Mathematische Annalen manuscript No.
(will be inserted by the editor)

The spherical part of the local and global Springer actions

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Received: date / Revised version: date –

Abstract. The affine Weyl group acts on the cohomology (with compact support) of affine Springer fibers (local Springer theory) and of parabolic Hitchin fibers (global Springer theory). In this paper, we show that in both situations, the action of the center of the group algebra of the affine Weyl group (the spherical part) factors through the action of the component group of the relevant centralizers. In the situation of affine Springer fibers, this partially verifies a conjecture of Goresky-Kottwitz-MacPherson and Bezrukavnikov-Varshavsky.

We first prove this result for the global Springer action, and then deduce from it the result for the local Springer action. This gives an application of global Springer theory to more classical problems.

Mathematics Subject Classification (2000): 14H60, 14M15, 20F55, 20G25

Contents

1. Introduction

Let k be an algebraically closed field. Let G be a reductive algebraic group over k. We assume either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k)$ is large with respect to G (see §1.4). Let $\mathfrak{g}$ be the Lie algebra of G. Let $\mathcal{B}$ be the flag variety of G. For $v \in \mathfrak{g}(k)$, the *Springer fiber* of v is the closed subvariety $\mathcal{B}_v \subset \mathcal{B}$ consisting of all Borel subgroups of G whose Lie algebras contain v. Classical Springer theory [**?**] gives an action of the Weyl group W of G on the cohomology of $\mathcal{B}_v$. One the other hand, the centralizer G_v of v in G acts on $\mathcal{B}_v$, hence on the cohomology of $\mathcal{B}_v$ via its component group $\pi_0(G_v)$. These two symmetries commute with each other:

$$W \curvearrowright \mathrm{H}^*(\mathcal{B}_v) \curvearrowleft \pi_0(G_v).$$

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However, there is no obvious way to recover the $\pi_0(G_v)$-action on $\mathrm{H}^*(\mathcal{B}_v)$ solely from the W-action.

Now we consider the affine situation. Let $F = k((\varpi))$ be the field of formal Laurent series in one variable and $\mathcal{O}_F = k[[\varpi]]$. Let LG be the loop group of G. This is an ind-scheme over k whose k-points are $G(F)$. Let $\mathrm{Fl}_G = LG/\mathbf{I}$ be the affine flag variety of G, where $\mathbf{I}$ is a fixed Iwahori subgroup of LG. This is an infinite union of projective varieties. For any regular semisimple element $\gamma \in \mathfrak{g}(F)$, Kazhdan and Lusztig [**?**] defined a closed sub-ind-scheme $\mathrm{Spr}_\gamma \subset \mathrm{Fl}_G$, called the *affine Springer fiber* of γ. The set $\mathrm{Spr}_\gamma(k)$ consists of those Iwahori subgroups whose Lie algebras contain γ. The ind-scheme Spr_γ is a possibly infinite union of projective varieties of dimension expressible in terms of γ (see [**?**]).

In [**?**], Lusztig defined an action of the affine Weyl group W_{aff} on the homology of affine Springer fibers Spr_γ. We will review this construction in §2, and extend it to an action of the extended affine Weyl group $\widetilde{W} = \mathbb{X}_*(T) \rtimes W$ on both $\mathrm{H}_*(\mathrm{Spr}_\gamma)$ and $\mathrm{H}^*_c(\mathrm{Spr}_\gamma)$. On the other hand, the centralizer group LG_γ (the centralizer of γ in the loop group LG) acts on Spr_γ, hence induces an action of its component group $\pi_0(LG_\gamma)$ on the homology of Spr_γ. These two symmetries on $\mathrm{H}_*(\mathrm{Spr}_\gamma)$ again commute with each other:

$$\widetilde{W} \curvearrowright \mathrm{H}_*(\mathrm{Spr}_\gamma) \curvearrowleft \pi_0(LG_\gamma).$$

Similar statement holds for $\mathrm{H}^*_c(\mathrm{Spr}_\gamma)$.

A priori, the definition of the $\widetilde{W}$-action and the $\pi_0(LG_\gamma)$-action has nothing to do with each other. However, as opposed to the situation in classical Springer theory, we expect that the $\pi_0(LG_\gamma)$-action be completely determined by the "central character" of the $\widetilde{W}$-action. The center of the group algebra $\mathbb{Q}_\ell[\widetilde{W}]$ is $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ (superscript W means taking W-invariants). If one views $\mathbb{Q}_\ell[\widetilde{W}]$ as the specialization of the affine Hecke algebra at $q = 1$, then $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ is the specialization of the spherical Hecke algebra at $q = 1$. For this reason, we shall call $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ the *spherical part* of the group algebra $\mathbb{Q}_\ell[\widetilde{W}]$. We shall see that there is a canonical algebra homomorphism (see §2.7)

$$\sigma_\gamma : \mathbb{Q}_\ell[\mathbb{X}_*(T)]^W \to \mathbb{Q}_\ell[\pi_0(LG_\gamma)]. \tag{1}$$

Conjecture 1 (Goresky, Kottwitz and MacPherson [?]; independently Bezrukavnikov and Varshavsky [?]). *For any regular semisimple element $\gamma \in \mathfrak{g}(F)$ and any $i \in \mathbb{Z}_{\geq 0}$, the spherical part of the $\widetilde{W}$-action on $\mathrm{H}_i(\mathrm{Spr}_\gamma)$ and $\mathrm{H}^i_c(\mathrm{Spr}_\gamma)$*

$$\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W \to End(\mathrm{H}_i(\mathrm{Spr}_\gamma)), \quad \mathbb{Q}_\ell[\mathbb{X}_*(T)]^W \to End(\mathrm{H}^i_c(\mathrm{Spr}_\gamma))$$

factors through the action of $\pi_0(LG_\gamma)$ via the homomorphism (1).

In fact, the $\pi_0(LG_\gamma)$-action factors through a further quotient $\pi_0(P_{a(\gamma)})$, where $P_{a(\gamma)}$ is a certain quotient of LG_γ (see §2.6).

The difficulty in proving this conjecture lies in the fact that we do not know an effective way of computing the action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$: Lusztig's construction of the W_{aff}-action only tells us how each simple reflection acts, but elements in $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ are in general sums of complicated words in simple reflections.

The main purpose of this paper is to prove

Theorem 1 (Local Main Theorem). *Conjecture 1 holds for* $\mathrm{H}_c^i(\mathrm{Spr}_\gamma)$.

For the homology part of the conjecture, we prove a weaker statement.

Theorem 2. *Under the conditions of Conjecture 1, there exists a filtration* Fil^p *on* $\mathrm{H}_i(\mathrm{Spr}_\gamma)$*, stable under both* $\widetilde{W}$ *and* $\pi_0(LG_\gamma)$*, such that the action of* $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ *on* $Gr^p_{\mathrm{Fil}}\mathrm{H}_i(\mathrm{Spr}_\gamma)$ *factors through the action of* $\pi_0(LG_\gamma)$ *via the homomorphism* (1).

Moreover, one may choose Fil^p *such that* $Gr^p_{\mathrm{Fil}}\mathrm{H}_i(\mathrm{Spr}_\gamma) = 0$ *unless* $0 \le p \le r$*, where* r *is the split rank of the* F*-torus* $G(F)_\gamma$.

The above conjecture and results have a parahoric version. For each parahoric subgroup $\mathbf{P} \subset LG$ we have the affine partial flag variety $\mathrm{Fl}_{\mathbf{P}} = LG/\mathbf{P}$ and affine partial Springer fibers $\mathrm{Spr}_{\mathbf{P},\gamma}$. The subalgebra $\mathbf{1_P}\mathbb{Q}_\ell[\widetilde{W}]\mathbf{1_P} \cong \mathbb{Q}_\ell[\mathbb{X}_*(T)]^{W_{\mathbf{P}}}$ acts on $\mathrm{H}_*(\mathrm{Spr}_{\mathbf{P},\gamma})$ and $\mathrm{H}_c^*(\mathrm{Spr}_{\mathbf{P},\gamma})$ (see §2.5).

Proposition 1. *Let* $\mathbf{P} \subset LG$ *be any parahoric subgroup. Let* $\gamma \in \mathfrak{g}(F)$ *be a regular semisimple element. If Conjecture 1 holds for* $\mathrm{H}_i(\mathrm{Spr}_\gamma)$ *(resp.* $\mathrm{H}_c^i(\mathrm{Spr}_\gamma)$*), then the action of* $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W \subset \mathbf{1_P}\mathbb{Q}_\ell[\widetilde{W}]\mathbf{1_P}$ *on* $\mathrm{H}_i(\mathrm{Spr}_{\mathbf{P},\gamma})$ *(resp.* $\mathrm{H}_c^i(\mathrm{Spr}_{\mathbf{P},\gamma})$*) factors through the action of* $\pi_0(LG_\gamma)$ *via the homomorphism* (1).

In fact, the natural projection $\mathrm{Spr}_\gamma \to \mathrm{Spr}_{\mathbf{P},\gamma}$ induces a *surjection* $\mathrm{H}_*(\mathrm{Spr}_\gamma) \twoheadrightarrow \mathrm{H}_*(\mathrm{Spr}_{\mathbf{P},\gamma})$ and an *injection* $\mathrm{H}_c^*(\mathrm{Spr}_{\mathbf{P},\gamma}) \hookrightarrow \mathrm{H}_c^*(\mathrm{Spr}_\gamma)$, which are easily seen to be equivariant under both $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ and $\pi_0(LG_\gamma)$ by construction in §2.5.

