THE BOUSFIELD-KUHN FUNCTOR AND TOPOLOGICAL ANDRÉ-QUILLEN COHOMOLOGY

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1. Introduction

Fix a prime p and a height h. Let K denote the Morava K-theory $K(h)$, and $(-)_K$ denotes localization with respect to K. Let E denote the Morava E-theory spectrum E_h associated to the Lubin-Tate universal deformation $\mathbb{G}$ of the Honda height h formal group $\mathbb{G}_0/\mathbb{F}_{p^h}$. We shall always take $E_*(-)$ to denotes completed E-homology

$$E_*Y = \pi_*(E \wedge Y)_K.$$

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Bousfield and Kuhn (see, for example, [Kuh08]) constructed a functor $\Phi = \Phi_h$ from pointed spaces to spectra with the properties that

$$\begin{aligned}\Phi(\Omega^\infty Y) &\simeq Y_K, \\ \pi_*\Phi(X) &\cong v_h^{-1}\pi_* X.\end{aligned}$$

In this paper we construct a natural transformation between functors from pointed spaces to K-local spectra

$$c^{S_K} : \Phi(X) \to \mathrm{TAQ}_{S_K}(S_K^{X_+})$$

(the "comparison map") which relates $\Phi(X)$ with the topological André-Quillen cohomology of the augmented S_K-algebra $S_K^{X_+}$. Our main theorem (Theorem 8.1) states that this map is an equivalence when $X = S^q$ for q odd. This likely implies that there is some "good" class of spaces for which the comparison map is an equivalence, though we do not pursue this here. The case of $h = 0$ is essentially a special case of Quillen's theory [Qui69]. The case of $h = 1$ is closely related to the thesis of Jennifer French [Fre10].

We apply our main theorem to understand the v_h-periodic Goodwillie tower of the identity evaluated on odd spheres. This constitutes a step in the program begun by Arone and Mahowald [AM99] to generalize the Mahowald-Thompson approach to unstable v_1-periodic homotopy groups of spheres [Mah82], [Tho90]. Work of Arone-Mahowald [AM99] and Arone-Dwyer [AD01] shows that applying Φ to the Goodwillie tower of the identity evaluated on S^q (q odd) gives a (finite) resolution

$$\Phi(S^q) \to (L(0)_q)_K \to (L(1)_q)_K \to (L(2)_q)_K \to \cdots \to (L(h)_q)_K. \tag{1.1}$$

Here $L(k)_q$ denotes the Steinberg summand of the Thom spectrum of q copies of the reduced regular representation of $(\mathbb{Z}/p)^k$, as described in §5.

We show (Theorem 9.1) that the E-homology of the resolution (1.1) is isomorphic to the dual of the Koszul resolution of the (degree q) Dyer-Lashof algebra Δ^q for Morava E-theory. This results in a spectral sequence having the form

$$\mathrm{Ext}^s_{\Delta^q}(\widetilde{E}^q(S^q), \bar{E}_t) \Rightarrow E_{q+t-s}\Phi(S^q). \tag{1.2}$$

This is related to unstable v_h-periodic homotopy groups of spheres by the homotopy fixed point spectral sequence [DH04]

$$H^s_c(\mathbb{S}_n; E_t\Phi(S^q))^{Gal} \Rightarrow v_h^{-1}\pi_{t-s}(S^q).$$

In [Reza], the second author defined the *modular isogeny complex*, a cochain complex geometrically defined in terms of finite subgroups of the formal group $\mathbb{G}$, mimicking the structure of the building for $GL_h(\mathbb{F}_p)$. We show that the cohomology of the modular isogeny complex is the dual of the Koszul resolution for Δ^q. This gives a modular interpretation of the E_2-term of the spectral sequence (1.2).

Organization of the paper. In Section 2 we summarize the results about the Morava E-theory Dyer-Lashof algebra Δ^q we will need for the rest of the paper. In Section 3 we introduce a form of André-Quillen homology for unstable algebras over Δ^q, as well as a Grothendeick-type spectral sequence which relates this homology to $\mathrm{Tor}^{\Delta^q}_*$. In Section 4 we introduce a bar construction model for Kuhn's filtration on topological André-Quillen homology. The layers of this filtration, as well as the layers of the Goodwillie tower of the identity, are equivalent to the spectra $L(k)_q$. In Section 5 we show the E-homology of the spectrum $L(k)_q$ is dual to the k-th term of the Koszul resolution for Δ^q. In Section 6, we define the comparison map, and investigate its behavior on infinite loop spaces. This requires a technical result on H_∞-structures and the norm map, which is relegated to to Appendix A. In Section 7, we discuss a K-local analog of Weiss's orthogonal calculus. In Section 8, we prove that the comparison map is an equivalence on odd spheres, by using K-local Weiss calculus to play the Goodwillie tower off of the Kuhn filtration. In Section 9 we use the identification of the Goodwillie tower with the Kuhn filtration to compute the E-homology of the k-invariants of the Goodwillie tower. From these results we establish the spectral sequence (1.2). In Section 10, we give our modular description of the Koszul resolution for Δ^q, by showing that it is given by the cohomology of the modular isogeny complex.

Conventions. Here and throughout this paper, $(-)^\vee$ denotes the E_0-linear dual when applied to an E_0-module, and the Spanier-Whitehead dual when applied to a spectrum, We shall let Sp denote the category of symmetric spectra with the positive stable model structure [MMSS01], and shall simply refer to these as "spectra". If R is a commutative S-algebra, A is a commutative augmented R-algebra, and M is an R-module, we will let $\mathrm{TAQ}^R(A; M)$ denote topological André-Quillen homology of A (relative to R) with coefficients in M. Similarly we let $\mathrm{TAQ}_R(A; M)$ denote the corresponding topological André-Quillen cohomology. If $M = R$, we shall omit it from the TAQ-notation.

Acknowledgments. The first author learned many of the techniques employed in this paper through conversations with his Ph.D. student Jennifer French, and also benefited from many conversations with Jacob Lurie. Mike Hopkins suggested a version of the comparison map. In some sense much of this paper completes a project initially suggested and pursued by Matthew Ando, Paul Goerss, Mike Hopkins, and Neil Strickland, relating Δ^q to the Tits building for $GL_h(\mathbb{F}_p)$. Johann Sigurdsson and Neil Strickland have also studied the Morava E-homology of $L(k)$, but from a slightly different perspective than taken in this paper.

2. Recollections on the Dyer-Lashof algebra for Morava E-theory

Morava E-theory of symmetric groups. Strickland studied the Hopf ring

$$E^0(\amalg_n B\Sigma_n),$$

where the two products $\cdot$ and $*$ are given respectively by the cup product and transfers associated to the inclusions

$$\Sigma_n \times \Sigma_m \to \Sigma_{n+m} \tag{2.1}$$

and the coproduct is given by the restrictions associated to the above inclusions. Note that there are actually inclusions (2.1) for every partition of the set $\{1, \dots, n\}$ into two pieces. We shall refer to the stabilizers of such partitions as *partition subgroups.*

Strickland [Str98] proved that $E^0(\amalg_n B\Sigma_n)$ is a formal power series ring with indecomposables

$$\prod_{k\geq 0} E^0(B\Sigma_{p^k})/\operatorname{Tr}(\text{proper partition subgroups}).$$

Let

$$\mathrm{Sub}_{p^k}(\mathbb{G}) = \mathrm{Spf}(\mathcal{S}_{p^k})$$

be the (affine) formal scheme of subgroups of $\mathbb{G}$ of order p^k. For a noetherian complete local E_0-algebra R, the R-points of $\mathrm{Sub}_{p^k}(\mathbb{G})$ are given by

$$\mathrm{Sub}_{p^k}(\mathbb{G})(R) = \{H < \mathbb{G} \times_{\mathrm{Spf}(E_0)} \mathrm{Spf}(R) \ : \ |H| = p^k\}.$$

Strickland also shows that there is a canonical isomorphism:

$$E^0(B\Sigma_{p^k})/\operatorname{Tr}(\text{proper partition subgroups}) \cong \mathcal{S}_{p^k}. \tag{2.2}$$

This E_0-module is free of finite rank.

Let

$$s : E_0 \to \mathcal{S}_{p^k}$$

be the map which gives $\mathcal{S}_{p^k}$ its (aforementioned) E_0-algebra-structure, induced topologically from the map

$$B\Sigma_{p^k} \to *.$$

We regard $\mathcal{S}_{p^k}$ as a left module over E_0 by the module structure induced by s. Give the ring $\mathcal{S}_{p^k}$ the structure of a right E_0-module via the ring map

$$t : E_0 \to \mathcal{S}_{p^k} \tag{2.3}$$

which associates to an R-point $H < \mathbb{G} \times_{\mathrm{Spf}(E_0)} \mathrm{Spf}(R)$ the deformation

$$(\mathbb{G} \times_{\mathrm{Spf}(E_0)} \mathrm{Spf}(R))/H.$$

The map t arises topologically from the total power operation

$$E^0(*) \to E^0(B\Sigma_{p^k})$$

coming from the E_∞ structure of E.

Morava E-theory of extended powers. For an E-module Y, define

$$\mathbb{P}_E(Y) := \bigvee_{i\geq 0} Y^{\wedge_E i}_{h\Sigma_i}.$$

In [Rez09, §4], the second author defined a monad

$$\mathbb{T} : \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$$

and a natural transformation

$$\mathbb{T}\pi_* Y \to \pi_*(\mathbb{P}_E Y)_K$$

which induces an isomorphism

$$[\mathbb{T}\pi_*(Y)]^\wedge_{\mathfrak{m}} \xrightarrow{\cong} \pi_*(\mathbb{P}_E Y)_K \tag{2.4}$$

if $\pi_* Y$ is flat as an E_*-module [Rez09, Prop. 4.9]. Here, $\mathfrak{m}$ denotes the maximal ideal of E_0. There is a decomposition

$$\mathbb{T} = \bigoplus_{i\geq 0} \mathbb{T}\langle i\rangle$$

so that if $\pi_* Y$ is finite and flat, we have

$$\pi_*[Y^{\wedge_E i}_{h\Sigma_i}]_K \cong \mathbb{T}\langle i\rangle \pi_* Y_K. \tag{2.5}$$

The monad $\mathbb{T}$ comes equipped [Rez09, Prop. 4.7] with natural isomorphisms

$$\mathbb{T}(M) \otimes_{E_*} \mathbb{T}(N) \xrightarrow{\cong} \mathbb{T}(M \oplus N). \tag{2.6}$$

In particular, if A is a $\mathbb{T}$-algebra, then A is a graded-commutative E_*-algebra in the following strong sense: not only do element of odd degree anticommute, but also elements of odd degree square to 0. (See [Rez09, Prop. 3.4] for an explanation of this phenomenon.)

A convenient summary of the most important properties of the $\mathbb{T}$ construction is given in Section 3.2 of [Rezb]. In particular, we note that if R is a $K(n)$-local commutative E-algebra, then $\pi_* R$ canonically admits the structure of a $\mathbb{T}$-algebra.

Lemma 2.7. *If M is a free E_*-module, then $\mathbb{T}M$ is a free graded commutative E_*-algebra in the above sense.*

Proof. The rank 1 cases $M = E_*$ and $M = \Sigma E_*$ are discussed in the proof of Proposition 7.2 of [Rez09]. The general case then follows from 2.6. □

Specializing to the case where $Y = \Sigma^q E$ (for $q \in \mathbb{Z}$), and $i = p^k$, we have

$$[\mathbb{T}\langle p^k\rangle E_*(S^q)]_q = [E_* S^{qp^k}_{h\Sigma_{p^k}}]_q = E_0(B\Sigma_{p^k})^{q\bar{\rho}_k}$$

where $\bar{\rho}_k$ denotes the reduced standard real representation of Σ_{p^k}, and $(B\Sigma_{p^k})^{q\bar{\rho}_k}$ denotes the associated Thom spectrum.

Consider the sub- and quotient modules

$$\mathrm{Prim}_q[k] \hookrightarrow E_0(B\Sigma_{p^k})^{q\bar{\rho}_k} \twoheadrightarrow \mathrm{Ind}_q[k]$$

where $\mathrm{Prim}_q[k]$ denotes the intersection of the kernels of transfers to proper partition subgroups, and $\mathrm{Ind}_q[k]$ denotes quotient by the sum of the images of the restrictions from proper partition subgroups. Both $\mathrm{Prim}_q[k]$ and $\mathrm{Ind}_q[k]$ are finite free E_0-modules, and (2.2) implies that there is a canonical isomorphism

$$\mathrm{Prim}_0[k] \cong \mathcal{S}^\vee_{p^k}.$$

Let ι_q denote the composite

$$\iota_q : \mathrm{Prim}_q[k] \to \mathrm{Ind}_q[k].$$

The suspension σ is shown in [Rez09] to fit these modules together to give a diagram
(2.8)

$$\begin{tikzcd}
 & \mathrm{Prim}_{2m-1}[k] \arrow[d, "\cong"', "\iota_{2m-1}"] & \mathrm{Prim}_{2m}[k] \arrow[d, hook, "\iota_{2m}"] & \mathrm{Prim}_{2m+1}[k] \arrow[d, "\cong"', "\iota_{2m+1}"] & \cdots \\
\cdots \arrow[ur, "\sigma", "\cong"'] & \mathrm{Ind}_{2m-1}[k] \arrow[ur, "\sigma", "\cong"'] & \mathrm{Ind}_{2m}[k] \arrow[ur, "\sigma", "\cong"'] & \mathrm{Ind}_{2m+1}[k] \arrow[ur, "\sigma", "\cong"'] &
\end{tikzcd}$$

where ι_q is an isomorphism for q odd, and an inclusion with torsion cokernel for q even.

The Dyer-Lashof algebra for Morava E-theory. The algebra of additive power operations acting on cohomological degree q is given by

$$\Gamma^q = \bigoplus_k \mathrm{Prim}_{-q}[k].$$

This is contained (via the map ι_{-q}) in the larger algebra of indecomposable power operations

$$\Delta^q = \bigoplus_k \mathrm{Ind}_{-q}[k].$$

In both rings, the ring E_0 is not central, and thus Γ^q and Δ^q have distinct left and right E_0-module structures. In the case of Γ^0, these left and right module structures are induced respectively from the left and right module structures of E_0 on $\mathcal{S}_{p^k}$ under the isomorphism

$$\Gamma^0[k] \cong \mathcal{S}^\vee_{p^k}. \tag{2.9}$$

The algebra Γ^q is the algebra of natural endomorphisms of the functor

$$\begin{gathered} U^q : \mathrm{Alg}_{\mathbb{T}} \to \mathrm{Mod}_{E_0}, \\ A_* \mapsto A_{-q}; \end{gathered}$$

see [Rezb, §3.8]. It follows that the underlying E_*-module of a $\mathbb{T}$-algebra carries the structure of a graded Γ^*-module. The morphism (2.6) gives this Γ^*-module the structure of a graded-commutative Γ^*-algebra. The functors U^q thus assemble to give a functor

$$U^* : \mathrm{Alg}_{\mathbb{T}} \to \mathrm{Alg}_{\Gamma^*}.$$

The algebra Δ^q is the algebra of natural endomorphisms of the functor

$$\begin{gathered} V^q : \mathrm{Alg}_{\mathbb{T}} \downarrow E_* \to \mathrm{Mod}_{E_0}, \\ A_* \mapsto [I(A)/I(A)^2]_{-q}.; \end{gathered}$$

see [Rezb, §3.10].

