2[v_0]}{(v_0^\infty)} \Rightarrow \mathrm{Ext}^{*,*}(M_0^2 BP_*).$$

Put a linear order on the monomials $v_0^k v_1^j$ in $\mathbb{F}_2[v_0^k, v_1^j]$ by left lexicographical ordering on the sequence of exponents (k, j). With respect to this ordering, the actual primitives correspond to elements

$$\frac{v_2^i}{2^k v_1^j} + \text{terms with smaller denominators.}$$

The main theorem of this section is the following.

Theorem 4.2.2. The map

$$\mathrm{Ext}^0(M_0^2 BP_*) \to H^0(M_0^2 C^*_{tot}(Q(3)))$$

is an isomorphism.

Remark 4.2.3. It was observed by Mahowald and Rezk [14] that the map

$$\mathrm{Ext}^0(M_1^1 BP_*) \to H^0(M_1^1 C^*_{tot}(Q(3)))$$

is an isomorphism.

However, the same cannot hold for $Q(5)$. Indeed, the following theorem implies it does not even hold on the level of M_1^1.

Theorem 4.2.4. The map

$$\mathrm{Ext}^0(M_1^1 BP_*) \to H^0(M_1^1 C^*_{tot}(Q(5)))$$

is *not* an isomorphism.

4.3. **Leibniz and doubling formulas.** The group $H^0(M_0^2 C^*_{tot}(Q(\ell)))$ is the kernel of the map

$$M_0^2 C^0_{tot}(Q(\ell)) \xrightarrow{d_0 - d_1} M_0^2 C^1_{tot}(Q(\ell))$$

where d_0 and d_1 are the cosimplicial coface maps of the total complex. Explicitly, we are applying M_0^2 to the map

$$D_{tot} : A_{(2)} \xrightarrow{(\eta_R - \eta_L) \oplus (q^* - f^*) \oplus (\psi^\ell - 1)} \Gamma_{(2)} \oplus B^1(\ell)_{(2)} \oplus A_{(2)}.$$

The projection of D_{tot} onto the last component is very easy to understand; it is given by

$$\psi^\ell - 1 : A \to A.$$

As long as ℓ generates $\mathbb{Z}_2^\times/\{\pm 1\}$, in degree $2t$ the map $\psi^\ell - 1$, up to a unit in $\mathbb{Z}_{(2)}^\times$, corresponds to multiplication by a factor of $2^{k(t)}$. It therefore suffices to understand the composite D of D_{tot} with the projection onto the first two components:

$$D : A_{(2)} \xrightarrow{(\eta_R - \eta_L) \oplus (q^* - f^*)} \Gamma_{(2)} \oplus B^1(\ell)_{(2)}.$$

We shall make repeated use of the following lemma about this map D.

Lemma 4.3.1. The map D satisfies the following two identities.

$$D(xy) = D(x)\eta_R(y) + xD(y), \tag{4.3.2}$$

$$D(x^2) = 2xD(x) + D(x)^2. \tag{4.3.3}$$

Here, Γ is given the A-module structure induced by the map η_L, and $B^1(3)$ is given the A-module structure induced from the map f^*. Consequently, we have

$$D(xy) \equiv xD(y) \mod (D(x)). \tag{4.3.4}$$

Proof. These identities hold for any map $D = d_0 - d_1 : R^0 \to R^1$, the difference of two ring maps:

$$\begin{aligned} D(xy) &= d_0(x)d_0(y) - d_1(x)d_1(y) \\ &= d_1(x)(d_0(y) - d_1(y)) + (d_0(x) - d_1(x))d_0(y) \\ &= d_1(x)D(y) + D(x)d_0(y). \\ D(x^2) &= d_0(x)^2 - d_1(x)^2 \\ &= (d_0(x) - d_1(x))^2 + 2d_0(x)d_1(x) - 2d_1(x)^2 \\ &= D(x)^2 + 2d_1(x)D(x). \end{aligned}$$

□

Observe that using the fact that $a_1 = v_1$, there are isomorphisms

$$\begin{aligned} \Gamma_{(2)} &\cong \mathbb{Z}_{(2)}[v_1][a_2, a_3, a_4, a_6, r, s, t][\Delta^{-1}], \\ B^1(3)_{(2)} &\cong \mathbb{Z}_{(2)}[v_1][a_3][\Delta^{-1}], \\ B^1(5)_{(2)} &\cong \mathbb{Z}_{(2)}[v_1][u][\Delta^{-1}]. \end{aligned}$$

Express elements of $\Gamma_{(2)}$ (respectively, $B^1(3)_{(2)}$, $B^1(5)_2$) "$(2, v_1)$-adically" so that every element is expressed as a power of the discriminant times a sum of terms

$$\Delta^\ell \sum_{k \geq 0} \sum_{j \geq 0} 2^k v_1^j c_{k,j}$$

for $\ell \in \mathbb{Z}$ and $c_{k,j} \in \mathbb{F}_2[a_2, a_3, a_4, a_6, r, s, t]$ (respectively $\mathbb{F}_2[a_3]$), $\mathbb{F}_2[u]$). We shall compare terms by saying that

$$2^k v_1^j c_{j,k} \quad \textit{is larger than} \quad 2^{k'} v_1^{j'} c_{j',k'}$$

if (k, j) is larger than (k', j') with respect to left lexicographical ordering. We shall be concerned with ordered sums of monomials of the form:

$$\begin{aligned} & v_1^{i_0} c_{0,i_0} + \text{terms of the form } v_1^j c_{0,j} \text{ with } j > i_0 \\ & +2v_1^{i_1} c_{1,i_1} + \text{terms of the form } 2v_1^j c_{1,j} \text{ with } j > i_1 \\ & +4v_1^{i_2} c_{2,i_2} + \text{terms of the form } 4v_1^j c_{2,j} \text{ with } j > i_2 \\ & + \cdots \\ & +2^n v_1^{i_n} c_{n,i_n} + \text{larger terms} \end{aligned}$$

for $(i_0 > i_1 > \cdots > i_n)$ and $n \geq 1$. Note that we permit the coefficients c_{k,i_k} to be zero. We shall abbreviate such expressions as

$$\begin{aligned} & v_1^{i_0} c_{0,i_0} + \cdots \\ & \quad + 2v_1^{i_1} c_{1,i_1} + \cdots \\ & \quad\quad + 4v_1^{i_2} c_{2,i_2} + \cdots \\ & \quad\quad\quad + \cdots \\ & \quad\quad\quad\quad + 2^n v_1^{i_n} c_{n,i_n} + \cdots . \end{aligned}$$

The following observation justifies considering such representations.

Lemma 4.3.5. Suppose that $x \in A_{(2)}$ satisfies

$$\begin{aligned} D(x) = & v_1^{i_0} c_{0,i_0} + \cdots \\ & + 2v_1^{i_1} c_{1,i_1} + \cdots \\ & + 4v_1^{i_2} c_{2,i_2} + \cdots \\ & + \cdots \\ & + 2^n v_1^{i_n} c_{n,i_n} + \cdots \end{aligned}$$

Then we have

$$\begin{aligned} D(x^2) = & v_1^{2i_0} c_{0,i_0}^2 + \cdots \\ & + 2v_1^{i_0} c_{0,i_0} x + \cdots \\ & + 4v_1^{i_1} c_{1,i_1} x + \cdots \\ & + 8v_1^{i_2} c_{2,i_2} x + \cdots \\ & + \cdots \\ & + 2^{n+1} v_1^{i_n} c_{n,i_n} x + \cdots \end{aligned} \tag{4.3.6}$$

and for m odd we have

$$\begin{aligned} D(x^m) = & v_1^{i_0} c_{0,i_0} x^{m-1} + \cdots \\ & + 2v_1^{i_1} c_{1,i_1} x^{m-1} + \cdots \\ & + 4v_1^{i_2} c_{2,i_2} x^{m-1} + \cdots \\ & + \cdots \\ & + 2^n v_1^{i_n} c_{n,i_n} x^{m-1} + \cdots . \end{aligned} \tag{4.3.7}$$