Surprisingly, Theorem 1 is deduced from its global counterpart, which we state next. Fix a connected smooth projective algebraic curve X over k. In [**?**, Definition 2.1.1] we have defined the parabolic Hitchin moduli stack $\mathcal{M}^{\mathrm{par}}$ classifying quadruples $(x, \mathcal{E}, \varphi, \mathcal{E}_x^B)$ where $x \in X$, $\mathcal{E}$ is a G-bundle over X, φ is a section of the twisted adjoint bundle $\mathrm{Ad}(\mathcal{E}) \otimes \mathcal{O}_X(D)$ (D is a fixed divisor on X with large degree) and $\mathcal{E}_x^B$ is a Borel reduction of $\mathcal{E}$ at x preserved by the Higgs field φ. We have the parabolic Hitchin fibration

$$f^{\mathrm{par}} : \mathcal{M}^{\mathrm{par}} \to \mathcal{A}^{\mathrm{Hit}} \times X$$

recording the characteristic polynomial of φ and the point x. One of the main results of [**?**] is that there exists a natural action of the extended affine Weyl group $\widetilde{W}$ on the derived direct image complex $\mathbf{R}f^{\mathrm{par}}_{!}\mathbb{Q}_\ell|_{\mathcal{A}^{\mathrm{ani}}\times X}$, where $\mathcal{A}^{\mathrm{ani}}\subset\mathcal{A}^{\mathrm{Hit}}$ is the *anisotropic* locus (see [**?**, §4.10.5]). In this paper, we will extend this construction to a larger locus $\mathcal{A}^{\heartsuit}\times X$, where $\mathcal{A}^{\heartsuit}\subset\mathcal{A}^{\mathrm{Hit}}$ is the *hyperbolic* locus (see [**?**, §4.5]), containing $\mathcal{A}^{\mathrm{ani}}$ as an open subset.

On the other hand, Ngô defined a Picard stack $\mathcal{P}$ over $\mathcal{A}^{\heartsuit}$ which acts on $\mathcal{M}^{\mathrm{par}}$ fiber-wise over $\mathcal{A}^{\heartsuit}\times X$. This action induces an action of the sheaf of groups $\pi_0(\mathcal{P}/\mathcal{A}^{\heartsuit})$ (which interpolates the component groups $\pi_0(\mathcal{P}_a)$ for $a\in\mathcal{A}^{\heartsuit}$) on $\mathbf{R}f^{\mathrm{par}}_{!}\mathbb{Q}_\ell$. The study of this action in the case of the usual Hitchin moduli space $\mathcal{M}^{\mathrm{Hit}}$ leads to the geometric theory of endoscopy, which plays a crucial role in Ngô's proof of the Fundamental Lemma [**?**]. The idea of relating the $\pi_0(\mathcal{P}/\mathcal{A}^{\heartsuit})$-action and the $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$-action on $\mathbf{R}f^{\mathrm{par}}_{!}\mathbb{Q}_\ell$ was also suggested to the author by Ngô.

Theorem 3 (Global Main Theorem). *For any $i\in\mathbb{Z}_{\geq0}$, the spherical part of the $\widetilde{W}$-action on the cohomology sheaves of the parabolic Hitchin complex $\mathbf{R}f^{\mathrm{par}}_{!}\mathbb{Q}_\ell$:*

$$\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W\to End(\mathbf{R}^if^{\mathrm{par}}_{!}\mathbb{Q}_\ell|_{(\mathcal{A}^{\heartsuit}\times X)'})$$

factors through the action of the sheaf $\pi_0(\mathcal{P}/\mathcal{A}^{\heartsuit})$ via a natural homomorphism of sheaves of algebras on $\mathcal{A}^{\heartsuit}$:

$$\sigma:\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W\otimes\mathbb{Q}_{\ell,\mathcal{A}^{\heartsuit}}\to\mathbb{Q}_\ell[\pi_0(\mathcal{P}/\mathcal{A}^{\heartsuit})].\tag{2}$$

Here, $(\mathcal{A}^{\heartsuit}\times X)'\subset\mathcal{A}^{\heartsuit}\times X$ is any open subset on which a certain codimension estimate holds (see [**?**, Proposition 2.6.3, Remark 2.6.4]). If char(k)=0, we may take $(\mathcal{A}^{\heartsuit}\times X)'=\mathcal{A}^{\heartsuit}\times X$ (see [**?**, p.4]).

A consequence of Theorem 3 is

Corollary 1. *For any geometric point $(a,x)\in(\mathcal{A}^{\heartsuit}\times X)'$, the action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ on $\mathrm{H}^i_c(\mathcal{M}^{\mathrm{par}}_{a,x})$ factors through the action of $\pi_0(\mathcal{P}_a)$ via the stalk of the homomorphism* (2) *at a.*

We also have a version of Theorem 3 for parahoric Hitchin fibrations. Let us spell out the case of the usual Hitchin fibration $f^{\mathrm{Hit}}:\mathcal{M}^{\mathrm{Hit}}\to\mathcal{A}^{\mathrm{Hit}}$. In [**?**, Theorem 6.6.1], we constructed an action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ on the restriction of complex $\mathbf{R}f^{\mathrm{Hit}}_{!}\mathbb{Q}_\ell\boxtimes\mathbb{Q}_{\ell,X}$ to $\mathcal{A}^{\mathrm{ani}}\times X$. We will also extend this action to an action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ on the same complex over $\mathcal{A}^{\heartsuit}\times X$.

Theorem 4. *The action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ on the sheaf $\mathbf{R}^if^{\mathrm{Hit}}_{!}\mathbb{Q}_\ell\boxtimes\mathbb{Q}_{\ell,X}$ factors through the action of the sheaf of algebras $\mathbb{Q}_\ell[\pi_0(\mathcal{P}/\mathcal{A}^{\heartsuit})]$ via the homomorphism* (2).

Note that this theorem is valid on the whole of $\mathcal{A}^{\heartsuit} \times X$, rather than just $(\mathcal{A}^{\heartsuit} \times X)'$ as in Theorem 3. In the preprint [**?**], Theorem 3 and 4 were proved over $(\mathcal{A}^{\mathrm{ani}} \times X)'$.

1.1. Application

In recent work of Bezrukavnikov, Kazhdan and Varshavsky [**?**], they construct new examples of stable distributions on p-adic groups using an affine analog of the relation between character sheaves and the center of the Hecke algebra. When the local field is a function field, a key step in checking the stability of their distributions is Theorem 1 and Theorem 2, see [**?**, Theorem 5.4].

1.2. Idea of the proof

The main idea of proving Theorem 1 is to view $\mathcal{O}_F$ as the completed local ring of an algebraic curve at one point x, and try to deform the point x along the curve X. For this, we also need to extend $\gamma \in \mathfrak{g}(F)$ to a $\mathfrak{g}$-valued meromorphic function on X. This naturally leads to the consideration of the (parabolic) Hitchin moduli stack, hence leading to Theorem 3.

To prove Theorem 3, we first prove Theorem 4. The $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$-action on $\mathbf{R}f_!^{\mathrm{Hit}}\mathbb{Q}_\ell \boxtimes \mathbb{Q}_{\ell,X}$ can be thought of as a family of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$-actions on $\mathbf{R}f_!^{\mathrm{Hit}}\mathbb{Q}_\ell$ indexed by $x \in X$. Homotopy invariance guarantees that the effect of this action on $\mathbf{R}^i f_!^{\mathrm{Hit}}\mathbb{Q}_\ell$ is independent of x. Then we only need to check that the $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$-action at a general point $x \in X$ does factor through $\pi_0(\mathcal{P}/\mathcal{A}^{\heartsuit})$, which is clear from the construction in [**?**].

We then deduce Theorem 3 from a variant of Theorem 4, i.e., Proposition 4. We simultaneously deform the point of Borel reduction (which is contained in the moduli problem of $\mathcal{M}^{\mathrm{par}}$) and the point of Hecke modification (which gives the $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$-action). This way we get a result about the $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$-action on $\mathbf{R}f_!^{\mathrm{par}}\mathbb{Q}_\ell \boxtimes \mathbb{Q}_{\ell,X}$ analogous to Theorem 4, but this time our complex lives on $\mathcal{A}^{\heartsuit} \times X^2$. Restricting to the diagonal $\mathcal{A}^{\heartsuit} \times \Delta(X)$, we get the desired factorization in Theorem 3. The idea behind this argument is reminiscent of Gaitsgory's construction of the center of the affine Hecke algebra via nearby cycles (see [**?**]).

Finally, we deduce Theorem 1 from Theorem 3. We argue that for every affine Springer fiber Spr_γ, its compactly supported cohomology appears inside the compactly supported cohomology of a certain rigidified parabolic Hitchin fiber, and this inclusion respects the various symmetries. This fact follows from a detailed analysis of Ngô's product formula.

We remark that using a parabolic version of Ngô's Support Theorem (see [**?**]), one can deduce a weaker version of Theorem 3, namely the *semisimplification* of the $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$-action factors through the $\pi_0(\mathcal{P}/\mathcal{A}^{\mathrm{ani}})$-action on $\mathbf{R}f^{\mathrm{par}}_!\mathbb{Q}_\ell|_{\mathcal{A}^{\mathrm{ani}}\times X}$.