The non-canonical natural isomorphisms $U^q \cong U^{q+2}$ and $V^q \cong V^{q+2}$ given by multiplication by a unit in E_{-2} induce non-canonical isomorphisms of algebras

$$\Gamma^q \cong \Gamma^{q+2}, \tag{2.10}$$

$$\Delta^q \cong \Delta^{q+2}. \tag{2.11}$$

The suspension induces canonical isomorphisms of algebras

$$\sigma : \Delta^q \xrightarrow{\cong} \Gamma^{q-1}. \tag{2.12}$$

In particular, all of the E_0-algebras Γ^q and Δ^q, for all q, are non-canonically isomorphic to each other.

The Koszul resolution. Observe that the augmentation

$$\epsilon : \Delta^q = \bigoplus_{k\geq 0} \Delta^q[k] \to \Delta^q[0] = E_0$$

endows E_0 with the structure of a Δ^q bi-module: we shall use the notation $\overline{E}_0$ to denote this Δ^q-bimodule. Let $\widetilde{\Delta}^q$ denote the kernel of the augmentation ϵ; it is likewise a Δ^q-bimodule.

Consider the normalized bar complex $B_*(\bar{E}_0, \widetilde{\Delta}^q, \bar{E}_0)$ with

$$B_s(\bar{E}_0, \widetilde{\Delta}^q, \bar{E}_0) = \bar{E}_0 \otimes_{E_0} (\widetilde{\Delta}^q)^{\otimes_{E_0} s} \otimes_{E_0} \bar{E}_0 \cong (\widetilde{\Delta}^q)^{\otimes_{E_0} s}.$$

Endow the complex $B_*(\bar{E}_0, \widetilde{\Delta}^q, \bar{E}_0)$ with a grading by setting

$$B_s(\bar{E}_0, \widetilde{\Delta}^q, \bar{E}_0)[k] := \bigoplus_{\substack{k=k_1+\cdots+k_s \\ k_i>0}} \Delta^q[k_1] \otimes_{E_0} \cdots \otimes_{E_0} \Delta^q[k_s].$$

Observe that since $\Delta^q[k]$ acts trivially on $\bar{E}_0$ for $k > 0$, the differential in the bar complex preserves this grading. Thus there is a decomposition of chain complexes

$$B_*(\bar{E}_0, \widetilde{\Delta}^q, \bar{E}_0) = \bigoplus_{k\geq 0} B_*(\bar{E}_0, \widetilde{\Delta}^q, \bar{E}_0)[k].$$

In [Rezb], the second author proved that the algebras Δ^q are Koszul, as summarized in the following theorem.

Theorem 2.13 ([Rezb], Prop. 4.6)**.** *For each k, the kth graded part of the chain complex has homology concentrated in top degree:*

$$H_s(B_*(\bar{E}_0, \widetilde{\Delta}^q, \bar{E}_0)[k]) = \begin{cases} C[k]_{-q}, & s = k, \\ 0, & s \neq k, \end{cases}$$

where each $C[k]_{-q}$ is finitely generated and free as a right E_0-module; furthermore, $C[k]_{-q} = 0$ if $k < h$.

Thus we have

$$\begin{aligned} \mathrm{Tor}^k_{\Delta^q}(\bar{E}_0, \bar{E}_0) &\cong C[k]_{-q}, \\ \mathrm{Ext}^k_{\Delta^q}(\bar{E}_0, \bar{E}_0) &\cong C[k]^\vee_{-q}. \end{aligned}$$

Remark 2.14. Actually, in [Rezb], it is proven that Δ^0 is Koszul, but using the isomorphisms (2.11) and (2.12), there are non-canonical isomorphisms $\Delta^0 \cong \Delta^q$. Therefore Δ^q is also Koszul.

If M is a Δ^q-module, then the *Koszul complex* $C^{\Delta^q}_*(M)$ associated to M is the chain complex

$$C^{\Delta^q}_*(M) = \left(C[0]_{-q} \otimes_{E_0} M \xleftarrow{\delta_0} C[1]_{-q} \otimes_{E_0} M \xleftarrow{\delta_1} \cdots\right)$$

with differentials δ_k induced from the following diagram.

$$\begin{tikzcd}
B_{k+1}(\bar{E}_0, \Delta^q, M)[k+1] \arrow[r, "d_{k+1}"] \arrow[d, equal] & B_k(\bar{E}_0, \Delta^q, M)[k] \arrow[d, equal] \\
\Delta^q[1]^{\otimes(k+1)} \otimes_{E_0} M & \Delta^q[1]^{\otimes k} \otimes_{E_0} M \\
C[k+1]_{-q} \otimes_{\bar{E}_0} M \arrow[r, "\delta_k"'] \arrow[u, hook] & C[k]_{-q} \otimes_{\bar{E}_0} M \arrow[u, hook]
\end{tikzcd} \tag{2.15}$$

Here, the map d_{k+1} is the last face map in the bar complex $B_\bullet(\bar{E}_0, \Delta^q, M)$. We have

$$H_s(C_*^{\Delta^q}(M)) \cong \mathrm{Tor}_s^{\Delta^q}(\bar{E}_0, M).$$

Recall that the E_0-modules $C[k]_{-q}$ are projective. It follows that if M is projective as an E_0-module, the dual cochain complex computes Ext:

$$H^s(C_*^{\Delta^q}(M)^\vee) \cong \mathrm{Ext}^s_{\Delta^q}(M, \bar{E}_0).$$

3. Barr-Beck homology

Augmented $\mathbb{T}$-algebras. Consider the adjunction

$$\mathbb{T} : \mathrm{Mod}_{E_*} \leftrightarrows \mathrm{Alg}_{\mathbb{T}} : U^*.$$

A free $\mathbb{T}$-algebra $\mathbb{T}M$ is augmented over E_* by the map

$$\mathbb{T}M \to \mathbb{T}\langle 0 \rangle M = E_*.$$

Thus the above adjunction restricts to an adjunction for augmented $\mathbb{T}$-algebras

$$\mathbb{T} : \mathrm{Mod}_{E_*} \leftrightarrows \mathrm{Alg}_{\mathbb{T}} \downarrow E_* : I(-)$$

where $I(-)$ is the kernel of the augmentation. The monad $\mathbb{T}$ contains a "non-unital" summand

$$\bar{\mathbb{T}} := \bigoplus_{i>0} \mathbb{T}\langle i \rangle.$$

Note that there is a natural isomorphism

$$I(\mathbb{T}M) \cong \bar{\mathbb{T}}M.$$

In particular, if A is an augmented $\mathbb{T}$-algebra, then $I(A)$ is a $\bar{\mathbb{T}}$-algebra.

Trivial $\mathbb{T}$-algebras. The monad $\bar{\mathbb{T}}$ is augmented over the identity functor via the projection

$$\bar{\mathbb{T}} \to \mathbb{T}\langle 1 \rangle = \mathrm{Id}.$$

If M is an E_*-module, then via the augmentation we can give M the *trivial* $\bar{\mathbb{T}}$-algebra structure. We shall denote the resulting $\bar{\mathbb{T}}$-algebra by $\overline{M}$.

If X is an E-module spectrum, write $\overline{X}$ for this spectrum endowed with the structure of a non-unital E-algebra spectrum with trivial multiplication. We have the following.

Proposition 3.1. *If X is a K-local E-module spectrum, the evident identification $\pi_*\overline{X} \approx \overline{\pi_* X}$ is an isomorphism of $\bar{\mathbb{T}}$-algebras.*

Cotriple homology. Suppose that we are given a functor $F : \mathrm{Alg}_{\mathbb{T}} \downarrow E_* \to \mathcal{A}$ for $\mathcal{A}$ an abelian category. Barr and Beck [BB69] define a "cotriple homology" associated to F relative to the comonad $\mathbb{T}I(-)$ on $\mathrm{Alg}_{\mathbb{T}} \downarrow E_*$, which we shall simply denote $\mathbb{L}_*F$, as it could be viewed as a kind of left derived functor. Explicitly it may be computed in terms of the monadic bar construction as

$$\mathbb{L}_s F(A) \cong H_s(F(B_*(\mathbb{T}, \bar{\mathbb{T}}, I(A)))).$$

Derived functors of $\mathbb{T}$-indecomposables. Consider the functor

$$\begin{aligned} \Omega^q_{\mathbb{T}/E_*} &: \mathrm{Alg}_{\mathbb{T}} \downarrow E_* \to \mathrm{Mod}_{E_0}, \\ A &\mapsto \bar{E}_0 \otimes_{\Delta^q} V^q(A). \end{aligned}$$

Combining (2.6), Lemma 2.7, and the definition of Δ^*, we have the following lemma.

Lemma 3.2. *Suppose that M is a free E_*-module. Then there is a natural isomorphism*

$$V^q(\mathbb{T}M) \approx \Delta^q \otimes_{E_0} M_{-q},$$

and hence a natural isomorphism

$$\Omega^q_{\mathbb{T}/E_*}\mathbb{T}M \cong M_{-q}.$$

Corollary 3.3. *If $A \in \mathrm{Alg}_{\mathbb{T}} \downarrow E_*$ is free as an E_*-module, then there is an isomorphism*

$$\mathbb{L}_s \Omega^q_{\mathbb{T}/E_*} A \cong H_s(B_*(\mathrm{Id}, \bar{\mathbb{T}}, I(A))_{-q}).$$

A Grothendeick spectral sequence.

Proposition 3.4. *Suppose that A is an augmented $\mathbb{T}$-algebra which is free as an E_*-module. Then there is a Grothendieck-type spectral sequence*

$$E^2_{s,t} = \mathrm{Tor}_s^{\Delta^q}(\bar{E}_0, \mathbb{L}_t V^q(A)) \Rightarrow \mathbb{L}_{s+t}\Omega^q_{\mathbb{T}/E_*}A.$$

Proof. Consider the double complex

$$C_{s,t} := B_s(\bar{E}^0, \Delta^q, V^q(B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A)))).$$

Computing the spectral sequence for the double complex by running s-homology, then t-homology, we have

$$H_t H_s C_{s,t} \Rightarrow H_{s+t} \operatorname{Tot} C_{s,t}.$$

Using (2.6), Lemma 2.7, and the definition of Δ^q, we have

$$\begin{aligned} H_t H_s C_{s,t} &= H_t H_s B_s(\bar{E}^0, \Delta^q, V^q(B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A)))) \\ &\cong H_t \operatorname{Tor}_s^{\Delta^q}(\bar{E}_0, V^q(B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A)))) \\ &\cong H_t \operatorname{Tor}_s^{\Delta^q}(\bar{E}_0, \Delta^q \otimes_{E_0} \bar{\mathbb{T}}^{\circ t} I(A)) \\ &\cong \begin{cases} H_t B_t(\mathrm{Id}, \bar{\mathbb{T}}, I(A)), & s = 0, \\ 0, & s \neq 0. \end{cases} \end{aligned}$$

The isomorphism of the second line uses the fact that $V^q(B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A)))$ is a free E_0-module when A is one, using Lemma 2.7. The isomorphism of the third line uses Lemma 3.2.

The spectral sequence therefore collapses to give an isomorphism

$$H_i \operatorname{Tot} C_{*,*} \cong \mathbb{L}_i \Omega^q_{\mathbb{T}/E_*} A.$$

Running t-homology followed by s-homology therefore gives a spectral sequence

$$H_s H_t C_{s,t} \Rightarrow \mathbb{L}_{s+t} \Omega^q_{\mathbb{T}/E_*} A.$$

Using the fact that Δ^q is free over E_0, we compute

$$\begin{aligned} H_s H_t C_{s,t} &= H_s H_t B_s(\bar{E}^0, \Delta^q, V^q(B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A)))) \\ &\cong H_s B_s(\bar{E}^0, \Delta^q, H_t V^q(B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A)))) \\ &\cong H_s B_s(\bar{E}^0, \Delta^q, \mathbb{L}_t V^q A) \\ &\cong \operatorname{Tor}_s^{\Delta^q}(\bar{E}^0, \mathbb{L}_t V^q A). \end{aligned}$$

□

The homology groups $\mathbb{L}_t V^q A$ appearing in the E^2-term of the Grothendieck spectral sequence are demystified by the following lemma. We write "$\mathbb{L}_* \Omega_{(-)/E_*}$" for the Andreé-Quillen homology of augmented graded commutative E_*-algebras, where as in §2 graded commutativity implies that odd degree elements square to 0.

Lemma 3.5. *Suppose that $A \in \mathrm{Alg}_{\mathbb{T}} \downarrow E_*$ is free as an E_*-module. Then there are isomorphisms*

$$\mathbb{L}_i V^q A \cong [\mathbb{L}_i \Omega_{A_*/E_*}]_{-q}.$$

Proof. By Lemma 2.7, the bar resolution

$$B_\bullet(\mathbb{T}, \bar{\mathbb{T}}, I(A)) \to A$$

is a simplicial resolution of A by free graded commutative algebras. Since $V^*(-) = I(-)/I(-)^2$, the result follows. □

Corollary 3.6. *Suppose that $A \in \mathrm{Alg}_{\mathbb{T}} \downarrow E_*$ is free as an augmented graded commutative E_*-algebra. Then the Grothendieck spectral sequence collapses to give an isomorphism*

$$\mathbb{L}_s \Omega^q_{\mathbb{T}/E_*} A \cong \operatorname{Tor}_s^{\Delta^q}(\bar{E}_0, V^q(A)).$$

Linearization. The definition of Δ^* gives rise to natural transformations

$$\Delta^* \otimes_{E_*} M \to V^*(\mathbb{T}M) = \bar{\mathbb{T}}(M)/(\bar{\mathbb{T}}(M))^2 \leftarrow \bar{\mathbb{T}}(M)$$

of functors. We have noted (Lemma 3.2) that if M is a free E_*-module, then $\Delta^* \otimes_{E*} M \to V^*(\mathbb{T}M)$ is an isomorphism. Hence, on the full subcategory of free E_*-modules we obtain a natural transformation of monads

$$\mathcal{L} : \bar{\mathbb{T}}M \to \Delta^* \otimes_{E_*} M$$

on Mod_{E_*}. (In [Rezb], this transformation is observed to be linearization for projective M, hence the notation $\mathcal{L}$.) For $A \in \mathrm{Alg}_{\mathbb{T}} \downarrow E_*$, the natural transformation $\mathcal{L}$ induces a map of chain complexes

$$\mathcal{L} : B(\mathrm{Id}, \bar{\mathbb{T}}, I(A))_{-q} \to B_*(\bar{E}_0, \Delta^q, V^q(A)) \tag{3.7}$$

and therefore a map

$$\mathcal{L} : \mathbb{L}_s \Omega^q_{\mathbb{T}/E_*} A \to \mathrm{Tor}_s^{\Delta^q}(\bar{E}_0, V^q(A)).$$

Lemma 3.8. *If A is free as a graded commutative E_*-algebra, the map (3.7) is a quasi-isomorphism.*

Proof. This essentially follows Corollary 3.6 from an identification of the map (3.7) with the edge homomorphism of the Grothendieck spectral sequence. Specifically, consider the following commutative diagram of maps of chain complexes.