Proof. The identity (4.3.6) follows immediately from (4.3.3). We prove (4.3.7) by induction on $m = 2j + 1$. Suppose that we know (4.3.7) for all odd $m' < m$. Write $j = 2^t s$ for s odd. Then by the inductive hypothesis, and repeated applications of (4.3.6), we deduce that

$$\begin{aligned} D(x^j) = & v_1^{i_0} c'_{0,i_0} + \cdots \\ & + 2v_1^{i_1} c'_{1,i_1} + \cdots \\ & + 4v_1^{i_2} c'_{2,i_2} + \cdots \\ & + \cdots \\ & + 2^n v_1^{i_n} c'_{n,i_n} + \cdots . \end{aligned}$$

Applying (4.3.6), we have

$$\begin{aligned} D(x^{2j}) = & v_1^{2i_0} (c'_{0,i_0})^2 + \cdots \\ & + 2v_1^{i_0} c'_{0,i_0} x^j + \cdots \\ & + 4v_1^{i_1} c'_{1,i_1} x^j + \cdots \\ & + 8v_1^{i_2} c'_{2,i_2} x^j + \cdots \\ & + \cdots \\ & + 2^{n+1} v_1^{i_n} c'_{n,i_n} x^j + \cdots . \end{aligned}$$

It follows from (4.3.4) that we have

$$\begin{aligned} D(x^{2j+1}) = D(x^{2j}x) = & v_1^{i_0} c_{0,i_0} x^{2j} + \cdots \\ & + 2v_1^{i_1} c_{1,i_1} x^{2j} + \cdots \\ & + 4v_1^{i_2} c_{2,i_2} x^{2j} + \cdots \\ & + \cdots \\ & + 2^n v_1^{i_n} c_{n,i_n} x^{2j} + \cdots . \end{aligned}$$

□

4.4. **Overview of the technique.** The technique for the proof of Theorem 4.2.2 is as follows (following [15] and [16]):

Step 1: Compute the differentials from the $s = 0$ to the $s = 1$-lines in the v_1-BSS

$$H^{s,*}(M_2^0 C^*_{tot}(Q(3))) \otimes \mathbb{F}_2[v_1]/(v_1^\infty) \Rightarrow H^{s,*}(M_1^1 C^*_{tot}(Q(3))). \tag{4.4.1}$$

This establishes the existence and v_1-divisibility of v_2^i/v_1^j in $H^{0,*}(C^*_{tot}(Q(3)))$.

Step 2: For i, j as above, demonstrate that $v_2^i/2^k v_1^j$ exists in $H^{0,*}(M_0^2 C^*_{tot}(Q(3)))$ by writing down an element

$$x_{i/j,k} = \frac{a_3^i}{2^k v_1^j} + \text{terms with smaller denominators} \in M_0^2 A$$

with $D_{tot}(x) = 0$.

Step 3: Given j, find the maximal k such that $x_{i/j,k}$ exists by using the exact sequence

$$H^{0,*}(M_0^2 C^*_{tot}(Q(3))) \xrightarrow{\cdot 2} H^{0,*}(M_0^2 C^*_{tot}(Q(3))) \xrightarrow{\partial} H^{1,*}(M_1^1 C^*_{tot}(Q(3))).$$

Specifically, the maximality of k is established by showing that $\partial(x_{i/j,k}) \neq 0$. The non-triviality of $\partial(x_{i/j,k})$ can be demonstrated by considering its image under the inclusion:

$$H^{1,*}(M_1^1 C^*_{tot}(Q(3))) \hookrightarrow \operatorname{Coker} M_1^1(D_{tot})$$

where $M_1^1(D_{tot})$ is the map

$$M_1^1(D_{tot}) : M_1^1 A \to M_1^1 \Gamma \oplus M_1^1 B^1(3) \oplus M_1^1 A$$

essentially computed in Step 1 by the computation of the differentials from $s = 0$ to $s = 1$ in the spectral sequence (4.6.1).

4.5. **Computation of $H^{*,*}(M_2^0 C^*_{tot}(Q(3)))$.** We have [2, Sec. 7]

$$H^{*,*}(M_2^0 C^*_\Gamma(A)) = \mathbb{F}_2[a_3^{\pm 1}, h_1, h_2, g]/(h_2^3 = a_3 h_1^3),$$
$$H^{*,*}(M_2^0 C^*_{\Lambda^1(3)}(B^1)) = \mathbb{F}_2[a_3^{\pm}, h_{2,1}]$$

with (s,t)-bidegrees

$$\begin{aligned} |a_3| &= (0,6), \\ |h_1| &= (1,2), \\ |h_2| &= (1,4), \\ |g| &= (4,24), \\ |h_{2,1}| &= (1,6), \end{aligned}$$

and $h_{2,1}^4 = g$. Moreover, the spectral sequence of the double complex gives

(4.5.1)
$$\left.\begin{array}{c} H^{s,t}(M_2^0C_\Gamma^*(A))\oplus \\ H^{s-1,t}(M_2^0C_\Gamma^*(A)) \oplus H^{s-1,t}(M_2^0C_{\Lambda^1(3)}^*(B^1))\oplus \\ H^{s-2,t}(M_2^0C_{\Lambda^1(3)}^*(B^1)) \end{array}\right\} \Rightarrow H^{s,t}(M_2^0C_{tot}^*(Q(3))).$$

In order to differentiate the terms x with the same name (such as a_3) occurring in the different groups in the E_1-term of spectral sequence (4.5.1), we shall employ the following notational convention:

(4.5.2)
$$\begin{aligned} x &\in C_\Gamma^*(A) \text{ on the 0-line,} \\ \bar{x} &\in C_\Gamma^*(A) \text{ on the 1-line,} \\ x' &\in C_{\Lambda^1(\ell)}^*(B^1) \text{ on the 1-line,} \\ \bar{x}' &\in C_{\Lambda^1(\ell)}^*(B^1) \text{ on the 2-line.} \end{aligned}$$

The formulas of Section 3.3 show that the only non-trivial d_1 differentials in spectral sequence (4.5.1) are

$$d_1(g^i(\bar{a}_3)^j) = h_{2,1}^{4i}(\bar{a}_3')^j.$$

Since g is the image of the element $g \in \mathrm{Ext}^{4,24}(BP_*)$ (the element that detects $\bar{\kappa}$ in the ANSS for the sphere), and the spectral sequence (4.5.1) is a spectral sequence of modules over $\mathrm{Ext}^{*,*}(BP_*)$, we deduce that there are no possible d_r-differentials for $r > 1$. We deduce that we have

$$\begin{aligned} H^{*,*}(M_2^0C_{tot}^*(Q(3))) = {} & \mathbb{F}_2[a_3^{\pm 1}, h_1, h_2, g]/(h_2^3 = a_3h_1^3) \\ & \oplus \mathbb{F}_2[\bar{a}_3^{\pm 1}, \bar{g}]\{\bar{h}_1, \bar{h}_2, \bar{h}_1^2, \bar{h}_2^2, \bar{h}_2^3 = \bar{a}_3\bar{h}_1^3\} \\ & \oplus \mathbb{F}_2[(a_3')^{\pm 1}, h_{2,1}'] \\ & \oplus \mathbb{F}_2[(\bar{a}_3')^{\pm 1}, \bar{g}']\{\bar{h}_{2,1}', (\bar{h}_{2,1}')^2, (\bar{h}_{2,1}')^3\}. \end{aligned}$$

Remark 4.5.3. Note that $H^{*,*}(M_2^0C_{tot}^*(Q(3)))$ is less than half of $\mathrm{Ext}^{*,*}(M_2^0BP_*)$. This indicates that $Q(3)$ cannot agree with 'half' of the proposed duality resolution of Goerss-Henn-Mahowald-Rezk at $p = 2$ [8], despite the fact that it is built from the same spectra. In particular, the fiber of the map

$$S_{K(2)} \to Q(3)_{K(2)}$$

cannot be the dual of $Q(3)_{K(2)}$.

4.6. **Computation of $H^{0,*}(M_1^1C_{tot}^*(Q(3)))$.** We now compute the differentials in the v_1-BSS

(4.6.1)
$$H^{s,*}(M_2^0C_{tot}^*(Q(3))) \otimes \mathbb{F}_2[v_1]/(v_1^\infty) \Rightarrow H^{s,*}(M_1^1C_{tot}^*(Q(3)))$$

from the $s = 0$-line to the $s = 1$-line. This computation was originally done by Mahowald and Rezk [14], but we redo it here to establish notation, and to motivate the rationale behind some of the computations to follow.