The proof described above shows that the global Springer theory developed in [**?**] can be useful in solving more classical problems about affine Springer fibers.

1.3. Convention

Throughout the paper, k will be an algebraically closed field. The stacks on which we talk about sheaves are of the form $[X/A]$, where X is an algebraic space, locally of finite type over k, and A is a linear algebraic group over k which acts on X. All complexes of sheaves will be objects in the derived category of $\mathbb{Q}_\ell$-complexes in the étale topology. See [**?**, §1.1] for the case of schemes of finite type and [**?**] for the case of stacks. See Appendix **??** for the convention for sheaves on algebraic spaces which are locally of finite type over k. **All sheaf-theoretic functors are understood to be derived without putting R or L in the front.** For a morphism $f:\mathcal{X}\to\mathcal{Y}$ between stacks, we use $\mathbb{D}_{\mathcal{X}/\mathcal{Y}}$ or $\mathbb{D}_f$ to denote the relative dualizing complex $f^!\mathbb{Q}_{\ell,\mathcal{Y}}$. The homology complex of f is defined as

$$\mathbf{H}_*(\mathcal{X}/\mathcal{Y}) := f_!\mathbb{D}_{\mathcal{X}/\mathcal{Y}}.$$

In particular, if $\mathcal{Y}=Speck$, we write $\mathrm{H}_*(\mathcal{X})$ for $\mathbf{H}_*(\mathcal{X}/Speck)$.

1.4. Notations for G

Let G be a reductive algebraic group over k. Fix a maximal torus T of G and a Borel B containing T. Let $\mathfrak{g},\mathfrak{b},\mathfrak{t}$ be the Lie algebras of G,B,T respectively. Let $(\mathbb{X}^*(T),\Phi,\mathbb{X}_*(T),\Phi^\vee)$ be the based root and coroot systems determined by (G,B,T). Let W be the Weyl group. Let $\mathfrak{c}=\mathfrak{g}/\!\!/G=\mathfrak{t}/\!\!/W$ be the adjoint quotient of $\mathfrak{g}$ in the GIT sense.

We now make precise the assumption on char (k). Let h be one plus the sum of coefficients of the highest root of G written in terms of simple roots. We assume either $\mathrm{char}(k)=0$ or $\mathrm{char}(k)>2h$. We impose this condition because we would like to make sure that the Kostant section $\epsilon:\mathfrak{c}\to\mathfrak{g}$ exists, see [**?**, §1.2].

The extended affine Weyl group and the affine Weyl group are defined as

$$\widetilde{W} := \mathbb{X}_*(T)\rtimes W; \qquad W_{\mathrm{aff}} := \mathbb{Z}\Phi^\vee\rtimes W$$

where $\mathbb{Z}\Phi^\vee\subset\mathbb{X}_*(T)$ is the coroot lattice. The affine Weyl group W_{aff} is a Coxeter group with the set of simple reflections Δ_{aff}. There is an exact sequence

$$1 \to W_{\mathrm{aff}} \to \widetilde{W} \to \Omega \to 1 \tag{3}$$

with $\Omega = \mathbb{X}_*(T)/\mathbb{Z}\Phi^\vee$ an abelian group (which is finite if G is semisimple).

2. Local Springer action

In this section, we will explain all the ingredients that go into the statement of Conjecture 1.

2.1. Loop group

Let $F = k((\varpi))$ and $\mathcal{O}_F = k[[\varpi]] \subset F$. For a scheme X over F, we use LX (the *formal loop space* of X) to denote the functor $\{k\text{-algebras}\} \to \{\text{Sets}\}$:

$$(LX)(R) = X(R((\varpi))).$$

Similarly, if $\mathcal{X}$ is defined over $\mathcal{O}_F$,we define $L^+\mathcal{X}$ (the *formal arc space* of $\mathcal{X}$) to be the functor

$$(L^+\mathcal{X})(R) = \mathcal{X}(R[[\varpi]]).$$

For notational brevity, we denote $L(G \otimes_k F)$ by LG. It is known that LG is represented by an ind-scheme over k, called the *loop group* of G (see [**?**, Discussion following Definition 1]). Similarly we denote $L^+(G \otimes_k \mathcal{O}_F)$ by L^+G.

For each parahoric subgroup $\mathbf{P} \subset LG$ there is a smooth $\mathcal{O}_F$-model $\mathcal{G}_\mathbf{P}$ of $G \otimes_k F$, the Bruhat-Tits group scheme, such that $\mathcal{G}_\mathbf{P}(\mathcal{O}_F) = \mathbf{P}$. We may form the functor $L^+\mathcal{G}_\mathbf{P}$, which is represented by a scheme. By abuse of notation, we denote this group scheme (of infinite type over k) by $\mathbf{P}$, and call them *parahoric subgroups* of LG. For example, L^+G is a parahoric subgroup of LG corresponding to the parahoric subgroup $G(\mathcal{O}_F)$ of LG. The standard Iwahori subgroup $\mathbf{I} \subset L^+G$ is the preimage of B under the evaluation map $L^+G \to G$. Standard parahoric subgroups $\mathbf{P} \supset \mathbf{I}$ are in bijection with proper subsets of Δ_{aff}.

The *Lie algebra Lie* $\mathbf{P}$ of $\mathbf{P}$ is, by definition, the Lie algebra of $\mathcal{G}_\mathbf{P}$, hence a finite free $\mathcal{O}_F$-module. Let $L_\mathbf{P}$ be maximal reductive quotient of the special fiber of $\mathcal{G}_\mathbf{P}$. This is a connected reductive group over k. The affine partial flag variety of type $\mathbf{P}$ is the ind-scheme

$$\mathrm{Fl}_\mathbf{P} = LG/\mathbf{P}.$$

Having fixed an Iwahori $\mathbf{I}$, the exact sequence (3) admits a section $\Omega \to \widetilde{W}$ whose image $\Omega_\mathbf{I}$ is the stabilizer of the fundamental alcove corresponding to $\mathbf{I}$ in the reduced building of LG. The group $\Omega_\mathbf{I} \subset \widetilde{W}$ can also be identified with $N(\mathbf{I})/\mathbf{I}$ where $N(\mathbf{I})$ is the normalizer of $\mathbf{I}$ in LG.

2.2. Affine Springer fibers

Let $\gamma \in \mathfrak{g}(F)$ be a regular semisimple element. The sub-ind-scheme $\widehat{\mathcal{S}\mathrm{pr}}_{\mathbf{P},\gamma} \subset LG$ is defined to have R-points

$$\widehat{\mathcal{S}\mathrm{pr}}_{\mathbf{P},\gamma}(R) = \{g \in G(R((\varpi)))|\mathrm{Ad}(g^{-1})\gamma \in R \otimes_k Lie\mathbf{P}\}.$$

The right $\mathbf{P}$-action on LG preserves $\widehat{\mathcal{S}\mathrm{pr}}_{\mathbf{P},\gamma}$, hence we can define the quotient

$$\mathcal{S}\mathrm{pr}_{\mathbf{P},\gamma} = \widehat{\mathcal{S}\mathrm{pr}}_{\mathbf{P},\gamma}/\mathbf{P} \subset \mathrm{Fl}_{\mathbf{P}}.$$

As defined above $\mathcal{S}\mathrm{pr}_{\mathbf{P},\gamma}$ is highly non-reduced. Its reduced structure $\mathrm{Spr}_{\mathbf{P},\gamma}$ is a scheme locally of finite type, see [**?**]. We call $\mathrm{Spr}_{\mathbf{P},\gamma}$ the *affine Springer fiber of* γ *with type* $\mathbf{P}$. When $\mathbf{P} = \mathbf{I}$, we often omit $\mathbf{I}$ from subscripts.

2.3. Lusztig's construction of the W_{aff}-action in [?]

Let $\mathfrak{l}_{\mathbf{P}}$ be the Lie algebra of $L_{\mathbf{P}}$ and $\widetilde{\mathfrak{l}}_{\mathbf{P}}$ be the Grothendieck simultaneous resolution of $\mathfrak{l}_{\mathbf{P}}$. We have a Cartesian diagram

$$\begin{tikzcd} \mathcal{S}\mathrm{pr}_{\gamma} \arrow[r, "ev_{\gamma}"] \arrow[d, "\nu_{\mathbf{P}}"] & {[\widetilde{\mathfrak{l}}_{\mathbf{P}}/L_{\mathbf{P}}]} \arrow[d, "\pi_{\mathbf{P}}"] \\ \mathcal{S}\mathrm{pr}_{\mathbf{P},\gamma} \arrow[r, "ev_{\mathbf{P},\gamma}"] & {[\mathfrak{l}_{\mathbf{P}}/L_{\mathbf{P}}]} \end{tikzcd} \tag{4}$$

Apply proper base change to the diagram (4), we get

$$\nu_{\mathbf{P}*}\mathbb{D}_{\mathrm{Spr}_{\gamma}} = ev^{!}_{\mathbf{P},\gamma}\pi_{\mathbf{P}*}\mathbb{D}_{[\widetilde{\mathfrak{l}}_{\mathbf{P}}/L_{\mathbf{P}}]}.$$

By classical Springer theory for the Lie algebra $\mathfrak{l}_{\mathbf{P}}$, there is a $W_{\mathbf{P}}$-action on $\pi_{\mathbf{P},*}\mathbb{D}_{[\widetilde{\mathfrak{l}}_{\mathbf{P}}/L_{\mathbf{P}}]}$ (to see this, we may identify $\mathbb{D}_{[\widetilde{\mathfrak{l}}_{\mathbf{P}}/L_{\mathbf{P}}]}$ with the constant sheaf on $[\widetilde{\mathfrak{l}}_{\mathbf{P}}/L_{\mathbf{P}}]$). Therefore $\nu_{\mathbf{P}*}\mathbb{D}_{\mathrm{Spr}_{\gamma}}$ also carries a $W_{\mathbf{P}}$-action. Since $\nu_{\mathbf{P}}$ is proper, $\mathrm{H}_*(\mathrm{Spr}_{\gamma}) = \mathbf{R}\Gamma_c(\mathrm{Spr}_{\mathbf{P},\gamma}, \nu_{\mathbf{P}*}\mathbb{D}_{\mathrm{Spr}_{\gamma}})$ also carries a $W_{\mathbf{P}}$-action.