(3.9)

$$\begin{tikzcd}
\bigoplus\limits_{s+t=n} B_s(\bar{E}_0, \Delta^q, V^q B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A))) \arrow[r, "\mathrm{aug}_t", "\simeq"'] \arrow[d, "\mathrm{aug}_s"', "\simeq"] \arrow[dr, "\mathcal{L}"] & B_n(\bar{E}_0, \Delta^q, V^q A) \\
\bar{E}_0 \otimes_{\Delta^q} V^q(B_n(\mathbb{T}, \bar{\mathbb{T}}, I(A))) \arrow[d, equal] & \bigoplus\limits_{s+t=n} B_s(\bar{E}_0, \Delta^q, B_t(\Delta^q, \Delta^q, V^q A)) \arrow[u, "\simeq", "\mathrm{aug}_t"'] \arrow[d, "\simeq"', "\mathrm{aug}_s"] \\
B_n(\mathrm{Id}, \bar{\mathbb{T}}, I(A))_{-q} \arrow[r, "\mathcal{L}"'] & B_n(\bar{E}_0, \Delta^q, V^q A)
\end{tikzcd}$$

Here the maps labeled aug_s and aug_t are the augmentations of the corresponding bar complexes, and $\mathcal{L}$ are the maps induced by linearization. All of the augmentation maps are edge homomorphisms of appropriate spectral sequences of double complexes, with E_2-terms:

$$\begin{aligned}
{}^{I}E^2_{s,t} &= H_t H_s B_s(\bar{E}_0, \Delta^q, V^q B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A))), \\
{}^{II}E^2_{s,t} &= H_s H_t B_s(\bar{E}_0, \Delta^q, V^q B_t(\mathbb{T}, \bar{\mathbb{T}}, I(A))), \\
{}^{III}E^2_{s,t} &= H_t H_s B_s(\bar{E}_0, \Delta^q, B_t(\Delta^q, \Delta^q, V^q A)), \\
{}^{IV}E^2_{s,t} &= H_s H_t B_s(\bar{E}_0, \Delta^q, B_t(\Delta^q, \Delta^q, V^q A)).
\end{aligned}$$

Each of these spectral sequences collapses: the case of $\{{}^{I}E^r_{s,t}\}$ is discussed in the proof of Proposition 3.4, the case of $\{{}^{II}E^r_{s,t}\}$, the Grothendieck spectral sequence, is handled by Corollary 3.6, and the spectral sequences $\{{}^{III}E^2_{s,t}\}$ and $\{{}^{IV}E^2_{s,t}\}$ collapse for trivial reasons. It follows that each of the augmentation maps in Diagram (3.9) are quasi-isomorphisms, as indicated. It follows that the bottom arrow in (3.9) is a quasi-isomorphism, as desired. □

4. Topological André-Quillen homology

Definitions. Suppose that R is a commutative S-algebra, and that A is an augmented commutative R-algebra. Topological André-Quillen homology of A (relative to R) was defined by Basterra [Bas99] as a suitably derived version of the cofiber of the multiplication map on the augmentation ideal:

$$\mathrm{TAQ}^R(A) = I(A)/I(A)^{\wedge 2}.$$

If M is an R-module, then Topological André-Quillen homology and cohomology of A with coefficients in M are defined respectively as

$$\begin{aligned}\mathrm{TAQ}^R(A; M) &= \mathrm{TAQ}^R(A) \wedge_R M,\\ \mathrm{TAQ}_R(A; M) &= F_R(\mathrm{TAQ}^R(A), M).\end{aligned}$$

As with TAQ^R, we let $\mathrm{TAQ}_R(A) = \mathrm{TAQ}_R(A; R)$.

The augmentation ideal functor gives an equivalence

$$I(-) : \mathrm{Alg}_R \downarrow R \xrightarrow{\simeq} \mathrm{Alg}_R^{nu}$$

between the category of augmented commutative R-algebras and the category of non-unital commutative R-algebras. These categories are tensored over pointed spaces. Basterra-McCarthy [BM02] show that $\mathrm{TAQ}^R(-)$ is the stabilization: there is an equivalence

$$\mathrm{TAQ}^R(A) \simeq \operatorname*{hocolim}_n \Omega^n(S^n \otimes IA).$$

The Kuhn filtration. Kuhn [Kuh04a] endows the topological André-Quillen homology $\mathrm{TAQ}^S(A)$ of an augmented commutative S-algebra A with an increasing filtration

$$F_1\,\mathrm{TAQ}^S(A) \to F_2\,\mathrm{TAQ}^S(A) \to \cdots . \tag{4.1}$$

We shall use the simplicial presentation of TAQ^S to give a point set level definition of a filtration which has identical properties to Kuhn's filtration.

Remark 4.2. While there is compelling evidence that the filtration we define in this section agrees with the filtration defined by Kuhn in [Kuh04a], the authors do not actually know a proof of this. We will nevertheless refer to the filtration defined in this section as the "Kuhn filtration" for the rest of the paper.

Let $\mathbb{P}$ denote the free E_∞-ring monad on Sp

$$\mathbb{P}(Y) := \bigvee_{n \ge 0} Y^{\wedge n}_{h\Sigma_n},$$

and let $\widetilde{\mathbb{P}}$ denotes the "non-unital" version

$$\widetilde{\mathbb{P}}(Y) := \bigvee_{n \ge 1} Y^{\wedge n}_{h\Sigma_n}.$$

Note that the monad $\widetilde{\mathbb{P}}$ is augmented over the identity. Basterra [Bas99, §5] shows that TAQ admits a simplicial presentation using the monadic bar construction:

$$\mathrm{TAQ}^S(A) \simeq \left| B_\bullet(\mathrm{Id}, \widetilde{\mathbb{P}}, I(A)) \right| . \tag{4.3}$$

For a non-unital operad $\mathcal{O}$ in Sp, let $\mathcal{F}_{\mathcal{O}}$ denote the free $\mathcal{O}$-algebra monad in Sp:

$$\mathcal{F}_{\mathcal{O}} Y := \bigvee_{n \geq 1} \mathcal{O}_n \wedge_{\Sigma_n} Y^{\wedge n}.$$

Let Comm denote the (non-unital) commutative operad in spectra, with

$$\mathrm{Comm}_n = S.$$

Viewed as an endofunctor of spectra, we have (using [MMSS01, Lem. 15.5])

$$\mathbb{L}\mathcal{F}_{\mathrm{Comm}} \simeq \widetilde{\mathbb{P}}.$$

We therefore have, for A positive cofibrant:

$$\mathrm{TAQ}^S(A) \simeq \left| B_\bullet(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, I(A)) \right|.$$

Observe for fixed s there is a splitting

$$\begin{aligned} B_s(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, I(A)) &= \mathcal{F}^s_{\mathrm{Comm}} I(A) \\ &\cong \mathcal{F}_{[\mathrm{Comm}^{\circ s}]} I(A) \\ &= \bigvee_{i \geq 1} [\mathrm{Comm}^{\circ s}]_i \wedge_{\Sigma_i} I(A)^{\wedge i}. \\ &=: \bigvee_{i \geq 1} B_s(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, I(A))\langle i \rangle. \end{aligned} \tag{4.4}$$

Here, $\circ$ denotes the composition product of symmetric sequences. Consider the filtration

$$F_n B_s(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, I(A)) = \bigvee_{1 \leq i \leq n} B_s(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, I(A))\langle i \rangle.$$

This filtration is compatible with the simplicial structure, and recovers the Kuhn filtration on TAQ^S for cofibrant R:

$$F_n \,\mathrm{TAQ}^S(A) \simeq \left| F_n B_\bullet(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, I(A)) \right|.$$

The layers of the Kuhn filtration. We now recall Kuhn's description of the layers of his filtration in this language. The (pointed) partition poset complex $\mathcal{P}(n)_\bullet$ is defined to be the pointed simplicial Σ_n-set whose set of s-simplices is the set

$$\left\{ \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_s \; : \; \begin{array}{l} \lambda_i \text{ is a partition of } \underline{n}, \\ \lambda_0 = \{1, \ldots, n\}, \\ \lambda_s = \{1\} \cdots \{n\} \end{array} \right\} \amalg \{*\}.$$

The face and degeneracy maps send the disjoint basepoint $*$ to the disjoint basepoint, and are given on the other elements by the formulas

$$d_i(\lambda_0 \leq \cdots \leq \lambda_s) = \begin{cases} \lambda_0 \leq \cdots \leq \widehat{\lambda_i} \leq \cdots \leq \lambda_s, & i \notin \{0, s\}, \\ *, & i \in \{0, s\}, \end{cases}$$

$$s_i(\lambda_0 \leq \cdots \leq \lambda_s) = \lambda_0 \leq \cdots \leq \lambda_i \leq \lambda_i \leq \cdots \leq \lambda_s.$$

Note that we have

$$\mathcal{P}(n)_0 = \begin{cases} \{\{1\}, *\}, & n = 1, \\ \{*\}, & n > 1. \end{cases}$$

Proposition 4.5 (Kuhn [Kuh04a]). *We have*

$$F_n \operatorname{TAQ}^S(A)/F_{n-1}\operatorname{TAQ}^S(A) \simeq |\mathcal{P}(n)_\bullet| \wedge_{h\Sigma_n} I(A)^{\wedge n}. \tag{4.6}$$

Proof. Let $\overline{I(A)}$ denote the spectrum $I(A)$ endowed with the trivial $\mathcal{F}_{\mathrm{Comm}}$-algebra structure. We have

$$\begin{aligned} F_n \operatorname{TAQ}^S(A)/F_{n-1}\operatorname{TAQ}^S(A) &\simeq B(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, \overline{I(A)})\langle n\rangle \\ &\cong |[\mathrm{Comm}^{\circ\bullet}]_n \wedge_{\Sigma_n} I(A)^{\wedge n}| \\ &\cong |B_\bullet(1, \mathrm{Comm}, 1)_n| \wedge_{\Sigma_n} I(A)^{\wedge n}. \end{aligned}$$

Here, 1 denotes the unit symmetric sequence. The lemma now follows from the isomorphism of simplicial Σ_n-spectra

$$B_\bullet(1, \mathrm{Comm}, 1)_n \cong \mathcal{P}(n)_\bullet$$

(see [Chi05]). ☐

5. The Morava E-theory of $L(k)$

$L(k)$-spectra. The spectrum $L(k)_q$ is defined as

$$L(k)_q := \epsilon_{st}(B\mathbb{F}_p^k)^{q\bar{\rho}_k}. \tag{5.1}$$

Here, $\bar{\rho}_k$ represents the reduced regular real representation of the elementary abelian p-group $\mathbb{F}_p^k$, and $(B\mathbb{F}_p^k)^{q\bar{\rho}_k}$ denotes the Thom spectrum of the q-fold direct sum of $\bar{\rho}_k$. We write ϵ_{st} for the Steinberg idempotent, acting on this spectrum, so that $L(k)_q$ is the Steinberg summand.

We shall let $L(k)$ denote the spectrum $L(k)_1$. Mitchell and Priddy show that there are equivalences

$$\mathrm{Sp}^{p^k}(S)/\mathrm{Sp}^{p^{k-1}}(S) \simeq \Sigma^k L(k)$$

where $\mathrm{Sp}^{p^k}(S)$ is the (p^k)th symmetric product of the sphere spectrum.

The Goodwillie derivatives of the identity functor

$$\mathrm{Id} : \mathrm{Top}_* \to \mathrm{Top}_*$$

are given by (see [AM99])

$$\partial_n(\mathrm{Id}) \simeq (\Sigma^\infty |\mathcal{P}(n)_\bullet|)^\vee. \tag{5.2}$$

Arone and Dwyer [AD01, Cor. 9.6] establish mod p equivalences (for q odd)

$$L(k)_q \simeq_p \Sigma^{k-q}[\partial_{p^k}(\mathrm{Id}) \wedge S^{qp^k}]_{h\Sigma_{p^k}} = \Sigma^{k-q}\mathbb{D}_{p^k}(\mathrm{Id})(S^q). \tag{5.3}$$

Here $\mathbb{D}_{p^k}(\mathrm{Id})$ is the infinite delooping of the (p^k)th layer of the Goodwillie tower of the identity functor on Top_*.

Remark 5.4. For the purposes of the rest of the paper, one could take (5.3) as the definition of the p-adic homotopy type of $L(k)_q$, instead of (5.1). All of the computations and properties of the spectra $L(k)_q$ in what follows are really aspects of the partition poset model of $\mathbb{D}_{p^k}(\mathrm{Id})(S^q)$.

The E-homology calculation. We now turn our attention to computing the E-homology of the spectra $L(k)_q$ using (5.3). We do this with a sequence of lemmas. Recall from §3 that for an E_*-module M, we write $\overline{M}$ for the $\bar{\mathbb{T}}$-algebra obtained by endowing M with the trivial action.

Lemma 5.5. *If Y is a spectrum with E_*Y finite and flat as an E_*-module, then there is an isomorphism of simplicial E_*-modules*

$$E_*(\mathcal{P}(n)_\bullet \wedge_{h\Sigma_n} Y^{\wedge n}) \cong B_\bullet(\mathrm{Id}, \bar{\mathbb{T}}, \overline{E_*Y})\langle n\rangle.$$

Proof. Replacing Y with a cofibrant replacement in the positive model structure for symmetric spectra, this follows immediately from applying (2.5) to the isomorphisms

$$\begin{aligned}\mathcal{P}(n)_\bullet \wedge_{h\Sigma_n} Y^{\wedge n} &\cong B_\bullet(1, \mathrm{Comm}, 1)_n \wedge_{\Sigma_n} Y^{\wedge n}\\ &\cong B_\bullet(\mathrm{Id}, \mathcal{F}_{\mathrm{Comm}}, \overline{Y})\langle n\rangle.\end{aligned}$$

□

Lemma 5.6. *For q odd, there is a canonical isomorphism*

$$E_0(\Sigma^{-k-q}\left|\mathcal{P}(p^k)_\bullet\right| \wedge_{h\Sigma_{p^k}} S^{qp^k}) \cong C[k]_q.$$

Proof. Consider the Bousfield-Kan spectral sequence:

$$E^1_{s,t} = E_t(\mathcal{P}(p^k)_s \wedge_{h\Sigma_{p^k}} S^{qp^k}) \Rightarrow E_{t+s}(\left|\mathcal{P}(p^k)_\bullet\right| \wedge_{h\Sigma_{p^k}} S^{qp^k}). \tag{5.7}$$

We compute, using Lemma 5.5 and Lemma 3.8

$$\begin{aligned}E_{q+*}(\mathcal{P}(p^k)_s \wedge_{h\Sigma_{p^k}} S^{qp^k}) &\cong E_* \otimes_{E_0} B_\bullet(\mathrm{Id}, \bar{\mathbb{T}}, \overline{E_*S^q})\langle p^k\rangle_q\\ &\xrightarrow[\simeq]{\mathcal{L}} E_* \otimes_{E_0} B_\bullet(\bar{E}_0, \Delta^{-q}, \bar{E}_0)[k].\end{aligned}$$

By Theorem 2.13, the spectral sequence (5.7) collapses to give the desired result. □

Theorem 5.8. *For q odd, there are canonical isomorphisms of E_*-modules*

$$E_0 L(k)_q \cong C[k]^\vee_{-q}$$

and

$$E^0 L(k)_q \cong C[k]_{-q}.$$

Proof. By (5.3) and (5.2) there are equivalences

$$\begin{aligned}L(k)_q &\simeq \Sigma^{k-q}\partial_{p^k}(\mathrm{Id}) \wedge_{h\Sigma_{p^k}} S^{qp^k}\\ &\simeq \Sigma^{k-q}\left|\mathcal{P}(p^k)_\bullet\right|^\vee \wedge_{h\Sigma_{p^k}} S^{qp^k}.\end{aligned}$$

Since $\mathcal{P}(p^k)_\bullet$ is a finite complex, the results of [Kuh04b] imply that there are K-local equivalences

$$\begin{aligned}[\Sigma^{k-q}\left|\mathcal{P}(p^k)_\bullet\right|^\vee \wedge S^{qp^k}]_{h\Sigma_{p^k}} &\xrightarrow[\text{norm}]{\simeq} [\Sigma^{k-q}\left|\mathcal{P}(p^k)_\bullet\right|^\vee \wedge S^{qp^k}]^{h\Sigma_{p^k}}\\ &\simeq [(\Sigma^{-k+q}\left|\mathcal{P}(p^k)_\bullet\right| \wedge S^{-qp^k})^\vee]^{h\Sigma_{p^k}}\\ &\simeq [(\Sigma^{-k+q}\left|\mathcal{P}(p^k)_\bullet\right| \wedge S^{-qp^k})_{h\Sigma_{p^k}}]^\vee.\end{aligned}$$

Now apply the universal coefficient theorem, using the fact that $C[k]_{-q}$ is free as a module over E_0, to deduce the result from Lemma 5.6. ☐

Remark 5.9. Arone and Dwyer actually give another identification of the spectrum $L(k)_q$, dual to (5.3); they also prove that there is an equivalence

$$L(k)_q \simeq \Sigma^{-k-q}[|\mathcal{P}(p^k)_\bullet| \wedge S^{qp^k}]_{h\Sigma_{p^k}}.$$

Thus Lemma 5.6 gives the following alternative to Theorem 5.8: for q odd we have

$$E_0 L(k)_q \cong C[k]_q.$$

This description of the E-homology of $L(k)_q$ is less well suited to the perspective of the present paper.