One computes using the formulas of Section 3.3:

$$
\begin{aligned}
D(x_0) &\equiv a_1 s^2 \mod (2, v_1^2), \\
D(x_1) &\equiv a_1^2 a_3 s \mod (2, v_1^3), \\
D(x_2) &\equiv (a_1')^6 (a_3')^2 \mod (2, v_1^7)
\end{aligned}
\tag{4.6.2}
$$

for

$$
\begin{aligned}
x_0 &:= a_3 + a_1 a_2 \equiv a_3 \mod (2, v_1) \\
x_1 &:= x_0^2 + a_1^2 a_4 + a_1^2 a_2^2 \equiv a_3^2 \mod (2, v_1), \\
x_2 &:= \Delta \equiv a_3^4 \mod (2, v_1).
\end{aligned}
$$

Remark 4.6.3. The above formulas for x_i were obtained by the following method. In the complex $M_2^0 C_\Gamma^*(A)$, we have

$$
\begin{aligned}
d(a_2) &= r + \cdots \\
d(a_4 + a_2^2) &= s^4 + \cdots \\
d(a_6) &= t^2 + \cdots .
\end{aligned}
$$

These are used in [2, Sec. 6] to produce a complex which is closely related to the cobar complex on the double of $A(1)_*$. To arrive at x_0 we calculate

$$D(a_3) = a_1 r + \cdots$$

which means that we need to add the correction term $a_1 a_2$ to arrive at x_0. The expression for x_1 was similarly produced. The definition Δ is a natural candidate for x_2, as it is an element of the form $a_3^4 + \cdots$ which is already known to be a cocycle in $C_\Gamma^0(A)$.

It follows from inductively applying (4.3.6) that we have

$$D(x_2^{2^{n-2}}) \equiv (a_1')^{3\cdot 2^{n-1}} (a_3')^{2^{n-1}} \mod (2, v_1^{3\cdot 2^{n-1}+1}).$$

It follows from (4.3.7) that for m odd, we have

$$
\begin{aligned}
D(x_0^m) &\equiv a_1 s^2 a_3^{m-1} \mod (2, v_1^2), \\
D(x_1^m) &\equiv a_1^2 a_3^{2m-1} s \mod (2, v_1^3), \\
D(x_2^{m 2^{n-2}}) &\equiv (a_1')^{3\cdot 2^{n-1}} (a_3')^{m2^n - 2^{n-1}} \mod (2, v_1^{3\cdot 2^{n-1}+1}).
\end{aligned}
$$

We deduce the following.

Lemma 4.6.4. The v_1-BSS differentials in (4.6.1) from the $(s = 0)$-line to the $(s = 1)$-line are given by

$$d_1\left(\frac{a_3^m}{v_1^j}\right) = \frac{a_3^{m-1}h_2}{v_1^{j-1}},$$
$$d_2\left(\frac{a_3^{2m}}{v_1^j}\right) = \frac{a_3^{2m-1}h_1}{v_1^{j-2}},$$
$$d_{3\cdot 2^{n-1}}\left(\frac{a_3^{m2^n}}{v_1^j}\right) = \frac{(a_3')^{m2^n-2^{n-1}}}{v_1^{j-3\cdot 2^{n-1}}}$$

where m is odd.

Corollary 4.6.5. The groups $H^{0,*}(M_1^1C_{tot}^*(Q(3)))$ are generated by the elements

$$\frac{a_3^{m2^n}}{v_1^j}$$

for m odd and $j \leq a(n)$.

4.7. **Computation of $H^{0,*}(M_0^2C_{tot}^*(Q(3)))$.** We now prove Theorem 4.2.2, which is more specifically stated below.

Theorem 4.7.1. The groups $H^{0,*}(M_0^2C_{tot}^*(Q(3)))$ are spanned by elements:

$$\frac{1}{2^kv_1^j}, \qquad j \geq 1 \text{ and } k \leq k(j);$$

$$\frac{a_3^{mp^n}}{2^kv_1^j}, \qquad 2 \nmid m,\ k \leq k(j), \text{ and}$$

$$j \leq \begin{cases} a(1), & k = 3, n = 2, \\ a(n-k+1), & \text{otherwise.} \end{cases}$$

In many cases, the bounds on 2-divisibility will follow from the following simple observation.

Lemma 4.7.2. Suppose the element

$$\frac{a_3^i}{2^kv_1^j} \in H^{0,2t}(M_0^2C_{tot}^*(Q(3)))$$

exists. Then $k \leq k(t)$.

Proof. The formula

$$(\psi^3 - 1)\frac{a_3^i}{2^kv_1^j} = (3^t - 1)\frac{\bar{a}_3^i}{2^kv_1^j}$$

implies that in order for

$$0 \neq D_{tot}\left(\frac{a_3^i}{2^kv_1^j}\right) \in M_0^2C_{tot}^1(Q(3))$$

we must have $k \leq \nu_2(3^t - 1)$. □

Proof of Theorem 4.7.1. Lemma 4.6.4 established that for m odd, $\frac{a_3^{m2^n}}{2v_1^j}$ exists for $1 \le j \le a(n)$. In order to prove the required 2-divisibility of these elements, we need to prove

$$D\left(\frac{a_3^{4m}}{8v_1^2} + \cdots\right) = 0, \tag{4.7.3}$$

$$D\left(\frac{a_3^{m2^n}}{4v_1^{2j}} + \cdots\right) = 0, \qquad 2j \le a(n-1), \tag{4.7.4}$$

$$D\left(\frac{a_3^{m2^n}}{2^k v_1^{j2^{k-2}}} + \cdots\right) = 0, \qquad k \ge 3,\ j2^{k-2} \le a(n-k+1). \tag{4.7.5}$$

In light of Lemma 4.7.2, to establish that these are the maximal 2-divisibilities of these elements, we need only check that

(4.7.6)
$$\partial\left(\frac{a_3^m}{2v_1} + \cdots\right) \not\equiv 0 \mod D(M_1^1 A),$$

(4.7.7)
$$\partial\left(\frac{a_3^{m2^n}}{2v_1^{2j}} + \cdots\right) \not\equiv 0 \mod D(M_1^1 A), \quad a(n-1) < 2j \le a(n),$$

(4.7.8)
$$\partial\left(\frac{a_3^{m2^n}}{2^{k-1} v_1^{j2^{k-2}}} + \cdots\right) \not\equiv 0 \mod D(M_1^1 A), \quad k \ge 2,\ a(n-k+1) < j2^{k-1} \le a(n-k+2).$$

Proof of (4.7.6). Using the formulas of Section 3.3, we have

$$\begin{aligned} D(x_0) = a_1 s^2 &+ \cdots \\ &+ 2(t + rs + s^3 + a_2 s) + \cdots \\ &+ 2a_3' + \cdots . \end{aligned} \tag{4.7.9}$$

It follows from (4.3.7) that we have for m odd

$$\begin{aligned} D(x_0^m) = a_1 a_3^{m-1} s^2 &+ \cdots \\ &+ 2a_3^{m-1}(t + rs + s^3 + a_2 s) + \cdots \\ &+ 2(a_3')^m + \cdots . \end{aligned} \tag{4.7.10}$$

Since we have

$$\eta_R(a_1) = a_1 + 2s \tag{4.7.11}$$

we deduce from (4.7.10) using (4.3.2):

$$\begin{aligned} D(x_0^m a_1) = a_1^2 a_3^{m-1} s^2 &+ \cdots \\ &+ 2a_3^m s + 2a_1 a_3^{m-1}(t + rs) + \cdots \\ &+ 2a_1'(a_3')^m + \cdots . \end{aligned} \tag{4.7.12}$$

Reducing modulo the invariant ideal $(4, v_1^2)$ we deduce

$$\partial\left(\frac{a_3^m}{2v_1} + \cdots\right) = \frac{a_3^m h_1}{v_1^2} + \cdots .$$