For $\mathbf{P} \subset \mathbf{Q}$, Lusztig then argues that the $W_{\mathbf{Q}}$-action on $\mathrm{H}_*(\mathrm{Spr}_{\gamma})$ restricts to the $W_{\mathbf{P}}$-action defined both as above. Therefore, these $W_{\mathbf{P}}$-actions generate a W_{aff}-action on $\mathrm{H}_*(\mathrm{Spr}_{\gamma})$.

Replacing the dualizing complexes by the constant sheaves in the above discussion, we obtain an action of W_{aff} on $\mathrm{H}^*_c(\mathrm{Spr}_{\gamma})$.

2.4. The $\Omega_{\mathbf{I}}$-action

Viewing $\Omega_{\mathbf{I}}$ as the quotient $N(\mathbf{I})/\mathbf{I}$, we get a natural action of $\Omega_{\mathbf{I}}$ on $\mathrm{Fl} = \mathrm{Fl}_{\mathbf{I}}$ by right multiplication. We denote this action by $\omega \mapsto R_\omega$ for $\omega \in \Omega_{\mathbf{I}}$. This action preserves Spr_γ (because $\mathrm{Ad}(g)\mathbf{I} = \mathrm{Ad}(g\omega)\mathbf{I}$ for any $\omega \in N(\mathbf{I})$), hence $\omega \in \Omega_{\mathbf{I}}$ acts on $\mathrm{H}_*(\mathrm{Spr}_\gamma)$ from the left:

$$R^{-1}_{\omega,*} : \mathrm{H}_*(\mathrm{Spr}_\gamma) \to \mathrm{H}_*(\mathrm{Spr}_\gamma).$$

Similarly, right multiplication by $\omega \in N(\mathbf{I})$ sends one standard parahoric $\mathbf{P}$ to another standard parahoric $\omega^{-1}\mathbf{P}\omega$, and gives an isomorphism $\mathrm{Fl}_{\mathbf{P},\gamma} \xrightarrow{\sim} \mathrm{Fl}_{\omega^{-1}\mathbf{P}\omega,\gamma}$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Spr}_\gamma \arrow[r, "R_\omega"] \arrow[d, "\nu_{\mathbf{P}}"] & \mathrm{Spr}_\gamma \arrow[d, "\nu_{\omega^{-1}\mathbf{P}\omega}"] \\
\mathrm{Spr}_{\mathbf{P},\gamma} \arrow[r, "R_\omega"] & \mathrm{Spr}_{\omega^{-1}\mathbf{P}\omega,\gamma}
\end{tikzcd} \tag{5}$$

for any $\omega \in \Omega_{\mathbf{I}}$. This implies that $R^{-1}_{\omega,*}$ intertwines the action of $W_{\omega^{-1}\mathbf{P}\omega}$ and of $W_{\mathbf{P}}$ on $\mathrm{H}_*(\mathrm{Spr}_\gamma)$, via the isomorphism $\mathrm{Ad}(\omega) : W_{\omega^{-1}\mathbf{P}\omega} \to W_{\mathbf{P}}$. Similar remarks apply to $\mathrm{H}^*_c(\mathrm{Spr}_\gamma)$. Summarizing, we get

Theorem 5. *Lusztig's construction in §2.3 and the $\Omega_{\mathbf{I}}$-action in §2.4 together generate a $\widetilde{W}$ action on both $\mathrm{H}_*(\mathrm{Spr}_\gamma)$ and $\mathrm{H}^*_c(\mathrm{Spr}_\gamma)$.*

2.5. The parahoric version

For each standard parahoric $\mathbf{P}$, let $W_{\mathbf{P}} \subset W_{\mathrm{aff}}$ be the finite Weyl group of the Levi quotient $L_{\mathbf{P}}$. The Cartesian diagram (4) and proper base change implies that

$$\mathrm{H}_*(\mathrm{Spr}_{\mathbf{P},\gamma}) = \mathrm{H}_*(\mathrm{Spr}_\gamma)^{W_{\mathbf{P}}}, \qquad \mathrm{H}^*_c(\mathrm{Spr}_{\mathbf{P},\gamma}) = \mathrm{H}^*_c(\mathrm{Spr}_\gamma)^{W_{\mathbf{P}}}.$$

In fact, the constant sheaf on $\mathfrak{t}_{\mathbf{P}}$ is the $W_{\mathbf{P}}$-invariants of the Springer sheaf $\pi_{\mathbf{P},*}\mathbb{Q}_{\ell,\widetilde{\mathfrak{l}}_{\mathbf{P}}}$. Let

$$\mathbf{1}_{\mathbf{P}} = \frac{1}{\#W_{\mathbf{P}}} \sum_{w \in W_{\mathbf{P}}} w \in \mathbb{Q}_\ell[\widetilde{W}]$$

be an idempotent. Then the subalgebra $\mathbf{1}_{\mathbf{P}}\mathbb{Q}_\ell[\widetilde{W}]\mathbf{1}_{\mathbf{P}}$ acts on $\mathrm{H}_*(\mathrm{Spr}_{\mathbf{P},\gamma})$ and $\mathrm{H}^*_c(\mathrm{Spr}_{\mathbf{P},\gamma})$.

2.6. Local Picard

We follow [?, §3.3] in this subsection. For $\gamma \in \mathfrak{g}(\mathcal{O}_F)$, let $a(\gamma)$ be its image in $\mathfrak{c}(\mathcal{O}_F)$. Let $G_{F,\gamma}$ be the centralizer group scheme of γ in $G \otimes_k F$. In particular, $G_{F,\gamma}$ is a torus over F. We denote the loop space of $G_{F,\gamma}$ by LG_γ, which is a subgroup of LG. From the definition of Spr_γ, we see that LG_γ acts on $\mathrm{Spr}_{\mathbf{P},\gamma}$ via its left translation action on $\mathrm{Fl}_{\mathbf{P}}$.

The F-torus $G_{F,\gamma}$ admits a smooth model $J_{a(\gamma)}$ over $Spec\mathcal{O}_F$ which canonically only depends on $a(\gamma)$. This is the regular centralizer group scheme defined in [?, §3]. Let $\mathcal{P}_{a(\gamma)}$ be the affine Grassmannian of the $Spec\mathcal{O}_F$-group scheme $J_{a(\gamma)}$ (see [?, §3.8]):

$$\mathcal{P}_{a(\gamma)} := LJ_{a(\gamma)}/L^+J_{a(\gamma)} = LG_\gamma/L^+J_{a(\gamma)}.$$

For a k-algebra R, $\mathcal{P}_{a(\gamma)}(R)$ is the set of isomorphism classes of $J_{a(\gamma)}$-torsors over $SpecR[[\varpi]]$ together with a trivialization over $SpecR((\varpi))$. Since $J_{a(\gamma)}$ is commutative, $\mathcal{P}_{a(\gamma)}$ has a group ind-scheme structure. The action of LG_γ on $\mathrm{Spr}_{\mathbf{P},\gamma}$ factors through $\mathcal{P}_{a(\gamma)}$.

We have the finite type Néron model $J^\flat_{a(\gamma)}$ of $J_{a(\gamma)}$ (see [?, §3.8]). We define $\mathcal{P}^\flat_{a(\gamma)}$ similarly using $J^\flat_{a(\gamma)}$ instead of $J_{a(\gamma)}$. By [?, Lemme 3.8.1], the reduced structure of $\mathcal{P}^\flat_{a(\gamma)}$ is a free abelian group $\Lambda_{a(\gamma)}$. Let $P_{a(\gamma)} \hookrightarrow \mathcal{P}_{a(\gamma)}$ be the preimage of $\Lambda_{a(\gamma)} \hookrightarrow \mathcal{P}^\flat_{a(\gamma)}$, then we have an exact sequence

$$1 \to R_{a(\gamma)} \to P_{a(\gamma)} \to \Lambda_{a(\gamma)} \to 1.$$

where the kernel $R_{a(\gamma)}$ is an affine commutative group scheme of finite type.