6. The Bousfield-Kuhn functor and the comparison map

The Bousfield-Kuhn functor. Let T denote any v_h-telescope on a type h finite complex. The Bousfield-Kuhn functor $\Phi^f = \Phi^f_h$ factors localization with respect to T. We are mainly interested in the composition Φ of Φ' with K-localization. Thus we have a diagram of functors commuting up to natural weak equivalence.

$$\begin{tikzcd}
\mathrm{Sp} \arrow[r, "(-)_T"] \arrow[rr, bend left, "(-)_K"] \arrow[d, "\Omega^\infty"'] & \mathrm{Sp} \arrow[r, "(-)_K"] & \mathrm{Sp} \\
\mathrm{Top}_* \arrow[ru, "\Phi^f"] \arrow[rru, "\Phi"'] & &
\end{tikzcd}$$

The unstable v_n-periodic homotopy groups of X are the homotopy groups of $\Phi^f(X)$:

$$v_n^{-1}\pi_*(X) \cong \pi_*\Phi^f(X).$$

When the telescope conjecture holds for all heights $\leq h$ (e.g., if $h = 1$), then $\Phi^f \approx \Phi$.

See [Kuh08] for a detailed summary of the construction and properties of these functors. The main additional property we will need is that Φ commutes with finite homotopy limits, and thus in particular $\Phi\Omega \to \Omega\Phi$ is a natural weak equivalence.

Applying Φ to the unit of the adjunction

$$X \to \Omega^\infty\Sigma^\infty X$$

we get a natural transformation

$$\eta_X : \Phi(X) \to (\Sigma^\infty X)_K.$$

The comparison map. Let R be a commutative S-algebra, and consider the functor

$$R^{(-)_+} : \mathrm{Top}_*^{op} \to \mathrm{Alg}_R \downarrow R.$$

Here, the R-algebra structure on R^{X_+} comes from the diagonal on X, with unit given by the map $X \to *$, and augmentation coming from the basepoint on X.

The augmentation ideal $I(R^{X_+})$ is identified with R^X, the R-module of maps from $\Sigma^\infty X$ to R. As the functor $\mathrm{Top}_*^{op} \to \mathrm{Alg}_R^{nu}$ given by $X \mapsto R^X$ is a pointed homotopy functor, there is are natural transformations

$$S^n \otimes R^X \to R^{\Omega^n X}.$$

Assume that R is K-local. We define a natural transformation

$$c_R \colon TAQ^R(R^{X_+}) \to R^{\Phi(X)}$$

of functors $\mathrm{Top}_*^{op} \to \mathrm{Mod}_R$ as folows.

$$\begin{aligned} c_R : \mathrm{TAQ}^R(R^{X_+}) &\simeq \operatorname*{hocolim}_n \Omega^n(S^n \otimes R^X) \\ &\to \operatorname*{hocolim}_n \Omega^n R^{\Omega^n X} \\ &\simeq \operatorname*{hocolim}_n \Omega^n R^{(\Sigma^\infty \Omega^n X)_K} \\ &\xrightarrow{\eta^*_{\Omega^n X}} \operatorname*{hocolim}_n \Omega^n R^{\Phi(\Omega^n X)} \\ &\simeq \operatorname*{hocolim}_n \Omega^n R^{\Sigma^{-n}\Phi(X)} \\ &\simeq R^{\Phi(X)}. \end{aligned}$$

Taking R-linear duals of c_R and composing with the evident map $R \wedge \Phi(X) \to \mathrm{Hom}_R(R^{\Phi(X)}, R)$ gives rise to a map

$$c^R : (R \wedge \Phi(X))_K \to \mathrm{TAQ}_R(R^{X_+}).$$

We shall refer to c_R and c^R as the *comparison maps.*

The comparison map on infinite loop spaces. Let Y be a spectrum. The counit of the adjunction

$$\epsilon : \Sigma^\infty \Omega^\infty \to \mathrm{Id}$$

induces a natural transformation

$$\epsilon^* : S_K^Y \to S_K^{\Omega^\infty Y}.$$

Regarding $S_K^{\Omega^\infty Y}$ as a non-unital commutative S_K-algebra, this induces a map of augmented commutative S_K-algebras

$$\widetilde{\epsilon}^* : \mathbb{P}_{S_K} S_K^Y \to S_K^{\Omega^\infty Y_+}.$$

The following property of $c_{S_K} \colon TAQ^{S_K}(S_K^{\Omega^\infty Y_+}) \to S_K^{\Phi(\Omega^\infty Y)}$ will be all that we need to know about it.

Lemma 6.1. *The composite*

$$S_K^Y \simeq \mathrm{TAQ}^{S_K}(\mathbb{P}_{S_K} S_K^Y) \xrightarrow{TAQ^{S_K}(\widetilde{\epsilon}^*)} \mathrm{TAQ}^{S_K}(S_K^{\Omega^\infty Y_+}) \xrightarrow{c_{S_K}} S_K^{\Phi(\Omega^\infty Y)} \simeq S_K^Y$$

is the identity.

Proof. The lemma is proved by the commutativity of the following diagram

$$\begin{tikzcd}
S^Y \arrow[r, "\simeq"] \arrow[d, hook] \arrow[ddddr, bend right, "\epsilon^*"'] & \mathrm{TAQ}^{S_K}(\mathbb{P}_{S_K} S_K^Y) \arrow[r, "\widetilde{\epsilon}^*"] \arrow[d, equal] & \mathrm{TAQ}^{S_K}(S_K^{\Omega^\infty Y_+}) \arrow[d, equal] \arrow[dddd, bend left, "c_{S_K}"] \\
\mathbb{P}_{S_K} S_K^Y \arrow[r] \arrow[dr] \arrow[dddr, "\widetilde{\epsilon}^*"'] & \varinjlim \Omega^n(S^n \otimes \widetilde{\mathbb{P}}_{S_K} S_K^Y) \arrow[r, "\widetilde{\epsilon}^*"'] \arrow[d] & \varinjlim \Omega^n(S^n \otimes S_K^{\Omega^\infty Y}) \arrow[d] \\
 & \varinjlim \Omega^n(\widetilde{\mathbb{P}}_{S_K} S_K^{\Sigma^{-n}Y}) \arrow[r, "\widetilde{\epsilon}^*"'] & \varinjlim \Omega^n(S_K^{\Omega^\infty \Sigma^{-n} Y}) \arrow[d, "\eta^*_{\Omega^\infty \Sigma^{-n} Y}"] \\
 & & \varinjlim \Omega^n(S_K^{\Sigma^{-n}Y}) \\
 & S_K^{\Omega^\infty Y} \arrow[r, "\eta^*_{\Omega^\infty Y}"'] \arrow[uur] & S_K^Y \arrow[u, "\simeq"']
\end{tikzcd}$$

together with the fact that $\eta^*_{\Omega^\infty Y} \circ \epsilon^* \simeq \mathrm{Id}$ [Kuh08, Sec. 7]. □

The comparison map on QX. The previous lemma allows us to deduce the following.

Proposition 6.2. *Suppose that $X \in \mathrm{Top}_*$ is connected and has finite free E-homology. Then the comparison map for QX:*

$$c_{S_K} : \mathrm{TAQ}^{S_K}(S_K^{QX_+}) \to S_K^{\Phi QX} \simeq S_K^X$$

is an equivalence.

Proof. We will argue that $\mathrm{TAQ}^{S_K}(S_K^{QX_+})$ has finite K-homology, of rank equal to the rank of the K-homology of S_K^X. The proposition then follows from Lemma 6.1. Observe that the Kahn splitting and our finiteness hypotheses gives rise to a sequence of equivalences

$$S_K^{QX_+} \simeq S_K^{\mathbb{P}(\Sigma^\infty X)} \simeq \prod_i [(S_K^X)^{\wedge_{S_K} i}]^{h\Sigma_i}.$$

The K-local norm equivalences give equivalences

$$\prod_i [(S_K^X)^{\wedge_{S_K} i}]^{h\Sigma_i} \simeq \prod [(S_K^X)^{\wedge_{S_K} i}_{h\Sigma_i}]_K =: \widehat{\mathbb{P}}_{S_K}(S_K^X).$$

In fact, the equivalence

$$S_K^{QX_+} \simeq \widehat{\mathbb{P}}_{S_K}(S_K^X)$$

is an equivalence of H_∞-ring spectra, where we give $\widehat{\mathbb{P}}_{S_K}(S_K^X)$ the H_∞-ring structure arising from the inverse limit of the localized truncated free algebras:

$$\widehat{\mathbb{P}}_{S_K}(S_K^X) \simeq \operatorname*{holim}_n \bigvee_{i \leq n} [(S_K^X)^{\wedge i}_{h\Sigma_i}]_K.$$

As the argument is somewhat technical, we defer the proof to Appendix A.

Associated to the simplicial presentation

$$\mathrm{TAQ}^{S_K}(S_K^{QX_+}) \simeq \left| B_\bullet(\mathrm{Id}, \widetilde{\mathbb{P}}_{S_K}, S_K^{QX}) \right|$$

is a Bousfield-Kan spectral sequence which takes the form

$$E^1_{s,t} = E_t B_s(\mathrm{Id}, \widetilde{\mathbb{P}}_{S_K}, S_K^{QX}) \Rightarrow E_{s+t}\,\mathrm{TAQ}^{S_K}(S_K^{QX_+}). \tag{6.3}$$

As our E-homology is implicitly completed E-homology, this spectral sequence only converges under very special circumstances (for instance, if it has finitely many lines on the E^r-page for some r). The E_1-term may be identified using (2.4):

$$E^1_{s,*} \cong B_s(\mathrm{Id}, \bar{\mathbb{T}}, E_* S_K^{QX})^\wedge_{\mathfrak{m}}.$$

To compute the E^2-page, we note that the homology of the uncompleted bar complex is given by:

$$H_s(B_*(\mathrm{Id}, \bar{\mathbb{T}}, E_* S_K^{QX})) = \mathbb{L}_s \Omega^*_{\mathbb{T}/E_*} E_* S_K^{QX_+}.$$

Since $S_K^{QX_+}$ is equivalent to $\widehat{\mathbb{P}}_{S_K}(S_K^X)$ as an H_∞-algebra, there is an isomorphism of augmented $\mathbb{T}$-algebras

$$E_* S_K^{QX_+} \cong E_* \widehat{\mathbb{P}}_{S_K}(S_K^X) \cong \widehat{\mathrm{Sym}}_{E_*}(\Delta^* \otimes_{E_*} \widetilde{E}^* X)$$

(the last isomorphism follows from Lemma 2.7 and the definition of Δ^*). Here $\widehat{\mathrm{Sym}}_{E_*}$ denotes the (graded commutative) power series algebra over E_*. The Grothendieck spectral sequence of Proposition 3.4

$$E^2_{s,t} = \mathrm{Tor}_s^{\Delta^*}(\bar{E}_0, \mathbb{L}_t V^*(\widehat{\mathrm{Sym}}_{E_*}(\Delta^* \otimes_{E_*} E^* X))) \Rightarrow \mathbb{L}_{s+t} \Omega^*_{\mathbb{T}/E_*} E_* S_K^{QX_+}$$

collapses to give

$$\mathbb{L}_s \Omega^*_{\mathbb{T}/E_*} E_* S_K^{QX_+} \cong \begin{cases} \widetilde{E}^* X, & s = 0, \\ 0, & s > 0. \end{cases}$$

In particular, this implies that $H_s(B_*(\mathrm{Id}, \bar{\mathbb{T}}, E_* S_K^{QX}))$ is free over E_*. Therefore, the higher derived functors of $\mathfrak{m}$-adic completion vanish on these homology groups [Rez09, Prop. 3.2], and it follows that we have

$$\begin{aligned} E^2_{s,*} &= H^s(B_*(\mathrm{Id}, \bar{\mathbb{T}}, E_* S_K^{QX})^\wedge_{\mathfrak{m}}) \\ &\cong H^s(B_*(\mathrm{Id}, \bar{\mathbb{T}}, E_* S_K^{QX}))^\wedge_{\mathfrak{m}} \\ &\cong \begin{cases} \widetilde{E}^* X, & s = 0, \\ 0, & s > 0. \end{cases} \end{aligned}$$

We conclude that spectral sequence (6.3) converges and collapses to give an isomorphism

$$E_*\,\mathrm{TAQ}^{S_K}(S_K^{QX_+}) \cong \widetilde{E}^* X.$$

In particular, there is an isomorphism

$$K_*\,\mathrm{TAQ}^{S_K}(S_K^{QX_+}) \cong K_* S_K^X.$$

□

7. Weiss towers

In this section we freely use the language of Weiss's orthogonal calculus [Wei95].

Definition 7.1. Let F be a reduced homotopy functor from complex vector spaces to K-local spectra. We shall say that a tower

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_1$$

of functors under F is a *finite K-local Weiss tower* if

(1) the fiber of $F_n \to F_{n-1}$ is equivalent to the K-localization of a homogeneous degree n functor from complex vector spaces to spectra, and
(2) The map $F \to F_n$ is an equivalence for $n \gg 0$.

Remark 7.2. Suppose that $\{F_n\}$ is a finite K-local Weiss tower for F. We record the following observations.

(1) The functor F_n are n-excisive. This is because the localization of a homogeneous degree n functor is n-excisive.
(2) If $\{G_n\}$ is a finite K-local Weiss tower for G, and $F \to G$ is a natural transformation, there is a homotopically unique induced compatible system of natural transformations

   $$F_n \to G_n.$$

   This is because if D_n is a homogeneous degree n functor which is K-locally equivalent to the fiber $F_n \to F_{n-1}$, the space of natural transformations

   $$\mathbf{Nat}((D_n)_K, G_m) \simeq \mathbf{Nat}(D_n, G_m)$$

   is contractible for $m < n$. It follows that the natural map

   $$\mathbf{Nat}(F_m, G_m) \xrightarrow{\simeq} \mathbf{Nat}(F, G_m)$$

   is an equivalence.
(3) It follows from (2) that if F admits a finite K-local Goodwillie tower, it is homotopically unique.

We will construct finite K-local Goodwillie towers of the following functors from complex vector spaces to spectra:

$$V \mapsto \Phi(\Sigma S^V)$$
$$V \mapsto \mathrm{TAQ}_{S_K}(S_K^{(\Sigma S^V)_+}).$$

In each of these cases, the towers will only have non-trivial layers in degrees p^k for $k \le h$.