Lemma 4.6.4 implies that this expression is not in $D(M_1^1 A)$ if $m \equiv 3 \mod 4$. However, if $m \equiv 1 \mod 4$, then Lemma 4.6.4 implies that $\frac{a_3^m h_1}{v_1^2}$ is killed in the v_1-BSS (4.6.1) by $d_2(\frac{a_3^{m+1}}{v_1^4})$. We compute using the formulas of Section 3.3:

$$\begin{aligned} D(x_1) &= a_1^2 a_3 s + a_1^3(t+rs) + \cdots \\ &\quad + 2a_1 a_3 s^2 + \cdots \\ &\quad + 2(a_1')^3 a_3' + \cdots . \end{aligned} \tag{4.7.13}$$

We deduce using (4.3.7) that for m odd we have:

$$\begin{aligned} D(x_1^m) &= a_1^2 a_3^{2m-1} s + a_1^3 a_3^{2m-2}(t+rs) + \cdots \\ &\quad + 2a_1 a_3^{2m-1} s^2 + \cdots \\ &\quad + 2(a_1')^3 (a_3')^{2m-1} + \cdots . \end{aligned} \tag{4.7.14}$$

We deduce that for $m \equiv 1 \mod 4$ we have

$$\begin{aligned} D(a_1^3 x_0^m + 2x_1^{\frac{m+1}{2}}) &= a_1^4 a_3^{m-1} s^2 + \cdots \\ &\quad + 2(a_1')^3 (a_3')^m + \cdots . \end{aligned}$$

Thus we have for $m \equiv 1 \mod 4$:

$$\partial\left(\frac{x_0^m}{2v_1} + \cdots\right) = \frac{(a_3')^m}{v_1} + \cdots$$

and Lemma 4.6.4 implies that this expression is not in $D(M_1^1 A)$. This establishes (4.7.6).

Proof of (4.7.7) for $n = 1$. Equation (4.7.14) implies that

$$\partial\left(\frac{a_3^{2m}}{v_1^2} + \cdots\right) = \frac{a_3^{2m-1} h_2}{v_1} + \cdots$$

which, by Lemma 4.6.4, is not in $D(M_1^1 A)$. This establishes (4.7.7) for $n = 1$.

Proof of (4.7.7) for $n = 2$. We compute using the formulas of Section 3.3

$$\begin{aligned} D(x_2) &= (a_1')^6 (a_3')^2 + \cdots \\ &\quad + 2(a_1')^3 (a_3')^3 + \cdots . \end{aligned} \tag{4.7.15}$$

Applying (4.3.7), we get for m odd:

$$\begin{aligned} D(x_2^m) &= (a_1')^6 (a_3')^{4m-2} + \cdots \\ &\quad + 2(a_1')^3 (a_3')^{4m-1} + \cdots . \end{aligned} \tag{4.7.16}$$

It follows that

$$\partial\left(\frac{x_2^m}{2v_1^{2j}}\right) = \frac{(a_3')^{4m-1}}{v_1^{2j-3}} + \cdots$$

for $a(1) < 2j \leq a(2)$, which is not in $D(M_1^1 A)$ by Lemma 4.6.4. This establishes (4.7.7) for $n = 2$.

Proof of (4.7.3). We deduce from (4.7.16) that $\frac{a_3^{4m}}{4v_1^2}$ exists. In order to understand its 2-divisibility, we compute $\partial(\frac{a_3^{4m}}{4v_1^2})$, which is the obstruction to divisibility. To do this we need to compute $D(\frac{x_2^m}{8v_1^2})$. Since $(8, v_1^4)$ is an invariant ideal, we compute this from $D(a_1^2 x_2^m)$. Since

$$D(a_1^2) = 4s^2 + 4sa_1 \tag{4.7.17}$$

and

$$x_2 \equiv a_3^4 + 2a_1^2 a_3^2 a_4 + a_3^3 a_1^3 \mod (4, v_1^4) \tag{4.7.18}$$

we deduce from (4.3.2) that

$$\begin{aligned} D(a_1^2 x_2^m) = (a_1')^8 (a_3')^{4m-2} + \cdots \\ + 2(a_1')^5 (a_3')^{4m-1} + \cdots \\ + 4a_3^{4m} s^2 + 4a_1 a_3^{4m} s + 4a_1^3 a_3^{4m-1} s^2 + \cdots \end{aligned} \tag{4.7.19}$$

which gives

$$D\left(\frac{x_2^m}{8v_1^2}\right) = \frac{a_3^{4m} s^2}{2v_1^4} + \frac{a_3^{4m} s}{2v_1^3} + \frac{a_3^{4m-1} s^2}{2v_1}.$$

Lemma 4.6.4 tells us that $\frac{a_3^{4m} h_2}{v_1^4}$ is killed by $\frac{x_0^{4m+1}}{v_1^5}$. We compute

$$D(x_0) \equiv a_1 s^2 + s a_1^2 \mod 2$$

and thus

$$D(x_0^4) \equiv a_1^4 s^8 \mod (2, v_1^5).$$

Using the fact that

$$x_0^{4m} \equiv a_3^{4m} \mod (2, v_1^4)$$

we have

$$D(x_0^{4m+1}) \equiv a_1 a_3^{4m} s^2 + a_1^2 a_3^{4m} s + a_1^4 a_3^{4m-3} s^8 \mod (2, v_1^5).$$

and thus

$$D\left(\frac{x_2^m}{8v_1^2} + \frac{x_0^{4m+1}}{2v_1^5}\right) = \frac{a_3^{4m-3} s^8}{2v_1} + \frac{a_3^{4m-1} s^2}{2v_1}.$$

Since $a_4 + a_2^2$ kills s^4 (see Remark 4.6.3), $(a_4 + a_2^2)^2$ kills s^8, and we compute

$$D((a_4 + a_2^2)^2) \equiv s^8 + a_3^2 s^2 \mod (2, v_1).$$

Therefore we have

$$D\left(\frac{x_2^m}{8v_1^2} + \frac{x_0^{4m+1}}{2v_1^5} + \frac{a_3^{4m-3}(a_4 + a_2^2)^2}{2v_1}\right) = 0. \tag{4.7.20}$$

This establishes (4.7.3).

Proof of (4.7.4). Iterated application of (4.3.3) to (4.7.16) yields

$$\begin{aligned}
D(x_2^{m2^{n-2}}) &= (a_1')^{3\cdot 2^{n-1}}(a_3')^{m2^n-2^{n-1}} + \cdots \\
&\quad + 2(a_1')^{3\cdot 2^{n-2}}(a_3')^{m2^n-2^{n-2}} + \cdots \\
&\quad + 4(a_1')^{3\cdot 2^{n-3}}(a_3')^{m2^n-2^{n-3}} + \cdots \\
&\quad + \cdots \\
&\quad + 2^{n-1}(a_1')^{3}(a_3')^{m2^n-1} + \cdots .
\end{aligned} \tag{4.7.21}$$

It follows that

$$D\left(\frac{x_2^{m2^{n-2}}}{4v_1^{2j}}\right) = 0, \quad 2j \le a(n-1).$$

This establishes (4.7.4).

Proof of (4.7.5). Suppose that j is even. Then the ideal $(2^k, v_1^{j2^{k-2}})$ is invariant, and reducing (4.7.21) modulo this invariant ideal gives

$$D\left(\frac{x_2^{m2^{n-2}}}{2^k v_1^{j2^{k-2}}}\right) = 0, \quad j2^{k-2} \le a(n-k+1).$$

This establishes (4.7.5) for j even.

