18.118 DECOUPLING
LECTURE 2

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1. Decoupling for interval

Let $\Omega = [0, N] = \cup_{j=1}^{N}[j-1, j]$ for some large integer N. Let θ_j denote the unit interval $[j-1, j]$. If supp$\widehat{f} \subset \Omega$, then $f = \sum_{j=1}^{N} f_j$ with $f_j(x) = \int_{\theta_j} e^{2\pi i w x}\widehat{f}(w)dw$.

Definition 1.1. *The decoupling constant $D_p(N)$ is the smallest constant such that for all f, supp$\widehat{f} \subset \Omega$,*

$$\|f\|_{L^p(\mathbb{R})} \leq D_p(N)(\sum_{j=1}^{N} \|f_j\|_{L^p(\mathbb{R})}^2)^{1/2}. \tag{1}$$

1.1. **Building blocks.** We consider f_1, supp$\widehat{f_1} \subset [0, 1]$.

Question 1.2. *Could the graph of $|f_1|$ look like several very narrow (say width $\frac{1}{10000}$) peaks and the rest is about zero (see Figure 1 in the attached picture)?*

Let η be a Schwartz function and $\eta = 1$ on $[0, 1]$. We have $\widehat{f_1} = \widehat{f_1} \cdot \eta$. Then $f_1 = f_1 * \check{\eta}$ and $\check{\eta}$ is also a Schwartz function.

Corollary 1.3. *If supp$\widehat{f_1} \subset [0, 1]$, then*

$$\|f_1\|_{L^\infty} \lesssim \|f_1\|_{L^1}.$$

Proof. By Young's inequality,

$$\begin{aligned} \|f_1\|_{L^\infty} &= \|f_1 * \check{\eta}\|_{L^\infty} \\ &\leq \|f_1\|_{L^1}\|\check{\eta}\|_{L^\infty} \\ &\lesssim \|f_1\|_{L^1}. \end{aligned}$$

□

The answer is NO by the above corollary. The height of peaks of $|f_1|$ is as much as $\|f_1\|_{L^\infty}$. However several very narrow peaks with limited height can not add up to the same L^1–norm.

Question 1.4. *How about if we add a flat low tail (see Figure 2) to the graph of $|f_1|$, such that $\|f_1\|_{L^1}$ is dominated by the flat part. Can the graph of $|f_1|$ still have very narrow peaks?*

The answer is still NO. We actually know more about $\check{\eta}$ other than its L^∞–norm. We know that $\check{\eta}$ is a Schwartz function:

$$|\check{\eta}(y)| \lesssim_M (\frac{1}{1+|y|})^M$$

for any large constant M. In fact, we almost know that $\|f_1\|_{L^\infty(I)} \lesssim \|f_1\|_{L^1(2I)}$ for any unit interval I, where $2I$ means that we stretch I to twice the length with the same center point.

Lemma 1.5. ***Locally Constant Lemma*** *If supp$\widehat{f_1} \subset [0,1]$, and I unit interval, then*

$$\|f_1\|_{L^\infty(I)} \lesssim \|f_1\|_{L^1(w_I)}.$$

The weighted L^1–norm is defined as $\|f_1\|_{L^1(w_I)} := \int_{\mathbb{R}} |f_1| w_I$. The weight function w_I has the following property:

- *$w_I \geq 0$.*
- *$w_I \sim 1$ on I.*
- *w_I is rapidly decaying off I.*
- *w_I is uniform in choice of I in the sense that $w_{I+a} = w_I(\cdot - a)$.*

Proof. For any $x \in I$,

$$\begin{aligned}|f_1(x)| &\leq \int |f_1(y)||\check{\eta}(x-y)|dy \\ &\leq \int |f_1(y)| \sup_{x\in I} |\check{\eta}(x-y)|dy\end{aligned}$$

We define $w_I(y) = \sup_{x\in I} |\check{\eta}(x-y)|$. □

One option of w_I is

$$w_I(y) = (\frac{1}{1+\operatorname{dist}(y,I)})^{50}.$$

The graph of $|f_1|$ should look like Figure 3: each peak should have width about 1.

Remark 1.6. *If supp$\widehat{f_j} \subset [j-1, j]$, then the Fourier transform of $e^{-2\pi i(j-1)x} f_j$ is supported in $[0,1]$. The Locally Constant Lemma still holds for f_j, $\forall j$.*

Example 1.7. *We consider f_1 a bump function with height ($\|f_1\|_{L^\infty}$) 1 concentrated on $[-1,1]$ and $f_1(0)$=1. Figure 4 is the graph of $Re f_1$.*

We take $f_j(x) = e^{2\pi i(j-1)x} f_1(x)$ *and we sum over* j: $f = \sum_{j=1}^{N} = f_j$. *Now* $f_j(0) = 1$ *and* $f(0) = N$.