2.7. Definition of the map σ_γ in (1)

Consider the following diagram

$$\begin{tikzcd} Spec\widetilde{\mathcal{O}}_F \arrow[r] \arrow[dr] & Spec\mathcal{O}_{F,a} \arrow[r] \arrow[d] & \mathfrak{t} \arrow[d] \\ & Spec\mathcal{O}_F \arrow[r, "a(\gamma)"] & \mathfrak{c} \end{tikzcd} \tag{6}$$

where the square on the right is Cartesian by definition. The morphism $Spec\mathcal{O}_{F,a} \to Spec\mathcal{O}_F$ is called the *local cameral cover*. The ring $\widetilde{\mathcal{O}}_F$ is the normalization of $\mathcal{O}_{F,a}$. Choose a component $Spec\widetilde{\mathcal{O}}^!_F \subset Spec\widetilde{\mathcal{O}}_F$. Let $W^! \subset W$ be the stabilizer of $Spec\widetilde{\mathcal{O}}^!_F$. According to [?, Prop. 3.9.2] (or rather its dual statement), the choice of $Spec\widetilde{\mathcal{O}}^!_F$ allows us to define a surjection

$$\mathbb{X}_*(T) \twoheadrightarrow \mathbb{X}_*(T)_{W^!} = \pi_0(LG_\gamma) \twoheadrightarrow \pi_0(P_{a(\gamma)}). \tag{7}$$

If we change the choice of $\widetilde{\mathcal{O}}^!_F$, the above map will differ by an action of W on $\mathbb{X}_*(T)$. In particular, taking the group algebras in (7) and restricting to $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$, the map

$$\sigma_\gamma : \mathbb{Q}_\ell[\mathbb{X}_*(T)]^W \to \mathbb{Q}_\ell[LG_\gamma] \to \mathbb{Q}_\ell[\pi_0(P_{a(\gamma)})]$$

is independent of any choice. The first of map above is the map in (1).

Proposition 2 (Local constancy). *Fix a regular semisimple $\gamma \in \mathfrak{g}(F)$ with $a(\gamma) \in \mathfrak{c}(\mathcal{O}_F)$. There is an integer $N > 0$ such that for any $\gamma' \in \mathfrak{g}(F)$ with $a(\gamma') \in \mathfrak{c}(\mathcal{O}_F)$ and $a(\gamma') \equiv a(\gamma) \mod \varpi^N$, there are isomorphisms*

$$\iota_{\mathcal{P}} : \mathcal{P}_\gamma \xrightarrow{\sim} \mathcal{P}_{\gamma'};$$
$$\iota : \mathcal{S}\mathrm{pr}_\gamma \xrightarrow{\sim} \mathcal{S}\mathrm{pr}_{\gamma'}$$

such that ι is equivariant under the $\mathcal{P}_\gamma$ and $\mathcal{P}_{\gamma'}$ actions via $\iota_{\mathcal{P}}$. Moreover, the isomorphism ι can be chosen so that both $\iota^* : \mathrm{H}^*_c(\mathrm{Spr}_\gamma) \to \mathrm{H}^*_c(\mathrm{Spr}_{\gamma'})$ and $\iota_* : \mathrm{H}_*(\mathrm{Spr}_\gamma) \xrightarrow{\sim} \mathrm{H}_*(\mathrm{Spr}_{\gamma'})$ are $\widetilde{W}$-equivariant.

Proof. We first deal with the case $a(\gamma) = a(\gamma')$. Since the field F has dimension ≤ 1, $\mathrm{H}^1(F, A) = 0$ for any torus A over F (see [**?**, Ch. X, end of §7]). In particular, if γ and γ' have the same image in $\mathfrak{c}^{rs}(F)$, they are conjugate by an element $g \in LG$, and the required isomorphisms are given by $\mathrm{Ad}(g)$.

By the above discussion, we may assume γ is the Kostant section of $a(\gamma)$ (see [**?**, §1.2]); similarly we may assume γ' is the constant section of $a(\gamma')$.

We need a variant of [**?**, Lemme 3.5.3] with $\mathfrak{g}(\mathcal{O}_F) = Lie\mathbf{G}$ replaced by $Lie\mathbf{I}$. For this, one only needs to use [**?**, Lemme 2.4.3] in place of [**?**, Lemme 2.1.1] in the argument. This variant of [**?**, Lemme 3.5.3] shows that $\mathcal{S}\mathrm{pr}_\gamma$ depends only on the centralizer $\mathcal{O}_F$-group scheme $G_{\mathcal{O}_F,\gamma} = J_{a(\gamma)}$ (recall $\gamma \in \mathfrak{g}(\mathcal{O}_F)$ comes from the Kostant section).

By [**?**, Lemme 3.5.2], there is an integer $N > 0$ such that the local cameral covers $\mathcal{O}_{F,a(\gamma)}$ and $\mathcal{O}_{F,a(\gamma')}$ are W-equivariantly isomorphic as $\mathcal{O}_F$-modules. By [**?**, Lemme 3.5.4], there exists $g \in G(\mathcal{O}_F)$ such that $\mathrm{Ad}(g)G_{\mathcal{O}_F,\gamma} = G_{\mathcal{O}_F,\gamma'}$ as subgroups of $G \otimes_k \mathcal{O}_F$. The isomorphism $\iota_{\mathcal{P}}$ is induced from $\mathrm{Ad}(g)$. The left translation $g : \mathrm{Fl} \to \mathrm{Fl}$ then induces an isomorphism $\iota : \mathrm{Spr}_\gamma \xrightarrow{\sim} \mathrm{Spr}_{\gamma'}$ intertwining the actions of $\mathcal{P}_{a(\gamma)}$ and $\mathcal{P}_{a(\gamma')}$. This proves the first statement of the Proposition.

To prove that ι_* and ι^* are $\widetilde{W}$-equivariant, one only needs to notice that under the left translation by $g \in G(\mathcal{O}_F)$, the diagrams (4) and (5) for γ map isomorphically to the corresponding diagrams for γ', at least after replacing the $\mathcal{S}$pr's by their reduced structures Spr. For the diagram (4), we remark that $\mathrm{Spr}_{\mathbf{P},\gamma}$ maps isomorphically to $\mathrm{Spr}_{\mathbf{P},\gamma'}$ under g, since they are images of Spr_γ and $\mathrm{Spr}_{\gamma'}$ under the projection $\mathrm{Fl} \to \mathrm{Fl}_{\mathbf{P}}$. Since these diagrams determine the $\widetilde{W}$-action by construction, the $\widetilde{W}$-equivariance follows.

3. Global Springer action: extension to the hyperbolic locus

In this section, we extend the $\widetilde{W}$-action on $f^{\mathrm{par}}_!\mathbb{Q}_\ell|_{\mathcal{A}^{\mathrm{ani}}\times X}$ constructed in [?] from the anisotropic locus $\mathcal{A}^{\mathrm{ani}}$ to the *hyperbolic locus* $\mathcal{A}^{\heartsuit}$.

3.1. The Hitchin moduli stack

We first recall the definition of the Hitchin moduli stack. Fix a divisor $D = 2D'$ on X with $\deg(D) \geq 2g_X$. The *Hitchin moduli stack* $\mathcal{M}^{\mathrm{Hit}} = \mathcal{M}^{\mathrm{Hit}}_{X,G,D}$ assigns to a k-scheme S the groupoid of *Hitchin pairs* $(\mathcal{E}, \varphi)$ where

- $\mathcal{E}$ is a (right) G-torsor over $X \times S$;
- φ is a section of the vector bundle $\mathrm{Ad}(\mathcal{E}) \otimes \mathcal{O}_X(D)$, where $\mathrm{Ad}(\mathcal{E}) = \mathcal{E} \overset{G}{\times} \mathfrak{g}$ is the adjoint bundle over $X \times S$.

It is well-known that $\mathcal{M}^{\mathrm{Hit}}$ is an algebraic stack.

Let $\mathfrak{c}_D$ be the affine space bundle $\mathrm{Tot}^\times(D) \overset{\mathbb{G}_m}{\times} \mathfrak{c}$ where $\mathrm{Tot}^\times(D)$ is the $\mathbb{G}_m$-torsor associated to the line bundle $\mathcal{O}_X(D)$. Let $\mathcal{A}^{\mathrm{Hit}} = \mathrm{H}^0(X, \mathfrak{c}_D)$ be the Hitchin base. We have the *Hitchin fibration*

$$f^{\mathrm{Hit}} : \mathcal{M}^{\mathrm{Hit}} \to \mathcal{A}^{\mathrm{Hit}}$$

which assigns $(\mathcal{E}, \varphi)$ the "invariant polynomials" of φ. Recall from [?, §4.5] that there is an open subset $\mathcal{A}^{\heartsuit} \subset \mathcal{A}^{\mathrm{Hit}}$ consisting of those sections $a : X \to \mathfrak{c}_D$ which generically lies in the regular semisimple locus $\mathfrak{c}^{rs}_D$. We call $\mathcal{A}^{\heartsuit}$ the *hyperbolic locus* of $\mathcal{A}^{\mathrm{Hit}}$.