Suppose now that j is odd. Then the ideal $(2^k, v_1^{j2^{k-2}+2^{k-2}})$ is invariant, and in order to compute $D(\frac{x_2^{m2^{n-2}}}{2^k v_1^{j2^{k-2}}})$ we must compute $D(a_1^{2^{k-2}} x_2^{m2^{n-2}})$ modulo $(2^k, v_1^{j2^{k-2}+2^{k-2}})$. Repeated application of (4.3.3) to (4.7.17) yields

$$D(a_1^{2^{k-2}}) \equiv 2^{k-1}a_1^{2^{k-2}-2}s^2 + 2^{k-1}a_1^{2^{k-2}-1}s \mod 2^k. \tag{4.7.22}$$

We also note that since

$$x_2 \equiv a_3^4 + a_1^3 a_3^3 + \cdots \mod 2$$

we have

$$\begin{aligned}
x_2^{m2^{n-2}} &\equiv a_3^{m2^n} + a_1^{3\cdot 2^{n-2}} a_3^{3\cdot 2^{n-2}+(m-1)2^{n-2}} + \cdots \mod 2 \\
&\equiv a_3^{m2^n} + a_1^{3\cdot 2^{n-2}} a_3^{2^{n-1}+m2^{n-2}} + \cdots \mod 2.
\end{aligned} \tag{4.7.23}$$

Applying (4.3.2) to (4.7.21), (4.7.22), and (4.7.23), we get

(4.7.24)

$$\begin{aligned}
D(a_1^{2^{k-2}} x_2^{m2^{n-2}}) &= (a_1')^{3\cdot 2^{n-1}+2^{k-2}}(a_3')^{m2^n-2^{n-1}} + \cdots \\
&\quad + 2(a_1')^{3\cdot 2^{n-2}+2^{k-2}}(a_3')^{m2^n-2^{n-2}} + \cdots \\
&\quad + 4(a_1')^{3\cdot 2^{n-3}+2^{k-2}}(a_3')^{m2^n-2^{n-3}} + \cdots \\
&\quad + \cdots \\
&\quad + 2^{k-1}(a_1')^{3\cdot 2^{n-k}+2^{k-2}}(a_3')^{m2^n-2^{n-k}} + \cdots \\
&\quad + 2^{k-1}a_1^{2^{k-2}-2}a_3^{m2^n}s^2 + 2^{k-1}a_1^{2^{k-2}-1}a_3^{m2^n}s + 2^{k-1}a_1^{3\cdot 2^{n-2}}a_3^{2^{n-1}+m2^{n-2}}s^2 + \cdots .
\end{aligned}$$

We deduce that for j odd and $j2^{k-2} \le a(n-k+1)$ we have

$$D\left(\frac{x_2^{m2^{n-2}}}{2^k v_1^{j2^{k-2}}}\right) = \frac{a_3^{m2^n} s^2}{2v_1^{j2^{k-2}+2}} + \frac{a_3^{m2^n} s}{2v_1^{j2^{k-2}+1}}.$$

However, Lemma 4.6.4 implies that $\frac{a_3^{m2^n} h_2}{v_1^{j2^{k-2}+2}}$ is killed by $\frac{a_3^{m2^n+1}}{v_1^{j2^{k-2}+3}}$. It follows from (4.7.9) that we have

$$D(x_0^{m2^n}) \equiv a_1^{m2^n} s^{m2^{n+1}} + \cdots \mod 2$$

and hence

$$D(x_0^{m2^n+1}) \equiv a_1 a_3^{m2^n} s^2 + a_1^2 a_3^{m2^n} s + a_1^{m2^n} a_3 s^{m2^{n+1}} + \cdots \mod 2.$$

This implies that we have

$$D\left(\frac{x_0^{m2^n+1}}{2v_1^{j2^{k-2}+3}}\right) = \frac{a_3^{m2^n} s^2}{2v_1^{j2^{k-2}+2}} + \frac{a_3^{m2^n} s}{2v_1^{j2^{k-2}+1}} \tag{4.7.25}$$

and therefore

$$D\left(\frac{x_2^{m2^{n-2}}}{2^k v_1^{j2^{k-2}}} + \frac{x_0^{m2^n+1}}{2v_1^{j2^{k-2}+3}}\right) = 0.$$

This establishes (4.7.5).

Proof of (4.7.7) for $n \ge 3$. It follows from (4.7.21) that we have for $a(n-1) < 2j \le a(n)$

$$D\left(\frac{x_2^{m2^{n-2}}}{4v_1^{2j}}\right) = \frac{(a_3')^{m2^n-2^{n-2}}}{2v_1^{2j-a(n-1)}} + \cdots$$

and hence

$$\partial\left(\frac{x_2^{m2^{n-2}}}{2v_1^{2j}}\right) = \frac{(a_3')^{m2^n-2^{n-2}}}{v_1^{2j-a(n-1)}} + \cdots.$$

This element is not in $D(M_1^1 A)$ by Lemma 4.6.4. This establishes (4.7.7).

Proof of (4.7.8). Suppose that j is even. Then the ideal $(2^k, v_1^{j2^{k-2}})$ is invariant, and reducing (4.7.21) modulo this invariant ideal gives

$$D\left(\frac{x_2^{m2^{n-2}}}{2^k v_1^{j2^{k-2}}}\right) = \frac{(a_3')^{m2^n-2^{n-k}}}{2v_1^{j2^{k-2}-a(n-k+1)}} + \cdots, \quad a(n-k+1) < j2^{k-2} \le a(n-k+2)$$

and therefore

$$\partial\left(\frac{x_2^{m2^{n-2}}}{2^{k-1} v_1^{j2^{k-2}}}\right) = \frac{(a_3')^{m2^n-2^{n-k}}}{v_1^{j2^{k-2}-a(n-k+1)}} + \cdots, \quad a(n-k+1) < j2^{k-2} \le a(n-k+2).$$

Since $k \ge 3$, this is not in $D(M_1^1 A)$ by Lemma 4.6.4. This establishes (4.7.5) for j even.

Suppose now that j is odd. Then the ideal $(2^k, v_1^{j2^{k-2}+2^{k-2}})$ is invariant, and in order to compute $D(\frac{x_2^{m2^{n-2}}}{2^k v_1^{j2^{k-2}}})$ we must compute $D(a_1^{2^{k-2}} x_2^{m2^{n-2}})$ modulo $(2^k, v_1^{j2^{k-2}+2^{k-2}})$.

It follows from (4.7.24) that for j odd and $a(n-k+1) < j2^{k-2} \le a(n-k+2)$ we have

$$D\left(\frac{x_2^{m2^{n-2}}}{2^k v_1^{j2^{k-2}}}\right) = \frac{a_3^{m2^n} s^2}{2v_1^{j2^{k-2}+2}} + \frac{a_3^{m2^n} s}{2v_1^{j2^{k-2}+1}} + \frac{(a_3')^{m2^n-2^{n-k}}}{2v_1^{j2^{k-2}-a(n-k+1)}} + \cdots .$$

Using (4.7.25), we have

$$D\left(\frac{x_2^{m2^{n-2}}}{2^k v_1^{j2^{k-2}}} + \frac{x_0^{m2^n+1}}{2v_1^{j2^{k-2}+3}}\right) = \frac{(a_3')^{m2^n-2^{n-k}}}{2v_1^{j2^{k-2}-a(n-k+1)}} + \cdots$$

and therefore

$$\partial\left(\frac{x_2^{m2^{n-2}}}{2^{k-1} v_1^{j2^{k-2}}}\right) = \frac{(a_3')^{m2^n-2^{n-k}}}{v_1^{j2^{k-2}-a(n-k+1)}} + \cdots$$

Since $k \ge 3$, this is not in $D(M_1^1 A)$ by Lemma 4.6.4. This establishes (4.7.5) for j odd. $\square$

4.8. **Computation of $H^{*,*}(M_2^0 C_{tot}^*(Q(5)))$.** We have (as before)

$$H^{*,*}(M_2^0 C_\Gamma^*(A)) = \mathbb{F}_2[a_3^{\pm 1}, h_1, h_2, g]/(h_2^3 = a_3 h_1^3),$$
$$H^{*,*}(M_2^0 C_{\Lambda^1(5)}^*(B^1)) = \mathbb{F}_2[u^{\pm}, h_{2,1}]$$

with (s,t)-bidegrees

$$\begin{aligned} |a_3| &= (0,6), \\ |h_1| &= (1,2), \\ |h_2| &= (1,4), \\ |g| &= (4,24), \\ |u| &= (0,2), \\ |h_{2,1}| &= (1,6), \end{aligned}$$

and $h_{2,1}^4 = g$. Moreover, the spectral sequence of the double complex gives
(4.8.1)

$$\left.\begin{array}{c} H^{s,t}(M_2^0 C_\Gamma^*(A)) \oplus \\ H^{s-1,t}(M_2^0 C_\Gamma^*(A)) \oplus H^{s-1,t}(M_2^0 C_{\Lambda^1(5)}^*(B^1)) \oplus \\ H^{s-2,t}(M_2^0 C_{\Lambda^1(5)}^*(B^1)) \end{array}\right\} \Rightarrow H^{s,t}(M_2^0 C_{tot}^*(Q(5))).$$