Proposition 7.3. *The tower $\{\Phi(P_n(\mathrm{Id})(\Sigma S^V))\}_n$ is a finite K-local Weiss tower for $\Phi(\Sigma S^V)$.*

Proof. The fibers of the tower $\{\Phi(P_n(\Sigma S^V))\}_n$ are given by

$$\mathbb{D}_n(\mathrm{Id})(\Sigma S^V)_K \to \Phi(P_n(\mathrm{Id})(\Sigma S^V)) \to \Phi(P_{n-1}(\mathrm{Id})(\Sigma S^V)).$$

By [Kuh08, Thm. 8.9], the map

$$\psi(V) \to \Phi(P_{p^n}(\mathrm{Id})(\Sigma S^V))$$

is an equivalence. $\square$

Proposition 7.4. *The tower* $\{F_n \, \mathrm{TAQ}_{S_K}(S_K^{(\Sigma S^V)_+})\}$ *obtained by taking the* K*-local Spanier-Whitehead dual of the Kuhn filtration* $\{F_n \, \mathrm{TAQ}^S(S^{(\Sigma S^V)_+})\}$ *is a finite* K*-local Weiss tower.*

Proof. By 4.6, the fibers of the tower are given by

$$\begin{aligned} F(\partial_n(\mathrm{Id})^\vee \wedge_{h\Sigma_n} (S^{\Sigma S^V})^{\wedge n}, S_K) &\simeq F(\partial_n(\mathrm{Id})^\vee \wedge (S^{\Sigma S^V})^{\wedge n}, S_K)^{h\Sigma_n} \\ &\simeq (F(\partial_n(\mathrm{Id})^\vee \wedge (S^{\Sigma S^V})^{\wedge n}, S_K)^{h\Sigma_n})_K \\ &\simeq (((\partial_n(\mathrm{Id}) \wedge S^n \wedge S^{nV})_K)^{h\Sigma_n})_K \\ &\simeq ((\partial_n(\mathrm{Id}) \wedge S^n \wedge S^{nV})_{h\Sigma_n})_K. \end{aligned}$$

Thus they are equivalent to K-localizations of homogeneous degree n functors. Since we have

$$\mathrm{TAQ}^S(S^{(\Sigma V)_+}) \simeq \operatorname*{hocolim}_n F_n \, \mathrm{TAQ}^S(S^{(\Sigma V)_+}),$$

we have

$$\mathrm{TAQ}_{S_K}(S_K^{(\Sigma S^V)_+}) \simeq \operatorname*{holim}_n F_n \, \mathrm{TAQ}_{S_K}(S_K^{(\Sigma S^V)_+}).$$

Since the layers are equivalent to $\mathbb{D}_n(\mathrm{Id})(\Sigma S^V)_K$, they are acyclic for $n > p^h$ [AM99]. $\square$

8. The comparison map on odd spheres

Fix q to be an odd positive integer. The main result of this section is the following theorem.

Theorem 8.1. *The comparison map*

$$c^{S_K} : \Phi(S^q) \to \mathrm{TAQ}_{S_K}(S_K^{S^q_+})$$

is an equivalence.

We shall begin with its dual, and establish the following weaker statement.

Lemma 8.2. *The natural transformation*

$$c_{S_K} : \mathrm{TAQ}^{S_K}(S_K^{(\Sigma S^V)_+}) \to S_K^{\Phi(\Sigma S^V)}$$

of functors from complex vector spaces to K*-local spectra has a weak section.*

Proof. Using work of Arone-Mahowald, Kuhn shows that the map

$$\Phi(X) \to \Phi(P_{p^h}(\mathrm{Id})(X))$$

is an equivalence [Kuh08, Thm. 8.9]. Let $X = \Sigma S^V$, and let

$$X \to Q^{\bullet+1} X$$

denote the Bousfield-Kan cosimplicial resolution. Consider the diagram:

(8.3)
$$\begin{tikzcd}
\mathrm{TAQ}^{S_K}(S_K^{X_+}) \arrow[r, "c_{S_K}"] & S_K^{\Phi(X)} \arrow[d, equal] \\
 & S_K^{\Phi P_{p^h}(\mathrm{Id})(X)} \\
\mathrm{TAQ}^{S_K}(S_K^{P_{p^h}(Q^{\bullet+1})(X)_+}) \arrow[uu] \arrow[r, "c_{S_K}"] & S_K^{\Phi P_{p^h}(Q^{\bullet+1})X} \arrow[u]
\end{tikzcd}$$

It is well known (see [AK98]) that there is an equivalence of cosimplicial Σ_n-spectra:

$$\partial_n(Q^{\bullet+1}) \simeq \Sigma^\infty \mathcal{P}(n)^\vee_\bullet$$

so that the induced map

$$\partial_n(\mathrm{Id}) \xrightarrow{\simeq} \mathrm{Tot}\, \partial_n(Q^{\bullet+1}) \simeq \mathrm{Tot}\, \Sigma^\infty \mathcal{P}(n)^\vee_\bullet \simeq \Sigma^\infty \left|\mathcal{P}(n)_\bullet\right|^\vee$$

is equivalence (5.2). For a fixed s, the iterated Snaith splitting implies that the Goodwillie tower for Q^{s+1} splits, giving an equivalence

$$P_{p^h}(Q^{\bullet+1})(X) \simeq \prod_{1 \le i \le p^h} Q(\mathcal{P}(i)_s \wedge_{h\Sigma_i} X^{\wedge i}).$$

In particular, the spaces above satisfy the hypotheses of Prop. 6.2, and the comparison map

$$\mathrm{TAQ}^{S_K}(S_K^{P_{p^h}(Q^{\bullet+1})(X)_+}) \xrightarrow{c_{S_K}} S_K^{\Phi P_{p^h}(Q^{\bullet+1})X}$$

is a levelwise equivalence of simplicial spectra. It follows from Diagram (8.3) that the natural map

$$\left| S_K^{\Phi P_{p^h}(Q^{\bullet+1})(X)} \right|_K \to S_K^{\Phi P_{p^h}(\mathrm{Id})(X)} \simeq S_K^{\Phi(X)}$$

factors through c_{S_K}:

$$\left| S_K^{\Phi(P_{p^h}(Q^{\bullet+1})(X))} \right|_K \to \mathrm{TAQ}^{S_K}(S_K^{X_+}) \xrightarrow{c_{S_K}} S_K^{\Phi(X)} \simeq S_K^{\Phi(P_{p^h}(\mathrm{Id})(X))}.$$

The lemma will be proven if we can show that the natural map

$$\left| S_K^{\Phi(P_{p^h}(Q^{\bullet+1})(X))} \right|_K \to S_K^{\Phi(P_{p^h}(\mathrm{Id})(X))}$$

is an equivalence. To do this, we will prove that for all n the composite

$$\left| S_K^{\Phi(P_n(Q^{\bullet+1})(X))} \right|_K \to S_K^{\Phi(P_n(\mathrm{Id})(X))}$$

is an equivalence, by induction on n. The map of fiber sequences

$$\begin{tikzcd}
D_n(\mathrm{Id})(X) \arrow[r] \arrow[d] & P_n(\mathrm{Id})(X) \arrow[r] \arrow[d] & P_{n-1}(\mathrm{Id})(X) \arrow[d] \\
D_n(Q^{\bullet+1})(X) \arrow[r] & P_n(Q^{\bullet+1})(X) \arrow[r] & P_{n-1}(Q^{\bullet+1})(X)
\end{tikzcd}$$

gives a map of fiber sequences

$$\begin{tikzcd}
\left|S_K^{\Phi(P_{n-1}(Q^{\bullet+1})(X))}\right|_K \arrow[r] \arrow[d] & \left|S_K^{\Phi(P_n(Q^{\bullet+1})(X))}\right|_K \arrow[r] \arrow[d] & \left|S_K^{\Phi(D_n(Q^{\bullet+1})(X))}\right|_K \arrow[d] \\
S_K^{\Phi(P_{n-1}(\mathrm{Id})(X))} \arrow[r] & S_K^{\Phi(P_n(\mathrm{Id})(X))} \arrow[r] & S_K^{\Phi(D_n(\mathrm{Id})(X))}
\end{tikzcd}$$

The induction on n therefore rests on proving that the natural map

$$\left|S_K^{\mathbb{D}_n(Q^{\bullet+1})(X)}\right|_K \simeq \left|S_K^{\Phi(D_n(Q^{\bullet+1})(X))}\right|_K \to S_K^{\Phi(D_n(\mathrm{Id})(X))} \simeq S_K^{\mathbb{D}_n(\mathrm{Id})(X)}$$

is an equivalence.

Using the finiteness of X and $\mathcal{P}(n)_\bullet$, together with the vanishing of K-local Tate spectra [Kuh04b], we have the following diagram of equivalences

$$\begin{tikzcd}
\left|S_K^{\mathbb{D}_n(Q^{\bullet+1})(X)}\right|_K \arrow[r] & S_K^{\mathbb{D}_n(\mathrm{Id}))(X)} \\
\left|S_K^{\Sigma^\infty\mathcal{P}(n)_\bullet^\vee\wedge_{h\Sigma_n}X^{\wedge n}}\right|_K \arrow[r] \arrow[u, "\simeq"] & S_K^{\Sigma^\infty|\mathcal{P}(n)_\bullet|^\vee\wedge_{h\Sigma_n}X^{\wedge n}} \arrow[u, "\simeq"] \\
\left|\left(S_K^{\Sigma^\infty\mathcal{P}(n)_\bullet^\vee\wedge X^{\wedge n}}\right)^{h\Sigma_n}\right|_K \arrow[r] \arrow[u, "\simeq"] & \left(S_K^{\Sigma^\infty|\mathcal{P}(n)_\bullet|^\vee\wedge X^{\wedge n}}\right)^{h\Sigma_n} \arrow[u, "\simeq"] \\
\left|\left(S_K^{\Sigma^\infty\mathcal{P}(n)_\bullet^\vee\wedge X^{\wedge n}}\right)_{h\Sigma_n}\right|_K \arrow[r] \arrow[u, "\simeq"] & \left[\left(S_K^{\Sigma^\infty|\mathcal{P}(n)_\bullet|^\vee\wedge X^{\wedge n}}\right)_{h\Sigma_n}\right]_K \arrow[u, "\simeq"] \\
\left|\left(\Sigma^\infty\mathcal{P}(n)_\bullet\wedge S_K^{X^{\wedge n}}\right)_{h\Sigma_n}\right|_K \arrow[r] \arrow[u, "\simeq"] & \left[\left(\Sigma^\infty\left|\mathcal{P}(n)_\bullet\right|\wedge S_K^{X^{\wedge n}}\right)_{h\Sigma_n}\right]_K \arrow[u, "\simeq"]
\end{tikzcd}$$

The bottom arrow in this diagram is an equivalence, since realizations commute past homotopy colimits and smash products. Therefore the top arrow in the diagram is an equivalence, as desired. ☐

The final ingredient we will need to prove Theorem 8.1 will be a result which will allow us to dualize Lemma 8.2.

Proposition 8.4. *The spectrum $\Phi(S^q)$ is K-locally dualizible.*

Proof. It suffices to show that its completed Morava E-homology is finitely generated [HS99]. Since $\Phi(S^q) \simeq \Phi(P_{p^h}(\mathrm{Id})(S^q))$ [Kuh08, Sec. 7], one can prove this by proving $\Phi(P_{p^k}(\mathrm{Id})(S^q))$ has finitely generated completed Morava E-homology by induction on k. This is done using the fiber sequences

$$\mathbb{D}_{p^k}(\mathrm{Id})(S^q)_K \to \Phi(P_{p^k}(\mathrm{Id})(S^q)) \to \Phi(P_{p^{k-1}}(\mathrm{Id})(S^q))$$

together with our computation $E_0L(k)_q \cong C[k]^\vee_{-q}$. Note that $C[k]^\vee_{-q}$ is finitely generated by [Rezb, Prop. 4.6]. □

Proof of Theorem 8.1. We can take the K-local Spanier-Whitehead dual of the retraction

$$S_K^{\Phi(\Sigma S^V)} \to \mathrm{TAQ}^{S_K}(S_K^{(\Sigma S^V)_+}) \xrightarrow{c_{S_K}} S_K^{\Phi(\Sigma S^V)}$$

provided by Lemma 8.2 to obtain a retraction of functors from complex vector spaces to K-local spectra:

$$\Phi(\Sigma S^V) \xrightarrow{c^{S_K}} \mathrm{TAQ}_{S_K}(S_K^{(\Sigma S^V)_+}) \to \Phi(\Sigma S^V).$$

We therefore get a retraction of the K-local Weiss towers of these functors (see Propositions 7.3 and 7.4)

$$\{\Phi(P_n(\mathrm{Id})(\Sigma S^V))\}_n \xrightarrow{c^{S_K}} \{F_n \, \mathrm{TAQ}_{S_K}(S_K^{(\Sigma S^V)_+})\}_n \to \{\Phi(P_n(\mathrm{Id})(\Sigma S^V))\}_n.$$

However, the layers of both of these towers are equivalent to the spectra $\mathbb{D}_n(\mathrm{Id})(\Sigma S^V)_K$. Since the Morava K-theory of these layers is finite, it follows that the map c^{S_K} induces an equivalence on the layers of the K-local Weiss towers. Since the K-local Weiss towers are themselves finite, we deduce that the natural transformation

$$\Phi(\Sigma S^V) \xrightarrow{c^{S_K}} \mathrm{TAQ}_{S_K}(S_K^{(\Sigma S^V)_+})$$

is an equivalence by inducting up the towers. □

Actually, the method of proof gives the following corollary, which allows us to compare Φ applied to the Goodwillie tower of the identity with the much easier to understand Kuhn tower.

Corollary 8.5. *The comparison map induces an equivalence of towers*

$$\{\Phi(P_n(\mathrm{Id})(S^q))\} \xrightarrow[\simeq]{c^{S_K}} \{F_n \, \mathrm{TAQ}_{S_K}(S_K^{S^q_+})\}.$$

9. The Morava E-homology of the Goodwillie attaching maps

Fix q to be an odd positive integer. Let α_k denote the attaching map connecting the p^k and p^{k+1}-layers of the Goodwillie tower for S^q.

$$\begin{tikzcd}
\alpha_k : D_{p^k}(\mathrm{Id})(S^q) \arrow[r] \arrow[d, equal] & BD_{p^{k+1}}(\mathrm{Id})(S^q) \arrow[d, equal] \\
\Sigma^{q-k}L(k)_q & \Sigma^{q-k}L(k)_q
\end{tikzcd}$$

Applying Φ and desuspending, we get a map

$$\Phi(\alpha_k) : (L(k)_q)_K \to (L(k+1)_q)_K$$

which should be regarded as the corresponding attaching map between consecutive non-trivial layers in the v_h-periodic Goodwillie tower of the identity.

Note that since $E^{S^q_+}$ is a commutative E-algebra, the reduced cohomology group

$$\widetilde{E}^q(S^q) = V^q\pi_*E^{S^q_+}$$

is a Δ^q-module. Under the isomorphisms

$$E_0 L(k)_q \cong \left(C[k]_{-q} \otimes_{E_0} \widetilde{E}^q(S^q)\right)^\vee$$

obtained by tensoring the isomorphism of Theorem 5.8 with the fundamental class in $\widetilde{E}^q(S^q)$, there is an induced map

$$E_0\Phi(\alpha_k) : \left(C[k]_{-q} \otimes_{E_0} \widetilde{E}^q(S^q)\right)^\vee \to \left(C[k+1]_{-q} \otimes_{E_0} \widetilde{E}^q(S^q)\right)^\vee .$$

We have the following more refined version of Theorem 5.8.