3.2. Rigidified Hitchin moduli space

Fix a point $z \in X(k)\backslash D$. Let $\mathcal{A}^z \subset \mathcal{A}^{\mathrm{Hit}}$ be the open subset consisting of sections $a : X \to \mathfrak{c}_D$ such that $a(z) \in \mathfrak{c}^{rs}$ (since $z \notin D$, $\mathcal{O}(D)$ is canonically trivialized at z and $a(z)$ is a well-defined element in $\mathfrak{c}$). Clearly $\mathcal{A}^z \subset \mathcal{A}^\heartsuit$. Evaluating at z gives a morphism $ev_z : \mathcal{A}^z \to \mathfrak{c}^{rs}$. Define the W-torsor $\mathcal{B} \to \mathcal{A}^z$ by the Cartesian diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r] \arrow[d] & \mathfrak{t}^{rs} \arrow[d] \\ \mathcal{A}^z \arrow[r, "ev_z"] & \mathfrak{c}^{rs} \end{tikzcd}$$

Let $\widehat{\mathcal{M}}^{\mathrm{Hit}}$ be the functor which assigns to a k-scheme S the groupoid of tuples $(\mathcal{E}, \varphi, t_z, \iota_z)$ where

- $(\mathcal{E}, \varphi) \in \mathcal{M}^{\mathrm{Hit}}(S)$;
- Let $a : X \times S \to \mathfrak{c}_D$ be the image of $(\mathcal{E}, \varphi)$ under f^{Hit}, and $a(z) : \{z\} \times S \to \mathfrak{c}$. We require that the image of $a(z)$ lies in $\mathfrak{c}^{rs}$. Moreover, $t_z : S \to \mathfrak{t}^{rs}$ is a lifting of $a(z)$.
- $\iota_z : (\mathcal{E}, \varphi)|_{\{z\}\times S} \xrightarrow{\sim} (G \times S, t_z)$ is an isomorphism of Hitchin pairs over $\{z\} \times S$.

Forgetting t_z and ι_z we get a morphism $\widehat{\mathcal{M}}^{\mathrm{Hit}} \to \mathcal{M}^{\mathrm{Hit}}|_{\mathcal{A}^z}$, which can be factored as

$$\widehat{\mathcal{M}}^{\mathrm{Hit}} \to \mathcal{B} \times_{\mathcal{A}^{\mathrm{Hit}}} \mathcal{M}^{\mathrm{Hit}} = \mathcal{B} \times_{\mathcal{A}^z} (\mathcal{M}^{\mathrm{Hit}}|_{\mathcal{A}^z}) \to \mathcal{M}^{\mathrm{Hit}}|_{\mathcal{A}^z}. \tag{8}$$

The space in the middle $\mathcal{B} \times_{\mathcal{A}^{\mathrm{Hit}}} \mathcal{M}^{\mathrm{Hit}}$ classifies triples $(\mathcal{E}, \varphi, t_z)$ as described in the definition of $\widehat{\mathcal{M}}^{\mathrm{Hit}}$. Therefore the first arrow in (8) is a T-torsor: T acts on $\widehat{\mathcal{M}}^{\mathrm{Hit}}$ by changing the trivialization ι_z, and the centralizer of t_z in G is exactly T. The last arrow in (8) is a W-torsor because $\mathcal{B} \to \mathcal{A}^z$ is. It is easy to see that $\widehat{\mathcal{M}}^{\mathrm{Hit}} \to \mathcal{M}^{\mathrm{Hit}}|_{\mathcal{A}^z}$ is a $N_G(T)$-torsor.

Lemma 1. *The functor $\widehat{\mathcal{M}}^{\mathrm{Hit}}$ is represented by an algebraic space which is locally of finite type and smooth over k.*

Proof. Since the forget morphism $\widehat{\mathcal{M}}^{\mathrm{Hit}} \to \mathcal{M}^{\mathrm{Hit}}|_{\mathcal{A}^z}$ is a $N_G(T)$-torsor, it is in particular of finite type. Since $\mathcal{M}^{\mathrm{Hit}}$ is an algebraic stack which is locally of finite type and smooth over k, so is $\widehat{\mathcal{M}}^{\mathrm{Hit}}$.

It remains to show that the automorphism group of any geometric point $(\mathcal{E}, \varphi, t_z, \iota_z) \in \widehat{\mathcal{M}}^{\mathrm{Hit}}(K)$ is trivial ($K \supset k$ being any algebraically closed field). In fact, in [**?**, Proposition 4.11.2] it is shown that

$$Aut(\mathcal{E}, \varphi) \subset \mathrm{H}^0(X_K, J_a^\flat).$$

where $J_a^\flat$ is the (finite type) Néron model of the regular centralizer group scheme J_a over X_K. Let $q_a : X_a \to X_K$ be the cameral cover of X_K, then by [**?**, Corollaire 4.8.1], we have

$$\mathrm{H}^0(X_K, J_a^\flat) = \mathrm{H}^0(X_a, T)^W \subset (q_a^{-1}(z) \times T)^W = J_{a,z}^\flat = Aut((\mathcal{E}, \varphi)_z)$$

the last equality holds because $a(z) \in \mathfrak{c}^{rs}(K)$. Therefore there is no non-trivial automorphism of $(\mathcal{E}, \varphi)$ which preserves ι_z. This shows that the automorphism group of the triple $(\mathcal{E}, \varphi, t_z, \iota_z)$ is trivial.

We still have the Hitchin fibration (which is no longer proper even over the anisotropic locus)

$$\widehat{f}^{\mathrm{Hit}} : \widehat{\mathcal{M}}^{\mathrm{Hit}} \to \mathcal{B}.$$

3.3. Rigidified parahoric Hitchin moduli space

For each standard parahoric subgroup $\mathbf{P} \subset LG$, we have defined in [**?**, Definition 4.3.3] an algebraic stack $\mathcal{M}_{\mathbf{P}}$ classifying Hitchin pairs $(\mathcal{E}, \varphi)$ together with a $\mathbf{P}$-level structure $\mathcal{E}_x^{\mathbf{P}}$ of $\mathcal{E}$ at a varying point $x \in X$ such that φ is compatible with $\mathcal{E}_x^{\mathbf{P}}$. For the precise meaning of "$\mathbf{P}$-level structure" and "compatible" we refer the readers to [**?**, §4.3].

A particular case is when $\mathbf{P} = \mathbf{I}$, then $\mathcal{M}^{\mathrm{par}} := \mathcal{M}_{\mathbf{I}}$ is called the *parabolic Hitchin moduli stack*, which classifies quadruples $(\mathcal{E}, \varphi, x, \mathcal{E}_x^B)$ where $(\mathcal{E}, \varphi) \in \mathcal{M}^{\mathrm{Hit}}$, $x \in X$ and $\mathcal{E}_x^B$ is a B-reduction of $\mathcal{E}$ at x such that $\varphi(x) \in \mathrm{Ad}(\mathcal{E}_x^B) \otimes \mathcal{O}(D)_x$. Another special case is when $\mathbf{P} = \mathbf{G}$, then $\mathcal{M}_{\mathbf{G}} = \mathcal{M}^{\mathrm{Hit}} \times X$. For each $\mathbf{P}$, we have the parahoric Hitchin fibration

$$f_{\mathbf{P}} : \mathcal{M}_{\mathbf{P}} \to \mathcal{A}^{\mathrm{Hit}} \times X.$$

We now introduce the rigidified version $\widehat{\mathcal{M}}_{\mathbf{P}}$ of $\mathcal{M}_{\mathbf{P}}$ similar to $\widehat{\mathcal{M}}^{\mathrm{Hit}}$. Let $\widehat{\mathcal{M}}_{\mathbf{P}}$ be the stack classifying data $(\mathcal{E}, \varphi, x, \mathcal{E}_x^{\mathbf{P}}, t_z, \iota_z)$ where

- $(\mathcal{E}, \varphi, x, \mathcal{E}_x^{\mathbf{P}}) \in \mathcal{M}_{\mathbf{P}}$;
- x is disjoint from z and $a(z) \in \mathfrak{c}^{rs}$ ($a = f^{\mathrm{Hit}}(\mathcal{E}, \varphi) \in \mathcal{A}^{\mathrm{Hit}}$), $t_z \in \mathfrak{t}^{rs}$ lifts $a(z)$;
- ι_z is an isomorphism $(\mathcal{E}, \varphi)|_z \xrightarrow{\sim} (G, t_z)$ of Hitchin pairs at $\{z\}$ (cf. the definition of $\widehat{\mathcal{M}}^{\mathrm{Hit}}$).

The forgetful morphism

$$\widehat{\mathcal{M}}_{\mathbf{P}} \to \mathcal{B} \times_{\mathcal{A}^z} (\mathcal{M}_{\mathbf{P}}|_{\mathcal{A}^z}) \to \mathcal{M}_{\mathbf{P}}|_{\mathcal{A}^z}$$

is again an $N_G(T)$-torsor. Parallel to Lemma 1, $\widehat{\mathcal{M}}_{\mathbf{P}}$ is represented by an algebraic space which is locally of finite type and smooth over k. We also have morphisms

$$\widehat{f}_{\mathbf{P}} : \widehat{\mathcal{M}}_{\mathbf{P}} \to \mathcal{B} \times X^z \tag{9}$$

where $X^z = X \backslash \{z\}$. When $\mathbf{P} = \mathbf{I}$, we usually write the morphism (9) as

$$\widehat{f}^{\mathrm{par}} : \widehat{\mathcal{M}}^{\mathrm{par}} \to \mathcal{B} \times X^z.$$

For two parahorics $\mathbf{Q} \subset \mathbf{P}$, we have a forgetful morphism over $\mathcal{B} \times X^z$:

$$\widehat{\mathrm{For}}_{\mathbf{Q}}^{\mathbf{P}} : \widehat{\mathcal{M}}_{\mathbf{Q}} \to \widehat{\mathcal{M}}_{\mathbf{P}}$$

3.4. Construction of the $\widetilde{W}$-action

In this subsection we construct a $\widetilde{W}$-action on the direct image complex $\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_{\ell}$. Since T acts on $\widehat{\mathcal{M}}^{\mathrm{par}}$ and $\widehat{f}^{\mathrm{par}}$ is T-invariant, we can view $\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_{\ell}$ as an object in ind $D^b_T(\mathcal{B}\times X^z)$, where $D^b_T(\mathcal{B}\times X^z)$ is the T-equivariant derived category of $\mathbb{Q}_{\ell}$-complexes on $\mathcal{B}\times X^z$ (with trivial T-action). The construction of the $\widetilde{W}$-action is completely parallel to the case of f^{par} treated in [**?**, §5.1] and the affine Springer fiber case in §2.3.