As before, we will differentiate the terms x with the same name occurring in the different groups in the E_1-term of spectral sequence (4.8.1), we shall employ the following notational convention:

(4.8.2)
$$\begin{aligned} &x \in C_\Gamma^*(A) \text{ on the 0-line}, \\ &\bar{x} \in C_\Gamma^*(A) \text{ on the 1-line}, \\ &y \in C_{\Lambda^1(\ell)}^*(B^1) \text{ on the 1-line}, \\ &\bar{y} \in C_{\Lambda^1(\ell)}^*(B^1) \text{ on the 2-line}. \end{aligned}$$

The formulas of Section 3.4 show that the only non-trivial d_1 differentials in spectral sequence (4.5.1) are

$$d_1(g^i \bar{a}_3^j) = h_{2,1}^{4i} \bar{u}^{3j}.$$

Since the spectral sequence (4.8.1) is a spectral sequence of modules over $\mathrm{Ext}^{*,*}(BP_*)$, we deduce that there are no possible d_r-differentials for $r > 1$. We deduce that we have

$$\begin{aligned} H^{*,*}(M_2^0 C^*_{tot}(Q(5))) = \ & \mathbb{F}_2[a_3^{\pm 1}, h_1, h_2, g]/(h_2^3 = a_3 h_1^3) \\ & \oplus \mathbb{F}_2[\bar{a}_3^{\pm 1}, \bar{g}]\{\bar{h}_1, \bar{h}_2, \bar{h}_1^2, \bar{h}_2^2, \bar{h}_2^3 = \bar{a}_3 \bar{h}_1^3\} \\ & \oplus \mathbb{F}_2[u^{\pm 1}, h_{2,1}] \\ & \oplus \mathbb{F}_2[\bar{u}^{\pm 3}, g]\{\bar{h}_{2,1}, (\bar{h}_{2,1})^2, (\bar{h}_{2,1})^3\} \\ & \oplus \mathbb{F}_2[\bar{u}^{\pm 3}, \bar{h}_{2,1}]\{\bar{u}, \bar{u}^2\}. \end{aligned}$$

4.9. **Computation of $H^{0,*}(M_1^1 C^*_{tot}(Q(5)))$.** We now compute the differentials in the v_1-BSS

$$H^{s,*}(M_2^0 C^*_{tot}(Q(5))) \otimes \mathbb{F}_2[v_1]/(v_1^\infty) \Rightarrow H^{s,*}(M_1^1 C^*_{tot}(Q(5))) \tag{4.9.1}$$

from the $s = 0$-line to the $s = 1$-line.

One computes using the formulas of Section 3.4:

$$\begin{aligned} D(x_0) &\equiv a_1 s^2 \mod (2, v_1^2), \\ D(x_1) &\equiv a_1^2 a_3 s \mod (2, v_1^3), \\ D(x_2) &\equiv a_1^8 u^4 \mod (2, v_1^9) \end{aligned} \tag{4.9.2}$$

for x_i as in Section 4.6. The formula for $D(x_2)$ already informs us that the v_1-BSS for $Q(5)$ differs from the v_1-BSS for $Q(3)$.

It follows from inductively applying (4.3.6) that we have

$$D(x_2^{2^{n-2}}) \equiv a_1^{2^{n+1}} u^{2^n} \mod (2, v_1^{2^{n+1}+1}).$$

It follows from (4.3.7) that for m odd, we have

$$\begin{aligned} D(x_0^m) &\equiv a_1 s^2 a_3^{m-1} \mod (2, v_1^2), \\ D(x_1^m) &\equiv a_1^2 a_3^{2m-1} s \mod (2, v_1^3), \\ D(x_2^{m2^{n-2}}) &\equiv a_1^{2^{n+1}} u^{3m2^n - 2^{n+1}} \mod (2, v_1^{2^{n+1}+1}). \end{aligned}$$

We deduce the following.

Lemma 4.9.3. The v_1-BSS differentials in (4.6.1) from the $(s = 0)$-line to the $(s = 1)$-line are given by

$$\begin{aligned} d_1\left(\frac{a_3^m}{v_1^j}\right) &= \frac{a_3^{m-1} h_2}{v_1^{j-1}}, \\ d_2\left(\frac{a_3^{2m}}{v_1^j}\right) &= \frac{a_3^{2m-1} h_1}{v_1^{j-2}}, \\ d_{2^{n+1}}\left(\frac{a_3^{m2^n}}{v_1^j}\right) &= \frac{u^{3m2^n - 2^{n+1}}}{v_1^{j-2^{n+1}}} \end{aligned}$$

where m is odd.

Corollary 4.9.4. The groups $H^{0,*}(M_1^1 C^*_{tot}(Q(5)))$ are generated by the elements

$$1/v_1^j, \qquad j \geq 1,$$

$$\frac{a_3^{m2^n}}{v_1^j}, \qquad m \text{ odd and } j \leq \begin{cases} 1, & n = 0, \\ 2, & n = 1, \\ 2^{n+1}, & n \geq 2. \end{cases}$$

In particular, the map

$$\mathrm{Ext}^{0,*}(M_1^1 BP_*) \to H^{0,*}(M_1^1 C^*_{tot}(Q(5)))$$

is *not* an isomorphism.

5. Low dimensional computations

In this section we explore the 2-primary homotopy $\pi_* Q(3)$ and $\pi_* Q(5)$ for $0 \leq * < 48$ (everything is implicitly 2-localized). In the case of $Q(3)$, Mark Mahowald has done similar computations, over a much vaster range, for the closely related Goerss-Henn-Mahowald-Rezk conjectural resolution of the 2-primary $K(2)$-local sphere — there is definitely some overlap here. In the case of $Q(5)$ the computations represent some genuinely unexplored territory, and give evidence that $Q(5)$ may detect more non-β-family v_2-periodic homotopy than $Q(3)$.

We do these low-dimensional computations in the most simple-minded manner, by computing the Bousfield-Kan spectral sequence

$$E_1^{s,t}(Q(\ell)) \Rightarrow \pi_{t-s} Q(\ell)$$

with

$$E_1^{s,t} = \begin{cases} \pi_t \,\mathrm{TMF}, & s = 0, \\ \pi_t \,\mathrm{TMF}_0(\ell) \oplus \pi_t \,\mathrm{TMF}, & s = 1, \\ \pi_t \,\mathrm{TMF}_0(\ell), & s = 2. \end{cases}$$

Actually, as the periodic versions of TMF typically have π_t of infinite rank, we only compute a certain "connective cover" of the spectral sequence — we only include holomorphic modular forms in this low dimensional computation (i.e. we do not invert Δ). Thus we are only computing a portion of the spectral sequence, which we shall refer to as the *holomorphic summand*. Note that the authors are not claiming that there exists a bounded below version of $Q(\ell)$ whose homotopy groups the holomorphic summand converges to (it remains an interesting open question how such connective versions of $Q(\ell)$ could be obtained by extending the semi-cosimplicial complex over the cusps).

In the following calculations, we employ a leading term algorithm, which basically amounts to only computing the leading terms of the differentials in row echelon form. Similarly to the previous section, we write everything 2-adically, and employ a lexicographical ordering on monomials

$$2^i v_1^j x.$$

Namely we say that $2^i v_1^j x$ is *lower* than $2^{i'} v_1^{j'} x'$ if $i < i'$, or if $i = i'$ and $j < j'$. We will write "leading term" differentials: the expression

$$x \mapsto y$$

indicates that

$$d_r(x + \text{higher terms}) = y + \text{higher terms}.$$

5.1. **The case of $Q(3)$.** In the case of $\mathrm{TMF}_0(3)$, recall that the modular forms of for $\Gamma_0(3)$ are spanned by those monomials $a_1^i a_3^j$ in $\mathbb{Z}[1/3, a_1, a_3]$ with $i + j$ even. In this section we will refer to a_1 as v_1 and a_3 as v_2, because that is what they correspond to under the complex orientation.