Theorem 9.1. *There is an isomorphism of cochain complexes*

$$(E_0 L(k)_q, E_0\Phi(\alpha_k)) \cong (C_k^{\Delta^q}(\widetilde{E}^q(S^q))^\vee, \delta_k^\vee)$$

where $C_k^{\Delta^q}(\widetilde{E}^q(S^q))$ *is the Koszul complex for the* Δ^q*-module* $\widetilde{E}^q(S^q)$.

Proof. By Corollary 8.5, it suffices to show that the E-homology of the attaching maps in the Kuhn tower

$$\alpha'_k : (L(k)_q)_K \simeq \left(\frac{F_{p^k}\,\mathrm{TAQ}^{S_K}(S_K^{S^q_+})}{F_{p^{k-1}}\,\mathrm{TAQ}^{S_K}(S_K^{S^q_+})} \right)^\vee \to \left(\frac{F_{p^{k+1}}\,\mathrm{TAQ}^{S_K}(S_K^{S^q_+})}{F_{p^k}\,\mathrm{TAQ}^{S_K}(S_K^{S^q_+})} \right)^\vee \simeq (L(k{+}1)_q)_K$$

have the desired description (here the $(-)^\vee$ notation above denotes the K*-local* Spanier-Whitehead dual). The result is obtained by dualizing the following diagram

$$\begin{tikzcd}
C[k+1]_{-q} \otimes_{E_0} \widetilde{E}^q(S^q) \arrow[r, "\delta_k"] \arrow[d, hook] & C[k]_{-q} \otimes_{E_0} \widetilde{E}^q(S^q) \arrow[d, hook] \\
B_{k+1}(\bar{E}_0, \Delta^q, \widetilde{E}^q(S^q))[k+1] \arrow[r, "d_{k+1}"] & B_k(\bar{E}_0, \Delta^q, \widetilde{E}^q(S^q))[k] \\
B_{k+1}(\mathrm{Id}, \bar{\mathbb{T}}, \widetilde{E}^*(S^q))\langle p^{k+1}\rangle_{-q} \arrow[r, "d_{k+1}\langle p^k\rangle"] \arrow[u, "\mathcal{L}"] \arrow[d, equal] & B_k(\mathrm{Id}, \bar{\mathbb{T}}, \widetilde{E}^q(S^q))\langle p^k\rangle_{-q} \arrow[u, "\mathcal{L}"'] \arrow[d, equal] \\
E_{-q}B_{k+1}(\mathrm{Id}, \widetilde{\mathbb{P}}, I(S^{S^q_+}))\langle p^{k+1}\rangle \arrow[r, "d_{k+1}\langle p^k\rangle"] & E_{-q}B_k(\mathrm{Id}, \widetilde{\mathbb{P}}, I(S^{S^q_+}))\langle p^k\rangle
\end{tikzcd}$$

which identifies the E-homology of the attaching map

$$\frac{F_{p^{k+1}}\,\mathrm{TAQ}^{S_K}(S_K^{S^q_+})}{F_{p^k}\,\mathrm{TAQ}^{S_K}(S_K^{S^q_+})} \to \frac{F_{p^k}\,\mathrm{TAQ}^{S_K}(S_K^{S^q_+})}{F_{p^{k-1}}\,\mathrm{TAQ}^{S_K}(S_K^{S^q_+})}.$$

In this diagram, the maps d_{k+1} are the last face maps in the corresponding bar complexes, and the maps $d_{k+1}\langle p^k\rangle$ are the projections of the face maps on to the $\langle p^k\rangle$-summands. □

Corollary 9.2. *The spectral sequence obtained by applying* E_* *to the tower* $\{\Phi(P_n(\mathrm{Id})(S^q))\}$ *takes the form*

$$\mathrm{Ext}^s_{\Delta^q}(\widetilde{E}^q(S^q), \bar{E}_t) \Rightarrow E_{q+t-s}\Phi(S^q).$$

10. A MODULAR DESCRIPTION OF THE KOSZUL COMPLEX

Reduction to the case of $q = 1$. In this section we give a modular interpretation of the Koszul complex $C_*^{\Delta^q}(\widetilde{E}^q(S^q))$ in the case of $q = 1$. Since the suspension gives inclusions of bar complexes (see (2.8))

$$B(\bar{E}_0, \Delta^q, \widetilde{E}^q(S^q)) \hookrightarrow B(\bar{E}_0, \Delta^1, \widetilde{E}^1(S^1))$$

we deduce that we have an induced map of Koszul complexes

$$\begin{tikzcd}
C[k]_{-q} \arrow[r, hook] \arrow[d, dotted, "\sigma^{q-1}"'] & \Delta^q[1]^{\otimes k} \arrow[d, hook] \\
C[k]_{-1} \arrow[r, hook] & \Delta^1[1]^{\otimes k}
\end{tikzcd}$$

Furthermore, the map σ^{q-1} above must be an inclusion. We deduce that there is an inclusion of Koszul complexes

$$\sigma^{q-1} : C_*^{\Delta^q}(\widetilde{E}^q(S^q)) \hookrightarrow C_*^{\Delta^1}(\widetilde{E}^1(S^1)).$$

It follows that the modular description of the Koszul complex we shall give for $q = 1$ will extend to a modular description for arbitrary odd q provided we have a good understanding of the inclusions of lattices

$$\Delta^q[1] \subseteq \Delta^1[1],$$
$$\Delta^q[2] \subseteq \Delta^1[2].$$

This amounts to having a concrete understanding of the second author's "Wilkerson Criterion" [Rez09].

The modular isogeny complex. We review the definition of the modular isogeny complex $\mathcal{K}_{p^k}^*$ of [Reza] associated to the formal group $\mathbb{G}$.

For $(k_1, \dots, k_s)$ a sequence of positive integers, let

$$\mathrm{Sub}_{p^{k_1}, \dots, p^{k_s}}(\mathbb{G}) = \mathrm{Spf}(\mathcal{S}_{p^{k_1}, \dots, p^{k_s}})$$

be the (affine) formal scheme whose R-points are given by

$$\mathrm{Sub}_{p^{k_1}, \dots, p^{k_s}}(\mathbb{G})(R) = \{H_1 < \cdots < H_s < \mathbb{G} \times_{\mathrm{Spf}(E_0)} \mathrm{Spf}(R) \; : \; |H_i / H_{i-1}| = p^{k_i}\}.$$

Lemma 10.1. *There is a canonical isomorphism of E_0-algebras*

$$\mathcal{S}_{p^{k_1}, \dots, p^{k_s}} \cong \mathcal{S}_{p^{k_1}} \otimes_{E_0} \cdots \otimes_{E_0} \mathcal{S}_{p^{k_s}}.$$

Proof. An R point of $\mathrm{Spf}(\mathcal{S}_{p^{k_1}, \dots, p^{k_s}})$ corresponds to a chain of finite subgroups

$$(H_1 < \cdots < H_s)$$

in $\mathbb{G}_1 := \mathbb{G} \times_{\mathrm{Spf}(E_0)} \mathrm{Spf}(R)$ with $|H_i / H_{i-1}| = p^i$. Define $\mathbb{G}_i := \mathbb{G}_1 / H_{i-1}$. Then, defining, $\widetilde{H}_i := H_i / H_{i-1}$, we get a collection of R-points

$$(\mathbb{G}_i, \widetilde{H}_i) \in \mathrm{Spf}(\mathcal{S}_{p^i})(R)$$

and isomorphisms $\mathbb{G}_i / \widetilde{H}_i \cong \mathbb{G}_{i+1}$. This is precisely the data of an R-point of

$$\mathrm{Spf}(\mathcal{S}_{p^{k_1}} \otimes_{E_0} \cdots \otimes_{E_0} \mathcal{S}_{p^{k_s}}).$$

Conversely, given such a sequence $(\mathbb{G}_i, \widetilde{H}_i)$ with isomorphisms $\mathbb{G}_i/\widetilde{H}_i \cong \mathbb{G}_{i+1}$, there is an associated chain of subgroups $(H_1, \ldots, H_s)$ of $\mathbb{G}_1$ obtained by pulling back the subgroup $\widetilde{H}_i$ over the isogeny:

$$\mathbb{G}_1 \to \mathbb{G}_1/\widetilde{H}_1 \cong \mathbb{G}_2 \to \mathbb{G}_2/\widetilde{H}_2 \cong \mathbb{G}_3 \to \cdots \to \mathbb{G}_{i-1}/\widetilde{H}_{i-1} \cong \mathbb{G}_i.$$

□

For $k > 0$ we define

$$\mathcal{K}^s_{p^k} = \begin{cases} \prod_{\substack{k_1+\cdots+k_s=k \\ k_i>0}} \mathcal{S}_{p^{k_1},\ldots,p^{k_s}}, & 1 \le s \le k, \\ 0, & \text{otherwise.} \end{cases}$$

We handle the case of $k = 0$ by defining

$$\mathcal{K}^s_1 = \begin{cases} E_0, & s = 0, \\ 0, & s > 0. \end{cases}$$

For $1 \le i \le s$ and a decomposition $k_i = k_i' + k_i''$ with $k_i', k_i'' > 0$, define maps

$$u_i : \mathcal{S}_{p^{k_1},\ldots,p^{k_s}} \to \mathcal{S}_{p^{k_1},\ldots,p^{k'_i},p^{k''_i},\ldots,p^{k_s}}$$

on R points by

$$u_i^* : (H_1 < \cdots < H_{s+1}) \mapsto (H_1 < \cdots < \widehat{H}_i < \cdots < H_{s+1}).$$

The maps u_i, under the isomorphism of Lemma 10.1, all arise from the maps

$$u_1 : \mathcal{S}_{p^{k'+k''}} \to \mathcal{S}_{p^{k'}} \otimes_{E_0} \mathcal{S}_{p^{k''}}.$$

In [Rez09], it is established that the maps u_1 above are dual to the algebra maps

$$\Gamma[k'] \otimes_{E_0} \Gamma[k''] \to \Gamma[k' + k''].$$

Taking a product over all possible such decompositions of $k_i = k_i' + k_i''$ gives a map

$$u_i : \mathcal{K}^s_{p^k} \to \mathcal{K}^{s+1}_{p^k}.$$

The differentials

$$\delta : \mathcal{K}^s_{p^k} \to \mathcal{K}^{s+1}_{p^k}, \quad 1 \le s < k$$

in the cochain complex $\mathcal{K}^*_{p^k}$ are given by

$$\delta(x) = \sum_{1 \le i \le s} (-1)^i u_i(x).$$

The cohomology of the modular isogeny complex. The key observation of this section is the following.

Proposition 10.2. *There is an isomorphism of cochain complexes*

$$B_*(\bar{E}_0, \widetilde{\Delta}^1, \bar{E}_0)[k]^\vee \cong \mathcal{K}^*_{p^k}.$$

It follows that we have

$$H^s(\mathcal{K}^*_{p^k}) \cong \begin{cases} C[k]^\vee_{-1}, & s = k, \\ 0, & s \ne k. \end{cases}$$

Proof. The suspension isomorphism (2.12)

$$\sigma : \Delta^1 \xrightarrow{\cong} \Gamma^0$$

induces an isomorphism of chain complexes

$$B_*(\bar{E}_0, \widetilde{\Delta}^1, \bar{E}_0)[k] \cong B_*(\bar{E}_0, \widetilde{\Gamma}^0, \bar{E}_0)[k].$$

The isomorphisms (2.9) together with those of Lemma 10.1 induce isomorphisms

$$\begin{aligned} B_s(\bar{E}_0, \widetilde{\Gamma}^0, \bar{E}_0)[k]^\vee &= \left(\bigoplus_{\substack{k_1+\cdots+k_s=k \\ k_i>0}} \widetilde{\Gamma}^0[k_1] \otimes_{E_0} \cdots \otimes_{E_0} \widetilde{\Gamma}^0[k_s] \right)^\vee \\ &\cong \bigoplus_{\substack{k_1+\cdots+k_s=k \\ k_i>0}} \widetilde{\Gamma}^0[k_1]^\vee \otimes_{E_0} \cdots \otimes_{E_0} \widetilde{\Gamma}^0[k_s]^\vee \\ &\cong \bigoplus_{\substack{k_1+\cdots+k_s=k \\ k_i>0}} \mathcal{S}_{p^{k_1}} \otimes_{E_0} \cdots \otimes_{E_0} \mathcal{S}_{p^{k_s}} \\ &\cong \prod_{\substack{k_1+\cdots+k_s=k \\ k_i>0}} \mathcal{S}_{p^{k_1},\ldots,p^{k_s}} \end{aligned}$$

since all of the E_0-modules involved are finite and free. Using the facts that $\Gamma^0[t]$ acts trivially on $\bar{E}_0$ for $t > 0$, and that the differential in the modular isogeny complex is an alternating sum of maps dual to the multiplication maps in Γ^0, our isomorphisms yield the desired isomorphism of cochain complexes

$$B_*(\bar{E}_0, \widetilde{\Delta}^1, \bar{E}_0)[k]^\vee \cong \mathcal{K}^*_{p^k}.$$

Again, appealing the the fact that these cochain complexes are free E_0-modules in each degree, and that the modules $C[k]_{-1}$ are free (see [Rezb, Prop. 4.6]), we have

$$\begin{aligned} H^s(\mathcal{K}^*_{p^k}) &\cong H^s(B_*(\bar{E}_0, \widetilde{\Delta}^1, \bar{E}_0)[k]^\vee) \\ &\cong H_s(B_*(\bar{E}_0, \widetilde{\Delta}^1, \bar{E}_0)[k])^\vee \\ &\cong \begin{cases} C[k]^\vee_{-1}, & s = k, \\ 0, & s \neq k. \end{cases} \end{aligned}$$

□

Modular description of the Koszul differentials. What remains is to give a modular description of the Koszul differentials

$$H^k(\mathcal{K}^*_{p^k}) \cong C^{\Delta^1}_k(\widetilde{E}^1(S^1))^\vee \xrightarrow{\delta_k^\vee} C^{\Delta^1}_{k+1}(\widetilde{E}^1(S^1))^\vee \cong H^{k+1}(\mathcal{K}^*_{p^{k+1}}).$$

Consider the map

$$u_{k+1} : \mathcal{S}_{\underbrace{p,\ldots,p}_{k}} \to \mathcal{S}_{\underbrace{p,\ldots,p}_{k+1}}$$

whose effect on R-points is given by

$$u^*_{k+1} : (H_1 < \cdots < H_{k+1} < \mathbb{G}) \mapsto (H_2/H_1 < \cdots < H_{k+1}/H_1 < \mathbb{G}/H_1).$$

Theorem 10.3. *The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{S}\underbrace{p, \dots, p}_{k} \arrow[r, "u_{k+1}"] \arrow[d, two heads] & \mathcal{S}\underbrace{p, \dots, p}_{k+1} \arrow[d, two heads] \\
C_k^{\Delta^1}(\widetilde{E}^1(S^1))^\vee \arrow[r, "\delta_k^\vee"'] & C_{k+1}^{\Delta^1}(\widetilde{E}^1(S^1))^\vee
\end{tikzcd}$$

Proof. Under the suspension isomorphism $\sigma : \Delta^1 \cong \Gamma^0$, the Δ^1-module $\widetilde{E}^1(S^1)$ is isomorphic to the Γ^0-module $\widetilde{E}^0(S^0) = E_0$. Moreover the action map

$$\Gamma^0[1] \cong \Gamma^0[1] \otimes_{E_0} E_0 \to E_0$$

is dual to the map t of (2.3)

$$t : E_0 \to \mathcal{S}_p$$

whose effect on R points is given by

$$t^* : (H < \mathbb{G}) \mapsto \mathbb{G}/H.$$

The result follows from the isomorphisms

$$\begin{aligned}
\mathcal{S}\underbrace{p, \dots, p}_{k} &\cong \underbrace{\mathcal{S}_p \otimes_{E_0} \cdots \otimes_{E_0} \mathcal{S}_p}_{k} \\
&\cong B_k(\bar{E}_0, \widetilde{\Gamma}^0, E_0)[k] \\
&\cong B_k(\bar{E}_0, \widetilde{\Delta}^0, \widetilde{E}^1(S^1))[k]
\end{aligned}$$

and (2.15). □

Appendix A. The H_∞ structure of $S_K^{QX_+}$

In this appendix we prove the following technical lemma needed in the proof of Proposition 6.2.