For each standard parahoric $\mathbf{P}$, we have a Cartesian diagram

$$\begin{tikzcd} \widehat{\mathcal{M}}^{\mathrm{par}} \arrow[r, "ev^{\mathbf{P}}_{\mathbf{I}}"] \arrow[d, "\widehat{\mathrm{For}}^{\mathbf{P}}_{\mathbf{I}}"] & {[\widetilde{\underline{\mathfrak{l}}}_{\mathbf{P}}/\underline{L}_{\mathbf{P}}]_D} \arrow[d] \\ \widehat{\mathcal{M}}_{\mathbf{P}} \arrow[r, "ev_{\mathbf{P}}"] & {[\underline{\mathfrak{l}}_{\mathbf{P}}/\underline{L}_{\mathbf{P}}]_D} \end{tikzcd} \tag{10}$$

Here $\underline{L}_{\mathbf{P}}$ is a group scheme over X which is an inner form of $L_{\mathbf{P}}$. For precise definition, see [**?**, Equation (4.1)]. Similarly, we have the twisted versions $\underline{\mathfrak{l}}_{\mathbf{P}}$ and $\widetilde{\underline{\mathfrak{l}}}_{\mathbf{P}}$ of the Lie algebra $\mathfrak{l}_{\mathbf{P}}$ and its Grothendieck resolution $\widetilde{\mathfrak{l}}_{\mathbf{P}}$. So $[\widetilde{\underline{\mathfrak{l}}}_{\mathbf{P}}/\underline{L}_{\mathbf{P}}]$ and $[\underline{\mathfrak{l}}_{\mathbf{P}}/\underline{L}_{\mathbf{P}}]$ are stacks over X with natural $\mathbb{G}_m$-actions by dilation. Adding a subscript D means applying the twisted product $(-)\overset{\mathbb{G}_m}{\times}_X \mathrm{Tot}^{\times}(D)$ to these stacks.

With this diagram, we can define a $W_{\mathbf{P}}$-action on $\widehat{\mathrm{For}}^{\mathbf{P}}_{\mathbf{I},*}\mathbb{Q}_{\ell}\in D^b(\widehat{\mathcal{M}}_{\mathbf{P}})$ similarly as in §2.3 or [**?**, Construction 5.1.1]. Therefore we get a $W_{\mathbf{P}}$-action on the ind-object $\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_{\ell}=\widehat{f}_{\mathbf{P},!}\widehat{\mathrm{For}}^{\mathbf{P}}_{\mathbf{I},*}\mathbb{Q}_{\ell}$. As in the proof of [**?**, Theorem 5.1.2], these actions for various $\mathbf{P}$ are compatible, and they together give an action of W_{aff} on $\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_{\ell}$.

On the other hand, $\Omega_{\mathbf{I}}$ still acts on $\widehat{\mathcal{M}}^{\mathrm{par}}$ on the right, lifting its action on $\mathcal{M}^{\mathrm{par}}$ in [**?**, Corollary 4.3.4]. This gives an $\Omega_{\mathbf{I}}$-action on $\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_{\ell}$. Putting together with the W_{aff}-action, we get a $\widetilde{W}$-action on $\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_{\ell}$.

The diagram (10) implies

$$\widehat{f}_{\mathbf{P},!}\mathbb{Q}_{\ell}=(\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_{\ell})^{W_{\mathbf{P}}}.$$

Therefore we get an action of $\mathbf{1}_{\mathbf{P}}\mathbb{Q}_{\ell}[\widetilde{W}]\mathbf{1}_{\mathbf{P}}$ on $\widehat{f}_{\mathbf{P},!}\mathbb{Q}_{\ell}$.

3.5. Hecke correspondences

In [**?**, §3], we also have a construction of the $\widetilde{W}$-action on $f^{\mathrm{par}}_{!}\mathbb{Q}_{\ell}$ via Hecke correspondences. Here we extend the construction to the case of $\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_{\ell}$.

Recall we have a Hecke correspondence $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$, which is a self-correspondence of $\mathcal{M}^{\mathrm{par}}$ over $\mathcal{A}^{\mathrm{Hit}} \times X$. Over the locus $(\mathcal{A}^{\heartsuit} \times X)^{rs}$, there is a $\widetilde{W}$-action on $\mathcal{M}^{\mathrm{par}}|_{(\mathcal{A}^{\heartsuit}\times X)^{rs}}$. For each $\widetilde{w} \in \widetilde{W}$, the closure $\mathcal{H}_{\widetilde{w}}$ of the graph of the $\widetilde{w}$-action is a closed subspace of $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$.

Let $\overrightarrow{h}$ be the second projection from $\mathcal{H}\mathrm{ecke}^{\mathrm{par}}$ or $\mathcal{H}_{\widetilde{w}}$ to $\mathcal{M}^{\mathrm{par}}$. Let

$$\widehat{\mathcal{H}}\mathrm{ecke}^{\mathrm{par}} = \mathcal{H}\mathrm{ecke}^{\mathrm{par}} \times_{\overrightarrow{h},\mathcal{M}^{\mathrm{par}}} \widehat{\mathcal{M}}^{\mathrm{par}};$$
$$\widehat{\mathcal{H}}_{\widetilde{w}} = \mathcal{H}_{\widetilde{w}} \times_{\overrightarrow{h},\mathcal{M}^{\mathrm{par}}} \widehat{\mathcal{M}}^{\mathrm{par}}.$$

Then $\widehat{\mathcal{H}}\mathrm{ecke}^{\mathrm{par}}$ and $\widehat{\mathcal{H}}_{\widetilde{w}}$ can be viewed as self-correspondences of $\widehat{\mathcal{M}}^{\mathrm{par}}$ over $\mathcal{B} \times X^z$. In fact, $\widehat{\mathcal{H}}\mathrm{ecke}^{\mathrm{par}}$ parametrizes two Hitchin pairs with Borel reductions at a point $x \neq z$, an isomorphism of these Hitchin pairs on $X\backslash\{x\}$ and a rigidification ι_z of the second Hitchin pair at z (which then automatically gives a rigidification of the first Hitchin pair at z).

3.6. The subset $(\mathcal{B} \times X^z)'$

On the scheme $\mathcal{A}^{\heartsuit} \times X$, we have an upper semi-continuous function δ given by the local δ-invariants $\delta(a, x)$, see [**?**, §2.6]. Let $(\mathcal{A}^{\heartsuit} \times X)_{\delta}$ be the level set of this function. By [**?**, Proposition 2.6.3], for each $\delta_0 \geq 0$, as long as $\deg(D)$ is large enough, we have

$$codim_{\mathcal{A}^{\heartsuit}\times X}(\mathcal{A}^{\heartsuit} \times X)_{\delta} \geq \delta + 1, \text{ for all } \delta \leq \delta_0.$$

Fix such a D (depending on δ_0) in the definition of $\mathcal{M}^{\mathrm{Hit}}$. Let $(\mathcal{A}^{\heartsuit} \times X)' = \bigsqcup_{\delta\leq\delta_0}(\mathcal{A}^{\heartsuit} \times X)_{\delta}$, which is an open subscheme of $\mathcal{A}^{\heartsuit} \times X$. Let $(\mathcal{B} \times X^z)' = (\mathcal{A}^{\heartsuit} \times X)' \times_{(\mathcal{A}^{\heartsuit}\times X)} (\mathcal{B} \times X^z)$.

We will need the notion of Property (G-2) of a correspondence, as defined in [**?**, Definition A.6.1] and recalled in Appendix **??**. The following fact is an easy consequence of [**?**, Lemma 3.1.4]

Lemma 2. *Any algebraic subspace $\widehat{\mathcal{H}}' \subset \widehat{\mathcal{H}}\mathrm{ecke}^{\mathrm{par}}|_{(\mathcal{B}\times X^z)'}$ which is of finite type over $\widehat{\mathcal{M}}^{\mathrm{par}}$ via both projections satisfies Property (G-2) with respect to $(\mathcal{B} \times X^z)^{rs}$, as a self-correspondence of $\widehat{\mathcal{M}}^{\mathrm{par}}|_{(\mathcal{B}\times X^z)'}$.*

Using the formalism of cohomological correspondences in Appendix **??**, the fundamental class of $\widehat{\mathcal{H}}_{\widetilde{w}}$ gives a map

$$[\widehat{\mathcal{H}}_{\widetilde{w}}]_{\#} : \widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_{\ell} \to \widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_{\ell}.$$

in the category ind $D^b_T(\mathcal{B} \times X^z)$ (with T acting trivially on $\mathcal{B} \times X^z$).