Notice that $Re f_j$ *oscillates with frequency about* $\frac{1}{j}$. *When* $|x| \leq \frac{1}{10N} \leq \frac{1}{10j}$, $|f_j(x) - 1| \leq \frac{1}{4}$, *so* $|f(x)| \sim N$.

This implies that $\|f\|_{L^p} \gtrsim N \cdot N^{-1/p} = N^{1-1/p}$.

For the right-hand side, $\|f_j\|_{L^p} \sim 1$ *and* $(\sum_j \|f_j\|_{L^p}^2)^{1/2} \sim N^{1/2}$. *This example gives a lower bound of the decoupling constant as defined in* 1

$$D_p(N) \gtrsim N^{1/2-1/p}.$$

Proposition 1.8. *If* $supp\widehat{f_j} \subset [j-1, j]$, $j = 1, \ldots, N$, *and* $f = \sum_j f_j$, *then for any* $2 \leq p \leq \infty$, *the decoupling constant defined in* 1

$$D_p(N) \lesssim N^{1/2-1/p}.$$

In particular, the example described above is sharp.

Remark 1.9. *For* $p = 2$ *and* $p = \infty$, *it is easy to estimate* $D_p(N)$.

When $p = 2$, *by Plancherel's inequality,*

$$\|f\|_{L^2}^2 = \sum_j \|f_j\|_{L^2}^2.$$

When $p = \infty$, *by triangle inequality,*

$$\|f\|_{L^\infty} \leq \sum_{j=1}^{N} \|f_j\|_{L^\infty} \leq N^{1/2} (\sum_{j=1}^{N} \|f_j\|_{L^\infty}^2)^{1/2}.$$

1.2. **Main Issue.** In this subsection, we discuss a hypothetical example for the issue might occur when $2 < p < \infty$. Let $p = 4$ for example, suppose $\forall j$, $|f_j|$ looks like a function that is 1 at $[0, 1]$ and $\frac{1}{N}$ at $[1, N^3]$ and zero elsewhere (See Figure 5 for the graph of $|f_j|$).

- $\|f_j\|_{L^2} \sim N^{1/2}$ is dominated by the short wide piece at interval $[1, N^3]$.
- $\|f_j\|_{L^4} \sim 1$ is dominated by the peak at $[0, 1]$.

We analyse f through the information provided by f_j.

- By orthogonality, $\|f\|_{L^2} \sim N$.
- By triangle inequality, $\|f\|_{L^\infty} \leq N$.

Question 1.10. *Could it happen that* $|f(x)| \sim N$ *for most* $x \in [0, 1]$*?*

Proposition 1.8 tells us that this is impossible.

$$\|f\|_{L^4} \lesssim N^{1/4} (\sum_{j=1}^{N} \|f_j\|_{L^4}^2)^{1/2} \sim N^{3/4}.$$

The following Local Orthogonality Lemma gives an even better estimate.

Lemma 1.11. ***Local Orthogonality*** *If I is a unit interval, $f = \sum_{j=1}^N f_j$ and supp$\widehat{f_j} \subset [j-1, j]$, then*

$$\|f\|^2_{L^2(I)} \lesssim \sum_j \|f_j\|^2_{L^2(w_I)}$$

for the weight function w_I with the same property as in the Locally Constant Lemma 1.5.

Proof. We choose $\eta(x)$ such that $|\eta| \sim 1$ on I, and supp$\widehat{\eta} \subset [-1, 1]$.

$$\begin{aligned}
\int_I |f|^2 &\le \int_{\mathbb{R}} |f\eta|^2 \\
&= \int_{\mathbb{R}} |\widehat{\eta} * \widehat{f}|^2 \\
&= \int_{\mathbb{R}} |\sum_j \widehat{\eta} * \widehat{f_j}|^2 \\
&\lesssim \sum_j \int_{\mathbb{R}} |\widehat{\eta} * \widehat{f_j}|^2 = \sum_j \int_{\mathbb{R}} |f_j|^2 |\eta|^2
\end{aligned}$$

Since supp$\widehat{\eta} \subset [-1, 1]$, the support of $\widehat{\eta} * \widehat{f_j}$ lies in $[j-2, j+1]$. Any frequency lies inside at most $O(1)$ intervals of the form $[j-2, j+1]$. We define $w_I = |\eta|^2$. □