Lemma A.1. *Suppose that X is a connected pointed space whose suspension spectrum is K-locally strongly dualizible. Then the equivalence*

$$S_K^{QX_+} \simeq \widehat{\mathbb{P}}_{S_K}(S_K^X)$$

in the proof of Proposition 6.2 is an equivalence of H_∞-ring spectra.

The equivalence in Lemma A.1 is given by the sequence of equivalences

$$\widehat{\mathbb{P}}_{S_K}(S_K^X) = \prod_i [(S_K^X)^{\wedge_{S_K} i}_{h\Sigma_i}]_K \xrightarrow[\simeq]{\prod N_{\Sigma_i}} \prod_i [(S_K^X)^{\wedge_{S_K} i}]^{h\Sigma_i} \simeq S_K^{\mathbb{P}(\Sigma^\infty X)} \xleftarrow[\simeq]{s_X^*} S_K^{QX_+}.$$

Here $s_X : \mathbb{P}\Sigma^\infty X \simeq \Sigma^\infty QX_+$ is the Kahn splitting, and N_{Σ_i} is the norm.

Observe that since K-localization commutes with products when the factors involved are E-local (see [BD10, Cor. 6.1.3]) there is an equivalence

$$\prod_i [(S_K^X)^{\wedge_{S_K} i}_{h\Sigma_i}]_K \simeq \left[\prod_i (S_K^X)^{\wedge_{S_K} i}_{h\Sigma_i} \right]_K.$$

Therefore, Lemma A.1 follows from the following slightly more general observation.

Lemma A.2. *Let R be a commutative S-algebra, X a connected pointed space, and suppose that $R \wedge X$ is strongly dualizible in the category of R-modules. Then the composite*

$$\widehat{\mathbb{P}}_R(R^X) := \prod_i (R^X)^{\wedge_R i}_{h\Sigma_i} \xrightarrow{\prod N_{\Sigma_i}} \prod_i [(R^X)^{\wedge_R i}]^{h\Sigma_i} \simeq R^{\mathbb{P}(\Sigma^\infty X)} \underset{\simeq}{\xleftarrow{s_X^*}} R^{QX_+}$$

is a map of H_∞-R-algebras.

The rest of the appendix will be devoted to a proof of this lemma. Observe that the dualizible hypothesis on X implies that the natural map

$$(R^X)^{\wedge_R i} \to R^{(X^{\wedge i})}$$

is an equivalence. We therefore may simply use R^{X^i} to unambiguously refer to either of these equivalent spectra.

Norm and transfer maps. The proof of Lemma A.2 will necessitate a detailed understanding of norm and transfer maps in the stable homotopy category, which we briefly review. The first author learned of this particular perspective on norms from some lectures of Jacob Lurie.

For a finite group G, let Sp_G denote the category of G-spectra (G-equivariant objects in Sp, with weak equivalences given by those equivariant maps which are equivalences on underlying non-equivariant spectra), and $\mathrm{Ho}(\mathrm{Sp}_G)$ the corresponding homotopy category.

Given a homomorphism $f : H \to G$, the associated restriction functor

$$f^* : \mathrm{Ho}(\mathrm{Sp}_G) \to \mathrm{Ho}(\mathrm{Sp}_H)$$

has a left adjoint

$$f_! : \mathrm{Ho}(\mathrm{Sp}_H) \to \mathrm{Ho}(\mathrm{Sp}_G)$$

and a right adjoint

$$f_* : \mathrm{Ho}(\mathrm{Sp}_H) \to \mathrm{Ho}(\mathrm{Sp}_G).$$

In the case where $f : H \to G$ is the inclusion of a subgroup, these functors are given by induction and coinduction

$$f_! Y = \mathrm{Ind}_H^G Y = G_+ \wedge_H Y,$$
$$f_* Y = \mathrm{CoInd}_H^G Y = \mathrm{Map}_H(G, Y).$$

In this special case, since finite products are equivalent to finite wedges in Sp, the natural map

$$f_! Y = \mathrm{Ind}_H^G Y \underset{\cong}{\xrightarrow{\psi_f}} \mathrm{CoInd}_H^G Y = f_* Y$$

is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_G)$, and thus $f_!$ is also right adjoint to f^*.

If f is the unique map to the trivial group $f : G \to 1$, then these functors are given by homotopy orbits and homotopy fixed points

$$\begin{aligned} f_! Y &= Y_{hG}, \\ f_* Y &= Y^{hG}. \end{aligned}$$

In general, these functors are compatible with composition:

$$\begin{aligned} (fg)^* &= g^* f^*, \\ (fg)_! &= f_! g_!, \\ (fg)_* &= f_* g_*. \end{aligned}$$

For Y_1 and Y_2 in Sp_G, let $Y_1 \wedge Y_2 \in \mathrm{Sp}_G$ denote the smash product with diagonal G-action. For $f : H \to G$, $Y \in \mathrm{Sp}_H$, and $Z \in \mathrm{Sp}_G$, there is a projection formula

$$Y \wedge (f_! Z) \cong f_!((f^* Y) \wedge Z).$$

Finally, if

$$\begin{tikzcd} H \arrow[r, "f"] \arrow[d, "g"'] & G \arrow[d, "g'"] \\ H' \arrow[r, "f'"'] & G' \end{tikzcd}$$

is a pullback, then for $Y \in \mathrm{Sp}_G$ there is an isomorphism

$$g_! f^* Y \cong (f')^* (g')_! Y.$$

For example, if $f : G \to G/N$ is a quotient, then for $Y \in \mathrm{Sp}_G$, $f_! Y$ is a G/N-equivariant model for Y_{hN}. Indeed, this can be seen formally by considering the following diagram.

$$\begin{tikzcd} N \arrow[r, "k"] \arrow[d, "i"'] & 1 \arrow[d, "j"] \\ G \arrow[r, "f"] \arrow[dr, "h"'] & G/N \arrow[d, "g"] \\ & 1 \end{tikzcd}$$

Since the square in the above diagram is a pullback, we deduce

$$j^* f_! Y \cong k_! i^* Y = Y_{hN}.$$

Furthermore, we get an iterated homotopy orbit theorem

$$(Y_{hN})_{hG/N} = g_! f_! Y = h_! Y = Y_{hG}.$$

In this language the norm and transfer maps have a particularly nice description. Suppose that H is a subgroup of G, and consider the diagram:

$$\begin{tikzcd}
H \arrow[rr, "i"] \arrow[dr, "f"'] & & G \arrow[dl, "g"] \\
& 1 &
\end{tikzcd}$$

For $Y \in \mathrm{Sp}_G$, the transfer is given by the composite

$$\mathrm{Tr}_H^G : Y_{hG} = g_! Y \to g_! i_* i^* Y \xrightarrow[\cong]{\psi_i^{-1}} g_! i_! i^* Y \cong f_! i^* Y = Y_{hH}.$$

If H is normal in G, there is a refinement of the transfer Tr_e^H which is G-equivariant. Consider the diagram

$$\begin{tikzcd}
G \arrow[dr, "\Delta"] \arrow[drr, "\mathrm{Id}"] \arrow[ddr, "\mathrm{Id}"'] & & \\
& G \times_{G/H} G \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & G \arrow[d, "f"] \\
& G \arrow[r, "f"'] & G/H
\end{tikzcd}$$

Using the fact that the square in the diagram is a pullback, we define the G-equivariant transfer to be the composite

$$\begin{aligned}\text{(A.3)} \quad \mathrm{Tr}_e^H : f^* Y_{hH} = f^* f_! Y = (\pi_1)_! (\pi_2)^* Y \to (\pi_1)_! \Delta_* \Delta^* (\pi_2)^* Y \\ \xrightarrow[\cong]{\psi_\Delta^{-1}} (\pi_1)_! \Delta_! \Delta^* (\pi_2)^* Y = Y.\end{aligned}$$

The adjoint of this map gives a G/H-equivariant norm map

$$N_H : Y_{hH} \to f_* Y = Y^{hH}.$$

The equivariant transfer maps (A.3) can be constructed more generally: for subgroups

$$K \le H \le G$$

with K and H normal in G we can construct the G/K equivariant transfer Tr_K^H as the composite

$$\mathrm{Tr}_K^H : Y_{hH} = (Y_{hK})_{hH/K} \xrightarrow{\mathrm{Tr}_e^{H/K}} Y_{hK}.$$

We end this section with a lemma which we will need to make use of later.

Lemma A.4. *Given $X, Y \in \mathrm{Sp}_G$, the following diagram commutes in* $\mathrm{Ho}(\mathrm{Sp}_G)$.

$$\begin{tikzcd}
X_{hG} \wedge Y_{hG} \arrow[r, "\cong"] \arrow[d, "\cong"'] & (X \wedge Y_{hG})_{hG} \arrow[d, "1 \wedge \mathrm{Tr}_e^G"] \\
(X_{hG} \wedge Y)_{hG} \arrow[r, "\mathrm{Tr}_e^G \wedge 1"'] & (X \wedge Y)_{hG}
\end{tikzcd}$$

Proof. With respect to the maps:

$$\begin{tikzcd}
G \arrow[dr, "\delta"] \arrow[drr, bend left, "="] \arrow[ddr, bend right, "="'] & & \\
& G \times G \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, "g"] & G \arrow[d, "f"] \\
& G \arrow[r, "f"'] & 1
\end{tikzcd}$$

the lemma follows from the following commutative diagram.

$$\begin{tikzcd}
f_{!}X \wedge f_{!}Y \arrow[rr, "\cong"] \arrow[dr, "\cong"] \arrow[dd, "\cong"'] & & f_{!}(X \wedge f^{*}f_{!}Y) \arrow[r, "\cong"] & f_{!}(X \wedge (\pi_1)_{!}\pi_2^{*}Y) \arrow[dd, bend left] \\
& g_{!}(\pi_1^{*}X \wedge \pi_2^{*}Y) \arrow[r, "\cong"] \arrow[d, "\cong"'] \arrow[dr] & f_{!}(\pi_1)_{!}(\pi_1^{*}X \wedge \pi_2^{*}Y) \arrow[ur, "\cong"'] & f_{!}(\pi_1)_{!}\delta_{!}(\delta^{*}\pi_1^{*}X \wedge \delta^{*}\pi_2^{*}Y) \arrow[d, "\cong"] \\
f_{!}(f^{*}f_{!}X \wedge Y) \arrow[d, "\cong"'] & f_{!}(\pi_2)_{!}(\pi_1^{*}X \wedge \pi_2^{*}Y) \arrow[dl, "\cong"'] & g_{!}\delta_{!}\delta^{*}(\pi_1^{*}X \wedge \pi_2^{*}Y) \arrow[ur, "\cong"] \arrow[dl, "\cong"'] & f_{!}(X \wedge (\pi_1)_{!}\delta_{!}\delta^{*}\pi_2^{*}Y) \arrow[d, "\cong"] \\
f_{!}((\pi_2)_{!}\pi_1^{*}X \wedge Y) \arrow[rr, bend right] & f_{!}(\pi_2)_{!}\delta_{!}(\delta^{*}\pi_1^{*}X \wedge \delta^{*}\pi_2^{*}Y) \arrow[r, "\cong"'] & f_{!}((\pi_2)_{!}\delta_{!}\delta^{*}\pi_1^{*}X \wedge Y) \arrow[r, "\cong"'] & f_{!}(X \wedge Y)
\end{tikzcd}$$

□

The H_∞ structure of R^{QX_+}. The H_∞ structure of R^{QX_+} comes from the Σ_i-equivariant diagonal maps

$$QX_+ \xrightarrow{\Delta} (QX_+)^{\wedge i}.$$

Recall the convenient point-set level description of the Kahn stable splitting of QX_+ given in [Kuh06].

Lemma A.5 ([Kuh06])**.** *The equivalence*

$$s_X : \mathbb{P}(X) \xrightarrow{\simeq} \Sigma^\infty QX_+$$

is the map of E_∞ ring spectra adjoint to the natural inclusion of spectra

$$\Sigma^\infty X \to \Sigma^\infty QX_+.$$

Lemma A.6. *The following diagram of Σ_k-spectra commutes*

$$\begin{tikzcd}
\mathbb{P}(\Sigma^\infty X) \arrow[rr, "\mathbb{P}(\Delta)"] \arrow[d, "s_X"', "\simeq"] & & \mathbb{P}((\Sigma^\infty X)^{\vee k}) \arrow[r, equal] & \mathbb{P}(\Sigma^\infty(X^{\vee k})) \arrow[d, "\simeq"', "s_{X^{\vee k}}"] \\
\Sigma^\infty QX_+ \arrow[r, "\Delta"'] & \Sigma^\infty(QX_+)^{\wedge k} \arrow[r, equal] & \Sigma^\infty((QX)^{\times k})_+ \arrow[r, equal] & \Sigma^\infty Q(X^{\vee k})_+
\end{tikzcd}$$

Proof. By adjointness, this follows from the commutativity of the diagram

$$\begin{tikzcd}
\Sigma^\infty X \arrow[rr, "\Delta"] \arrow[d] & & (\Sigma^\infty X)^{\vee k} \arrow[r, equal] & \Sigma^\infty(X^{\vee k}) \arrow[d] \\
\Sigma^\infty QX_+ \arrow[r, "\Delta"'] & \Sigma^\infty(QX_+)^{\wedge k} \arrow[r, equal] & \Sigma^\infty((QX)^{\times k})_+ \arrow[r, equal] & \Sigma^\infty Q(X^{\vee k})_+
\end{tikzcd}$$

□

For a sequence $I = (i_1, \dots, i_k)$ of non-negative integers, define $\|I\| := i_1 + \cdots + i_k$ and $|I| := k$. Define

$$\Sigma_I := \Sigma_{i_1} \times \cdots \times \Sigma_{i_k},$$

and let $\Sigma_{(I)}$ denote the subgroup of Σ_k which preserves the sequence I, and define $\Sigma_{[I]}$ to be the subgroup of Σ_i given by

$$\Sigma_{[I]} := \Sigma_{(I)} \ltimes \Sigma_I.$$

There are Σ_k-equivariant equivalences

$$\mathbb{P}(Y^{\vee k}) \simeq \mathbb{P}(Y)^{\wedge k} \simeq \bigvee_{|I|=k} (\Sigma_k)_+ \wedge_{\Sigma_{(I)}} Y^{\|I\|}_{h\Sigma_I}. \tag{A.7}$$

Let α_I denote the I-component of the above equivalence

$$\alpha_I : (Y^{\vee k})^{\wedge \|I\|}_{h\Sigma_{\|I\|}} \to (\Sigma_k)_+ \wedge_{\Sigma_{(I)}} Y^{\wedge \|I\|}_{h\Sigma_I}.$$

Since there is a Σ_k-equivariant equivalence

$$(\Sigma_k)_+ \wedge_{\Sigma_{(I)}} Y^{\wedge\|I\|}_{h\Sigma_I} = \mathrm{Ind}^{\Sigma_k}_{\Sigma_{(I)}} Y^{\wedge\|I\|}_{h\Sigma_I} \xrightarrow{\simeq} \mathrm{CoInd}^{\Sigma_k}_{\Sigma_{(I)}} Y^{\wedge\|I\|}_{h\Sigma_I}.$$

the Σ_k-equivariant map α_I determines and is determined by a $\Sigma_{(I)}$-equivariant map

$$\widetilde{\alpha}_I : (Y^{\vee k})^{\wedge\|I\|}_{h\Sigma_{\|I\|}} \to Y^{\wedge\|I\|}_{h\Sigma_I}.$$

The following is a consequence of [LMSM86, VII.1.10].