Figure 5.1.1 shows a low dimensional portion of the holomorphic summand of the spectral sequence $E_r^{s,t}(Q(3))$. There are many aspects of this chart that deserve explanation/remark.

- The copies of $\pi_* \mathrm{TMF}$ and $\pi_* \mathrm{TMF}_0(3)$ are separated by dotted lines. The bottom pattern is the $s = 0$ line of the spectral sequence ($\pi_* \mathrm{TMF}$). The next pattern up is the $\pi_* \mathrm{TMF}_0(3)$ summand of the $s = 1$ line, followed by the $\pi_* \mathrm{TMF}$ summand of the $s = 1$ line. The top pattern is the $s = 2$ line of the spectral sequence ($\pi_* \mathrm{TMF}_0(3)$). The spectral sequence is Adams-indexed, with the x-axis corresponding to the coordinate $t - s$.
- Dots indicate $\mathbb{Z}/2$'s. Boxes indicate $\mathbb{Z}_{(2)}$'s. The solid lines between the dots indicate 2-extensions, and η and ν multiplication.
- Horizontal dashed lines denote *bo*-patterns. Arrows indicate the *bo* patterns continue.
- There are two *bo*-patterns which are denoted "Im J". These *bo*-patterns (together with the *bo*-patterns which hit them with differentials) combine to form Im J patterns.
- Differentials are indicated with vertical curvy lines. All differentials displayed only indicate the leading terms of the differentials, as explained in the beginning of this section. For example, the d_1 differential from the 1-line to the 2-line showing

  $$v_1^2 v_2^2 \mapsto 2 v_2^2 v_1^2$$

  actually corresponds to a differential

  $$d_1(v_1^2 v_2^2 + v_1^5 v_2) = 2 v_2^2 v_1^2 + \text{higher terms}.$$

  The differentials on the torsion-free portions spanned by the modular forms are computed using the Mahowald-Rezk formulas.
- Differentials on the torsion summand can often be computed by noting that the maps f, t, q, and ψ^3 that define the coface maps of the semi-cosimplicial spectrum $Q(3)^\bullet$ are all maps of ring spectra, and in particular all have the same effect on elements in the Hurewicz image. There are a few notable exceptions, which we explain below.
- Dashed lines between layers indicate hidden extensions. These (probably) do not represent all hidden extensions: there are several possible hidden extensions which we have not resolved.

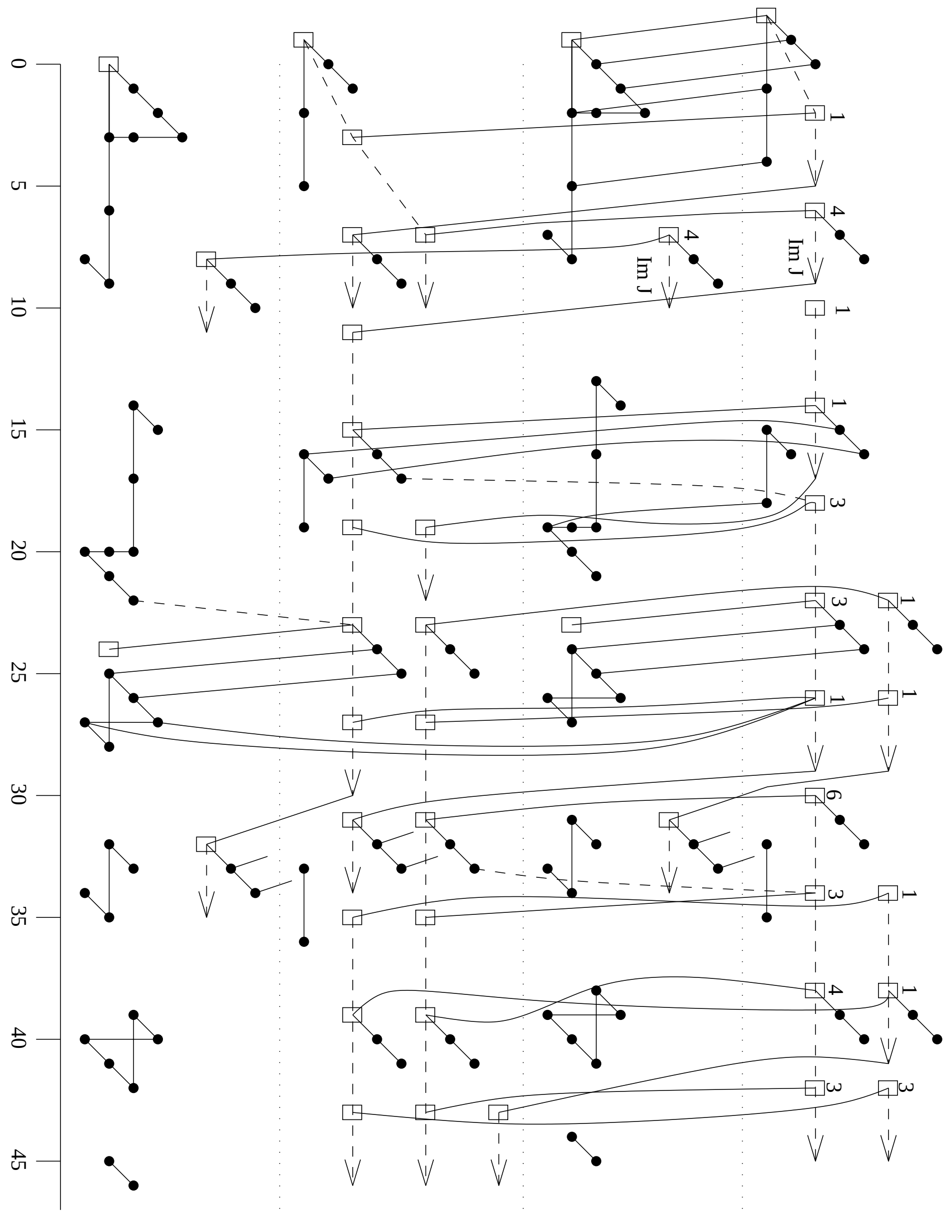


FIGURE 5.1.1. The holomorphic summand of the spectral sequence $E_r^{s,t}(Q(3))$ in low degrees.

- The differentials supported by the non-Hurewicz classes x and ηx in $\pi_{17}\,\mathrm{TMF}_0(3)$ and $\pi_{18}\,\mathrm{TMF}_0(3)$ are deduced because they kill the Hurewicz image of $\beta_{4/4}\eta$ and $\beta_{4/4}\eta^2$, which are zero in π_*S.
- The d_2-differentials are computed by observing that there is a (zero) hidden extension $\eta^3 v_1^6 v_2^2[1] = 4v_1^5 v_2^3[2]$ (where $[s]$ means s-line).

- Up to the natural deviations introduced by computing with the Bousfield-Kan spectral sequence, and not the Adams-Novikov spectral sequence, the divided β family is faithfully reproduced on the 2-line with the exception of the additional copy of Im J (there in fact should be infinitely many copies of such Im J summands) and one peculiar abnormality: the element $\beta_{8/8}$, detected by $32v_1v_2^5$ is 32-divisible. This extra divisibility does not contradict the results of Section 4 — the results there pertain to the monochromatic layer $M_2Q(3)$, and not $Q(3)$ directly.
- Boxes which are targets of differentials are labeled with numbers. A number n above a box indicates that after all differentials are run, you are left with a $\mathbb{Z}/2^n$.
- It is interesting to note that the permanent cycles on the zero line in this range are exactly the image of the TMF-Hurewicz homomorphism

We did not label the modular forms generating the boxes in the spectral sequence. In the case of $\pi_* \mathrm{TMF}$, the dimensions resolve this ambiguity. The remaining ambiguity is resolved by the following table, which indicates all of the leading terms of d_1 differentials between torsion free classes on the 1 and 2-lines of the spectral sequence.