Proposition 1.12. *(Local decoupling). If I is a unit interval, $2 \le p \le \infty$, f_j and f are defined as in Proposition* 1.8*, then*

$$\|f\|_{L^p(I)} \lesssim N^{1/2-1/p} (\sum_{j=1}^N \|f_j\|^2_{L^p(w_I)})^{1/2}.$$

Proof. By Local Orthogonality Lemma 1.11 and triangle inequality

$$\begin{aligned}
\int_I |f|^p &\le (\int_I |f|^2) \|f\|^{p-2}_{L^\infty(I)} \\
&\le (\sum_j \|f_j\|^2_{L^2(w_I)})(\sum_j \|f_j\|_{L^\infty(I)})^{p-2}.
\end{aligned}$$

By Locally Constant Lemma 1.5,

$$\|f_j\|_{L^\infty(I)} \lesssim \|f_j\|_{L^1(w_I)} \lesssim \|f_j\|_{L^2(w_I)}.$$

$$\begin{aligned}\int_I |f|^p &\leq (\sum_j \|f_j\|^2_{L^2(w_I)})(\sum_j \|f_j\|_{L^2(w_I)})^{p-2} \\ &\lesssim N^{\frac{p-2}{2}}(\sum_j \|f_j\|^2_{L^2(w_I)})^{\frac{p}{2}} \\ &\lesssim N^{\frac{p-2}{2}}(\sum_j \|f_j\|^2_{L^p(w_I)})^{\frac{p}{2}}\end{aligned}$$

The last inequality follows from Hölder's inequality. □

Remark 1.13. *Since w_I is a measure with total mass about 1. Hölder's inequality implies that $\|f\|_{L^p(w_I)} \lesssim \|f\|_{L^q(w_I)}$ if $p \leq q$ and for any function f.*

For f_j in particular, Locally Constant Lemma says $\|f_j\|_{L^\infty(I)} \lesssim \|f_j\|_{L^1(w_I)}$. Furthermore, we can show $\|f_j\|_{L^p(w_I)} \lesssim \|f_j\|_{L^q(w_I)}$ for any $1 \leq p, q$. It suffices to prove for $p > q$ since other cases are provided by Hölder's inequality. We consider a collection of unit intervals $\{I'\}$ that tiles $\mathbb{R}$.

$$\begin{aligned}\|f_j\|^p_{L^p(w_I)} &\leq \sum_{I'} c(I')\|f_j\|^p_{L^p(I')} \\ &\leq \sum_{I'} c(I')\|f_j\|^p_{L^\infty(I')} \\ &\lesssim \sum_{I'} c(I')\|f_j\|^p_{L^1(w_{I'})} \\ &\lesssim \sum_{I'} c(I')\|f_j\|^p_{L^q(w_{I'})} \\ &\lesssim \|f_j\|^p_{L^q(w_I)}\end{aligned}$$

$c(I') := \sup_{x\in I'} w_I(x)$. *Since w_I is a Schwartz function and satisfies the property listed in Lemma 1.5,*

$$\sum_{I'} c(I')^{q/p} w_{I'} \lesssim w_I.$$

In the end we used $l^q \geq l^p$ when $q \leq p$ to sum up $\|f_j\|^p_{L^q(w_{I'})}$.

1.3. **Parallel Decoupling Lemma.**

In this subsection, we prove a general Parallel Decoupling Lemma for general decoupling inequalities.

Lemma 1.14. *For some $p \geq 2$ and for any function g_j, $g = \sum_j g_j$ and any measure μ_i, w_i, $\mu = \sum_i \mu_i$, $w = \sum_i w_i$, if we know*

$$\|g\|_{L^p(\mu_i)} \leq D(\sum_j \|g_j\|^2_{L^p(w_i)})^{1/2}, \forall i,$$

then we have

$$\|g\|_{L^p(\mu)} \leq D(\sum_j \|g_j\|^2_{L^p(w_i)})^{1/2}$$

for the same decoupling constant D.

Proof. The proof is an application of Minkowski's inequality.

$$\begin{aligned}
\int |g|^p \mu &= \sum_i \int |g|^p \mu_i \\
&\leq D^p \sum_i (\sum_j \|g_j\|^2_{L^p(w_i)})^{p/2} \\
&= D^p \| \sum_j \|g_j\|^2_{L^p(w_i)} \|^{p/2}_{l_i^{p/2}} \\
&\leq D^p [\sum_j \| \|g_j\|^2_{L^p(w_i)} \|_{l_i^{p/2}}]^{p/2} \\
&= D^p [\sum_j (\sum_i \|g_j\|^p_{L^p(w_i)})^{2/p}]^{p/2} \\
&= D^p (\sum_j \|g_j\|^2_{L^p(w)})^{p/2}
\end{aligned}$$

□