Lemma A.8. *The composite*

$$Y^{\wedge\|I\|}_{h\Sigma_{\|I\|}} \xrightarrow{\Delta} (Y^{\vee k})^{\wedge\|I\|}_{h\Sigma_{\|I\|}} \xrightarrow{\widetilde{\alpha}_I} Y^{\wedge\|I\|}_{h\Sigma_I}$$

is equal to the transfer $\mathrm{Tr}^{\Sigma_{\|I\|}}_{\Sigma_I}$ *in* $\mathrm{Ho}(\mathrm{Sp}_{\Sigma_{(I)}})$.

The H_∞-R-algebra structure of R^{QX_+} is given by structure maps

$$\xi_k : (R^{QX_+})^{\wedge_R k}_{h\Sigma_k} \to R^{QX_+}$$

whose adjoints are given by the composites (see [BMMS86, Lem. II.3.3])

$$\begin{aligned}
\widetilde{\xi}_k : (R^{QX_+})^{\wedge_R k}_{h\Sigma_k} \wedge QX_+ &\simeq \big((R^{QX_+})^{\wedge_R k} \wedge QX_+\big)_{h\Sigma_k} \\
&\xrightarrow{1\wedge\Delta} \big((R^{QX_+})^{\wedge_R k} \wedge (QX_+)^{\wedge k}\big)_{h\Sigma_k} \xrightarrow{\mathrm{ev}^{\wedge k}} R^{\wedge_R k}_{h\Sigma_k} \xrightarrow{\mu_k} R.
\end{aligned}$$

Here μ_k comes from the H_∞-R-algebra structure of R itself: under the isomorphism $R^{\wedge_R k} \cong R$ the composite

$$R_{h\Sigma_k} = R^{\wedge_R k}_{h\Sigma_k} \xrightarrow{\mu_k} R$$

is the restriction coming from the map of groups $\Sigma_k \to 1$. Therefore the map $\widetilde{\xi}_k$ is also given by the composite

$$\begin{aligned}\widetilde{\xi}_k : (R^{QX_+})^{\wedge_R k}_{h\Sigma_k} \wedge QX_+ \to (R^{(QX_+)^{\wedge k}})_{h\Sigma_k} \wedge QX_+ &\simeq \left(R^{(QX_+)^{\wedge k}} \wedge QX_+\right)_{h\Sigma_k} \\ &\xrightarrow{1\wedge\Delta} \left(R^{(QX_+)^{\wedge k}} \wedge (QX_+)^{\wedge k}\right)_{h\Sigma_k} \xrightarrow{\mathrm{ev}} R_{h\Sigma_k} \xrightarrow{\mathrm{Res}^1_{\Sigma_k}} R.\end{aligned}$$

Using (A.7) there is a Σ_k-equivariant equivalence

$$R^{(QX_+)^{\wedge k}} \underset{\simeq}{\xrightarrow{s_X^*}} R^{(\mathbb{P}X)^{\wedge k}} \simeq \prod_{|I|=k} \mathrm{CoInd}^{\Sigma_k}_{\Sigma_{(I)}} R^{X^{\|I\|}_{h\Sigma_I}} \simeq \prod_{|I|=k} \mathrm{Ind}^{\Sigma_k}_{\Sigma_{(I)}} R^{X^{\|I\|}_{h\Sigma_I}}.$$

The following lemma is therefore obtained by combining the above description of $\widetilde{\xi}_k$ with Lemma A.6 and Lemma A.8.

Lemma A.9. *The map ξ_k is give by the composite*

$$\begin{aligned}\xi_k : (R^{QX_+})^{\wedge_R k}_{h\Sigma_k} &\to (R^{(QX_+)^{\wedge k}})_{h\Sigma_k} \\ &\simeq \left(\prod_{|I|=k} \mathrm{Ind}^{\Sigma_k}_{\Sigma_{(I)}} R^{X^{\|I\|}_{h\Sigma_I}}\right)_{h\Sigma_k} \\ &\to \prod_{|I|=k} \left(\mathrm{Ind}^{\Sigma_k}_{\Sigma_{(I)}} R^{X^{\|I\|}_{h\Sigma_I}}\right)_{h\Sigma_k} \\ &\simeq \prod_{|I|=k} (R^{X^{\|I\|}_{h\Sigma_I}})_{h\Sigma_{(I)}} \\ &\simeq \prod_i \bigvee_{\substack{\|I\|=i \\ |I|=k}} (R^{X^{i}_{h\Sigma_I}})_{h\Sigma_{(I)}} \\ &\xrightarrow{\prod_i \sum_I \xi_k^I} \prod_i R^{X^i_{h\Sigma_i}} \\ &\simeq R^{\mathbb{P}X} \\ &\simeq R^{QX_+}.\end{aligned}$$

where the maps ξ_k^I are adjoint to the composites

$$\begin{aligned}\widetilde{\xi}^I_k : (R^{X^{\|I\|}_{h\Sigma_I}})_{h\Sigma_{(I)}} \wedge X^{\|I\|}_{h\Sigma_{\|I\|}} &\simeq (R^{X^{\|I\|}_{h\Sigma_I}} \wedge X^{\|I\|}_{h\Sigma_{\|I\|}})_{h\Sigma_{(I)}} \\ &\xrightarrow{1\wedge \mathrm{Tr}^{\Sigma_{\|I\|}}_{\Sigma_I}} (R^{X^{\|I\|}_{h\Sigma_I}} \wedge X^{\|I\|}_{h\Sigma_I})_{h\Sigma_{(I)}} \xrightarrow{\mathrm{ev}} R_{h\Sigma_{(I)}} \xrightarrow{\mathrm{Res}^1_{\Sigma_{(I)}}} R.\end{aligned}$$

Completion of the proof of Lemma A.2. The H_∞-R-algebra structure of $\widehat{\mathbb{P}}_R(R^X)$ has structure maps

$$\zeta_k : \widehat{\mathbb{P}}_R(R^X)^{\wedge_R k}_{h\Sigma_k} \to \widehat{\mathbb{P}}_R(R^X)$$

which are given by the composites

$$\begin{aligned}
\zeta_k : \widehat{\mathbb{P}}_R(R^X)^{\wedge_R k}_{h\Sigma_k} &= \left(\prod_i (R^{X^i})_{h\Sigma_i} \right)^{\wedge_R k}_{h\Sigma_k} \\
&\to \left(\prod_{|I|=k} \mathrm{Ind}^{\Sigma_k}_{\Sigma_{(I)}} (R^{X^{\|I\|}})_{h\Sigma_I} \right)_{h\Sigma_k} \\
&\to \prod_{|I|=k} \left(\mathrm{Ind}^{\Sigma_k}_{\Sigma_{(I)}} (R^{X^{\|I\|}})_{h\Sigma_I} \right)_{h\Sigma_k} \\
&\simeq \prod_{|I|=k} ((R^{X^{\|I\|}})_{h\Sigma_I})_{h\Sigma_{(I)}} \\
&\simeq \prod_{|I|=k} (R^{X^{\|I\|}})_{h\Sigma_{[I]}} \\
&\simeq \prod_i \bigvee_{\substack{\|I\|=i \\ |I|=k}} (R^{X^i})_{h\Sigma_{[I]}} \\
&\xrightarrow{\prod_i \sum_I \mathrm{Res}^{\Sigma_i}_{\Sigma_{[I]}}} \prod_i (R^{X^i})_{h\Sigma_i} \\
&= \widehat{\mathbb{P}}_R(R^X).
\end{aligned}$$

Lemma A.2 therefore follows from the following lemma.

Lemma A.10. *Fix a partition I with $|I| = k$ and $\|I\| = i$. Then the following diagram commutes.*

$$\begin{tikzcd}
(R^{X^i})_{h\Sigma_{[I]}} \arrow[r, "\mathrm{Res}^{\Sigma_i}_{\Sigma_{[I]}}"] \arrow[d, "N_{\Sigma_I}"'] & (R^{X^i})_{h\Sigma_i} \arrow[d, "N_{\Sigma_i}"] \\
(R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \arrow[r, "\xi^I_k"'] & R^{X^i_{h\Sigma_i}}
\end{tikzcd}$$

Proof. The construction of the norm as the adjoint to the equivariant transfer implies that the following diagram commutes.

$$\begin{tikzcd}
(R^{X^i})_{h\Sigma_{[I]}} \arrow[rr, "\mathrm{Res}^{\Sigma_i}_{\Sigma_{[I]}}"] \arrow[d, "N_{\Sigma_I}"'] \arrow[dr, "N_{\Sigma_{[I]}}"] & & (R^{X^i})_{h\Sigma_i} \arrow[d, "N_{\Sigma_i}"] \\
(R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \arrow[r, "N_{\Sigma_{(I)}}"'] & R^{X^i_{h\Sigma_{[I]}}} \arrow[r, "\mathrm{Tr}^{\Sigma_i}_{\Sigma_{[I]}}"'] & R^{X^i_{h\Sigma_i}}
\end{tikzcd}$$

$\square$

Therefore it suffices to show the commutativity of the diagram below.

$$\begin{tikzcd}
(R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \arrow[r, "\xi^I_k"] \arrow[d, "N_{\Sigma_{(I)}}"'] & R^{X^i_{h\Sigma_i}} \\
R^{X^i_{h\Sigma_{[I]}}} \arrow[ur, "\mathrm{Tr}^{\Sigma_i}_{\Sigma_{[I]}}"'] &
\end{tikzcd}$$

By adjointness this is equivalent to showing the commutativity of the diagram:

$$\begin{tikzcd}
(R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \wedge X^i_{h\Sigma_i} \arrow[r, "\widetilde{\xi}^I_k"] \arrow[d, "N_{\Sigma_{(I)}} \wedge 1"'] & R \\
R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_i} \arrow[r, "\mathrm{Tr}^{\Sigma_i}_{\Sigma_{[I]}} \wedge 1"'] & R^{X^i_{h\Sigma_i}} \wedge X^i_{h\Sigma_i} \arrow[u, "\mathrm{ev}"']
\end{tikzcd}$$

By Lemma A.9, the commutativity of the above diagram is seen in the following commutative diagram.
(A.11)

$$\begin{tikzcd}
(R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \wedge X^i_{h\Sigma_i} \arrow[r, equal] \arrow[ddr, "1\wedge\mathrm{Tr}^{\Sigma_i}_{\Sigma_{[I]}}"'] \arrow[ddd, "N_{\Sigma_{(I)}}\wedge 1"'] & (R^{X^i_{h\Sigma_I}} \wedge X^i_{h\Sigma_i})_{h\Sigma_{(I)}} \arrow[r, "1\wedge\mathrm{Tr}^{\Sigma_i}_{\Sigma_I}"] \arrow[d, "1\wedge\mathrm{Tr}^{\Sigma_i}_{\Sigma_{[I]}}"'] & (R^{X^i_{h\Sigma_I}} \wedge X^i_{h\Sigma_I})_{h\Sigma_{(I)}} \arrow[r, "\mathrm{ev}"] & R_{h\Sigma_{(I)}} \arrow[r, "\mathrm{Res}^1_{\Sigma_{(I)}}"] & R \\
& (R^{X^i_{h\Sigma_I}} \wedge X^i_{h\Sigma_{[I]}})_{h\Sigma_{(I)}} \arrow[ur, "1\wedge\mathrm{Tr}^{\Sigma_{[I]}}_{\Sigma_I}"] \arrow[d, equal] \arrow[dr, phantom, "(1)"] & (R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_I})_{h\Sigma_{(I)}} \arrow[u, "\mathrm{Res}^{\Sigma_{[I]}}_{\Sigma_I}\wedge 1"'] \arrow[d, equal] & (2) & \\
& (R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \wedge X^i_{h\Sigma_{[I]}} \arrow[r, "N_{\Sigma_{(I)}}\wedge 1"] & R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_{[I]}} \arrow[uurr, "\mathrm{ev}"'] & & \\
R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_i} \arrow[urr, "1\wedge\mathrm{Tr}^{\Sigma_i}_{\Sigma_{[I]}}"'] \arrow[rrrr, "\mathrm{Tr}^{\Sigma_i}_{\Sigma_{[I]}}\wedge 1"'] & & & & R^{X^i_{h\Sigma_i}} \wedge X^i_{h\Sigma_i} \arrow[uuu, "\mathrm{ev}"']
\end{tikzcd}$$

With the exception of regions (1) and (2) in the above diagram, all of the faces of the diagram clearly commute.

The commutativity of (1) is seen below, making use of Lemma A.4.

$$\begin{tikzcd}
(R^{X^i_{h\Sigma_I}} \wedge (X^i_{h\Sigma_I})_{h\Sigma_{(I)}})_{h\Sigma_{(I)}} \arrow[rr, "1\wedge\mathrm{Tr}_1^{\Sigma_{(I)}}"] \arrow[dd, equal] & & (R^{X^i_{h\Sigma_I}} \wedge X^i_{h\Sigma_I})_{h\Sigma_{(I)}} \\
& ((R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \wedge X^i_{h\Sigma_I})_{h\Sigma_{(I)}} \arrow[ur, "\mathrm{Tr}_1^{\Sigma_{(I)}}\wedge 1"] \arrow[r, "N_{\Sigma_{(I)}}\wedge 1"] & (R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_I})_{h\Sigma_{(I)}} \arrow[u, "\mathrm{Res}^{\Sigma_{[I]}}_{\Sigma_I}\wedge 1"'] \arrow[d, equal] \\
(R^{X^i_{h\Sigma_I}})_{h\Sigma_{(I)}} \wedge X^i_{h\Sigma_{[I]}} \arrow[ur, equal] \arrow[rr, "N_{\Sigma_{(I)}}\wedge 1"'] & & R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_{[I]}}
\end{tikzcd}$$

By adjointness, the commutativity of region (2) in Diagram (A.11) is equivalent to the commutativity of the following diagram in $\mathrm{Ho}(\mathrm{Sp}_{\Sigma_{(I)}})$, which clearly commutes.

$$\begin{tikzcd}
R^{X^i_{h\Sigma_I}} \wedge X^i_{h\Sigma_I} \arrow[r, "\mathrm{ev}"] & R \\
R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_I} \arrow[u, "\mathrm{Res}^{\Sigma_{[I]}}_{\Sigma_I} \wedge 1"] \arrow[r, "1 \wedge \mathrm{Res}^{\Sigma_{[I]}}_{\Sigma_I}"'] & R^{X^i_{h\Sigma_{[I]}}} \wedge X^i_{h\Sigma_{[I]}} \arrow[u, "\mathrm{ev}"']
\end{tikzcd}$$

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