$v_1^2 \mapsto 2v_1^2$		
$v_1v_2 \mapsto v_1^4$	$v_1^4 \mapsto 16v_1v_2$	
$v_2^2 \mapsto 2v_2^2$		
$v_1^2v_2^2 \mapsto 2v_2^2v_1^2$		
$v_1v_2^3 \mapsto v_2^2v_1^4$	$v_1^4v_2^2 \mapsto 8v_2^3v_1$	
$8\Delta \mapsto 8v_2^3v_1^3$	$v_2^4 \mapsto 2v_2^4$	$v_1^4v_2^2 \mapsto 8v_2^3v_1$
$v_2^4 \mapsto 2v_2^4$		
$v_2^4v_1^2 \mapsto 2v_2^4v_1^2$	$v_1^8v_2^2 \mapsto 8v_2^3v_1^5$	
$c_4\Delta \mapsto v_2^4v_1^4$	$v_2^5v_1 \mapsto v_2^3v_1^7$	$v_1^4v_2^4 \mapsto 64v_2^5v_1$
$v_2^4v_1^6 \mapsto 8v_2^5v_1^3$	$v_2^6 \mapsto 2v_2^6$	
$v_2^4v_1^8 \mapsto 16v_2^5v_1^5$	$v_2^6v_1^2 \mapsto 2v_2^6v_1^2$	
$v_2^7v_1 \mapsto v_2^6v_1^4$	$v_2^6v_1^4 \mapsto 8v_2^7v_1$	$v_1^{10}v_2^4 \mapsto 8v_2^5v_1^7$

5.2. **The case of $Q(5)$.** Figure 5.2.1 displays the spectral sequence for $Q(5)$. Essentially all of the conventions and remarks for the $Q(3)$ computation above extend to the $Q(5)$ computation. Below is the corresponding table for leading terms of differentials from the torsion-free elements in the 1-line to those in the 2-line.

$b_2 \mapsto 4b_2$		
$b_4 \mapsto b_2^2$	$\delta \mapsto 2\delta$	$b_2^2 \mapsto 16b_4$
$b_2\delta \mapsto 4b_2\delta$		
$b_4\delta \mapsto b_2^2\delta$	$\delta^2 \mapsto 2\delta^2$	$b_2^2\delta \mapsto 32b_4\delta$

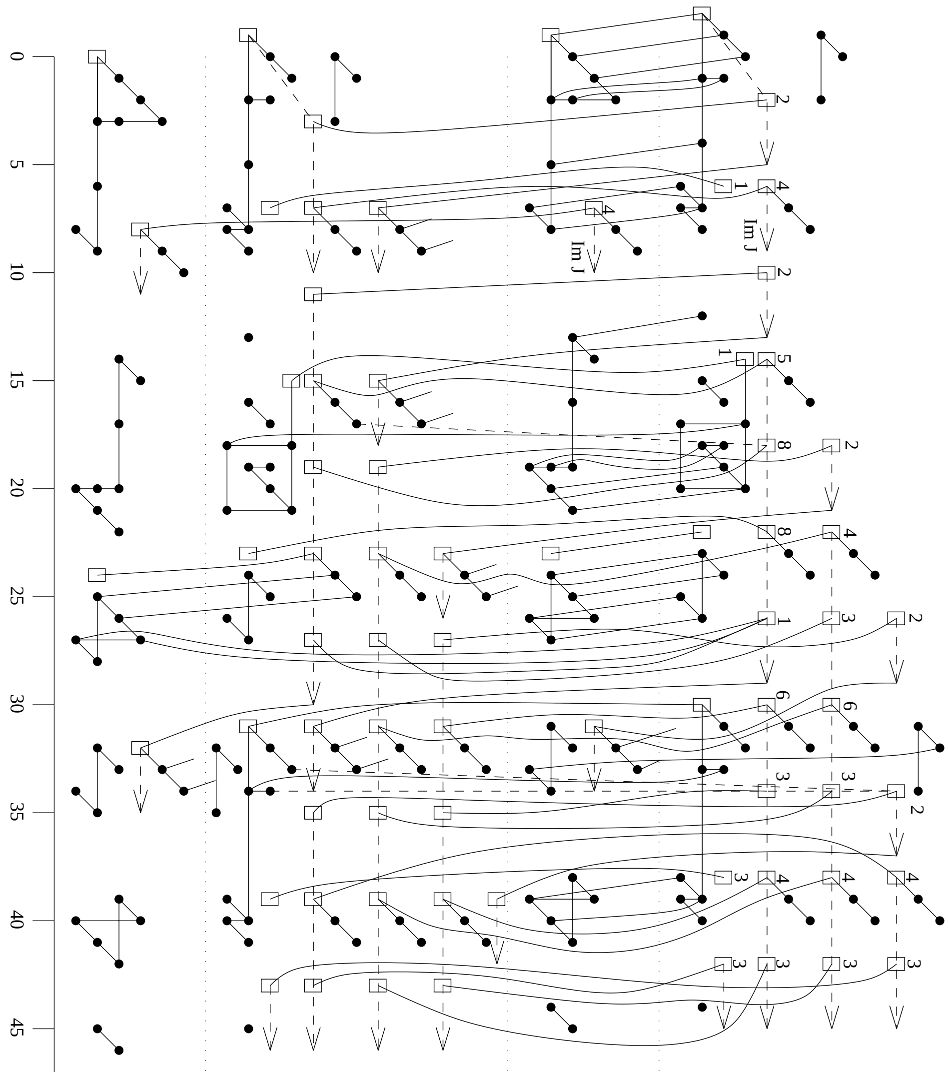


FIGURE 5.2.1. The holomorphic summand of the spectral sequence $E_r^{s,t}(Q(5))$ in low degrees.

$b_2\delta^2 \mapsto 4\delta^2 b_2$	$b_2^3\delta \mapsto 8b_2b_4\delta$			
$8\Delta \mapsto 8\delta^3$	$b_4\delta^2 \mapsto b_2^2\delta^2$	$4\delta^3 \mapsto 8b_2^2b_4\delta$	$b_2^2\delta^2 \mapsto 16b_4\delta^2$	
$b_2^5\delta \mapsto 8b_4\delta b_2^3$	$b_2\delta^3 \mapsto 4b_2\delta^3$	$b_2^3\delta^2 \mapsto 8b_4\delta^2 b_2$		
$c_4\Delta \mapsto b_2^2\delta^3$	$\delta^4 \mapsto 2\delta^4$	$b_4\delta^3 \mapsto b_2^4b_4\delta$	$b_2^2\delta^3 \mapsto 64b_4\delta^3$	$b_2^4\delta^2 \mapsto 64b_2^2b_4\delta^2$
$b_2\delta^4 \mapsto 4b_2\delta^4$	$b_2^3\delta^3 \mapsto 8b_2b_4\delta^3$	$b_2^5\delta^2 \mapsto 8b_2^3b_4\delta^2$		
$b_4\delta^4 \mapsto b_2^2\delta^4$	$4\delta^5 \mapsto 8\delta^5$	$b_2^4\delta^3 \mapsto 16b_2^2b_4\delta^3$	$b_2^2\delta^4 \mapsto 16b_4\delta^4$	$b_2^6\delta^2 \mapsto 16b_4\delta^2b_2^4$
$b_2\delta^5 \mapsto 4b_2b_4\delta^4$	$b_2^3\delta^4 \mapsto 8b_2\delta^5$	$b_2^5\delta^3 \mapsto 8\delta^3b_4b_2^6$	$b_2^7\delta^2 \mapsto 8b_2^5b_4\delta^2$	

We make the following remarks.

- The 2-line now bears little resemblance to the divided β-family. This is in sharp contrast with the situation with $Q(3)$. This fits well with our premise that while $Q(3)$ reproduces the divided β family almost flawlessly, $Q(5)$ does not.
- The much more robust torsion in $\pi_* \mathrm{TMF}_0(5)$ gives a significant source of homotopy in $\pi_* Q(5)$ which does not appear in $\pi_* Q(3)$. In particular, the elements
$$\nu\delta^4, \quad \nu^2\delta^4, \quad \epsilon\delta^4$$
seem like candidates to detect the elements in $\pi_* S$ with Adams spectral sequence names
$$h_5 h_2^2, \quad h_5 h_2^3, \quad h_5 h_3 h_1,$$
though the ambiguity resulting from the leading term algorithm makes it difficult to resolve this in the affirmative. These classes are *not* seen by $Q(3)$.
- Just as in the case of $Q(3)$, the permanent cycles on the zero line in this range are exactly the image of the TMF-Hurewicz homomorphism.

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