Completely parallel to [**?**, Theorem 3.3.3] and [**?**, Proposition 5.2.1], we have

Proposition 3. *The assignment* $\widetilde{w} \mapsto [\widehat{\mathcal{H}}_{\widetilde{w}}]_{\#}$ *for* $\widetilde{w} \in \widetilde{W}$ *gives a left action of* $\widetilde{W}$ *on the restriction* $\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell|_{(\mathcal{B}\times X^z)'}$. *Moreover, this action coincides with the action constructed in §3.4.*

3.7. Global Picard stack

We recall some facts from [**?**, §4.8]. For a point $a \in \mathcal{A}^{\heartsuit}(k)$, we have a smooth commutative group scheme J_a over X, called the *regular centralizer group scheme*. The global Picard stack $\mathcal{P}_a$ is defined as the moduli stack of J_a-torsors on X. It acts on both $\mathcal{M}^{\mathrm{Hit}}_a$ and $\mathcal{M}^{\mathrm{par}}_{a,x}$ (for any $x \in X(k)$).

Because we work with the rigidified moduli spaces, it is more relevant to consider the group subscheme $J^z_a \subset J_a$ which fits into the exact sequence

$$1 \to J^z_a \to J_a \to i_{z,*}J_{a,z} \to 1 \tag{11}$$

Here $J_{a,z}$ is the fiber of J_a at z and $i_z : \{z\} \to X$ is the inclusion. Let $\widehat{\mathcal{P}}_a$ be the Picard stack of J^z_a-torsors over X. One may also view $\widehat{\mathcal{P}}_a$ as classifying a J_a-torsor over X together with a trivialization at z. Similar argument as in Lemma 1 shows that $\widehat{\mathcal{P}}_a$ is in fact a group scheme, locally of finite type and smooth over k. The exact sequence (11) gives a homomorphism of group schemes $J_{a,z} \to \widehat{\mathcal{P}}_a$, and an isomorphism of Picard stacks

$$\mathcal{P}_a \cong [\widehat{\mathcal{P}}_a/J_{a,z}].$$

As a varies in $\mathcal{A}^z$, $\{\widehat{\mathcal{P}}_a\}$ form a group scheme $\widehat{\mathcal{P}}_{\mathcal{A}^z}$ over $\mathcal{A}^z$. Let

$$\widehat{\mathcal{P}} = \mathcal{B} \times_{\mathcal{A}^z} \widehat{\mathcal{P}}_{\mathcal{A}^z}.$$

For $b = (a, t_z) \in \mathcal{B}$, the choice of t_z gives an isomorphism $J_{a,z} \xrightarrow{\sim} T$. Therefore we have an isomorphism of Picard stacks over $\mathcal{B}$:

$$\mathcal{B} \times_{\mathcal{A}} \mathcal{P} \cong [\widehat{\mathcal{P}}/T]$$

The group scheme $\widehat{\mathcal{P}}$ acts on both $\widehat{\mathcal{M}}^{\mathrm{Hit}}$ and its parahoric variants $\widehat{\mathcal{M}}_{\mathbf{P}}$ over $\mathcal{B}$.

Let $J^\flat_a$ be the finite-type Néron model of J_a over X (see [**?**, §4.8]), then there is an exact sequence

$$1 \to J^z_a \to J^\flat_a \to J_{a,z} \times \prod_{x\in\mathrm{Sing}(a)} R_{a,x} \to 1.$$

Here $R_{a,x}$ is an affine group scheme of finite type over $Speck = Speck(x)$, and $\mathrm{Sing}(a) \subset X$ is the locus where $a(x) \notin \mathfrak{c}^{rs}_D$. Let $\mathcal{P}^\flat_a$ be the Picard stack of $J_a^\flat$-torsors on X, and let $P_a^\flat$ be its coarse moduli space. From the above sequence we deduce an exact sequence

$$1 \to \mathrm{H}^0(X, J_a^\flat) \to J_{a,z} \times \prod_{x \in \mathrm{Sing}(a)} R_{a,x} \to \widehat{\mathcal{P}}_a \to P_a^\flat \to 1. \tag{12}$$

3.8. Definition of the map σ in (2)

Recall from [**?**, Definition 2.2.2] we have the *universal cameral cover* defined by the Cartesian diagram

$$\begin{tikzcd} \widetilde{\mathcal{A}}^{\mathrm{Hit}} \arrow[r, "ev"] \arrow[d, "q"] & \mathfrak{t}_D \arrow[d, "q_{\mathfrak{t}}"] \\ \mathcal{A}^{\mathrm{Hit}} \times X \arrow[r, "ev"] & \mathfrak{c}_D \end{tikzcd}$$

For $a \in \mathcal{A}^{\mathrm{Hit}}(k)$, the preimage $X_a := q^{-1}(\{a\} \times X)$ is the *cameral curve* of a. Let $\widetilde{\mathcal{A}}^{rs} \subset \widetilde{\mathcal{A}}^{\mathrm{Hit}}$ (resp. $(\mathcal{A}^\heartsuit \times X)^{rs} \subset \mathcal{A}^{\mathrm{Hit}} \times X$) be the preimage of $\mathfrak{t}_D^{rs}$ (resp. $\mathfrak{c}_D^{rs}$). Then $q^{rs} : \widetilde{\mathcal{A}}^{rs} \to (\mathcal{A}^\heartsuit \times X)^{rs}$ is a W-torsor.

Recall from [**?**, Second line of the proof of Proposition 3.2.1] that for each $\lambda \in \mathbb{X}_*(T)$, there is a canonical morphism $s_\lambda : \widetilde{\mathcal{A}}^{rs} \to \mathcal{P}$ (in [**?**] this map was defined over a larger open subset $\widetilde{\mathcal{A}}^0$ but we do not need this fact). Putting the various $\{s_\lambda\}_{\lambda \in \mathbb{X}_*(T)}$ together we get a morphism

$$s : \mathbb{X}_*(T) \times \widetilde{\mathcal{A}}^{rs} \to \mathcal{P}. \tag{13}$$

This gives a push-forward map on homology complexes

$$s_* : \mathbb{Q}_\ell[\mathbb{X}_*(T)] \otimes \mathbf{H}_*(\widetilde{\mathcal{A}}^{rs}/\mathcal{A}^\heartsuit) \to \mathbf{H}_*(\mathcal{P}/\mathcal{A}^\heartsuit)$$

which is W-invariant (W acts diagonally on the two factors on the LHS and acts trivially on the RHS). Therefore, it factors through the W-coinvariants of $\mathbb{Q}_\ell[\mathbb{X}_*(T)] \otimes \mathbf{H}_*(\widetilde{\mathcal{A}}^{rs}/\mathcal{A})$. In particular, if we restrict to $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$, the map s_* factors through a map

$$s'_* : \mathbb{Q}_\ell[\mathbb{X}_*(T)]^W \otimes \mathbf{H}_*(\widetilde{\mathcal{A}}^{rs}/\mathcal{A}^\heartsuit)_W \to \mathbf{H}_*(\mathcal{P}/\mathcal{A}^\heartsuit) \tag{14}$$

Since q^{rs} is a W-torsor, we have $\mathbf{H}_*(\widetilde{\mathcal{A}}^{rs}/\mathcal{A}^\heartsuit)_W = \mathbf{H}_*((\mathcal{A}^\heartsuit \times X)^{rs}/\mathcal{A}^\heartsuit)$. Since $(\mathcal{A}^\heartsuit \times X)^{rs} \to \mathcal{A}^\heartsuit$ has connected fibers, we get

$$\mathbf{H}_0(\widetilde{\mathcal{A}}^{rs}/\mathcal{A}^\heartsuit)_W = \mathbf{H}_0((\mathcal{A}^\heartsuit \times X)^{rs}/\mathcal{A}^\heartsuit) = \mathbb{Q}_{\ell,\mathcal{A}^\heartsuit}. \tag{15}$$

On the other hand,

$$\mathbf{H}_0(\mathcal{P}/\mathcal{A}^\heartsuit) = \mathbb{Q}_\ell[\pi_0(\mathcal{P}/\mathcal{A}^\heartsuit)]. \tag{16}$$

Therefore, the degree zero part of (14) gives the desired map σ in (2).

3.9. Rigidified version of the map s

In [**?**], the map s in (13) was used to describe the action of $\mathbb{X}_*(T)$ on the regular semisimple locus $\mathcal{M}^{\mathrm{par},rs}$ explicitly. We shall in this subsection define a rigidified version of the map s and use it to describe the action of $\mathbb{X}_*(T)$ on the regular semisimple locus of $\widehat{\mathcal{M}}^{\mathrm{par}}$. This construction will be used in the proof of Theorem 4 in §4.1.

When we work with the rigidified versions of Hitchin moduli spaces, we may similarly define a rigidified version of s

$$\widehat{s}:\mathbb{X}_*(T)\times\widetilde{\mathcal{A}}^{z,rs}\xrightarrow{s}\mathcal{G}r_J|_{X^z}\to\widehat{\mathcal{P}},$$

The last map is defined as follows: a point in $\mathcal{G}r_J$ over $x\in X^z$ is a J-torsor over X with a trivialization over $X-\{x\}$; since $z\in X-\{x\}$, this trivialization restricts to a trivialization at z, and hence defines a point in $\widehat{\mathcal{P}}$. Analogous to the map σ in (2), we obtain from this a homomorphism

$$\sigma_{\mathcal{B}}:\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W\to\mathbb{Q}_\ell[\pi_0(\widehat{\mathcal{P}}/\mathcal{B})]. \tag{17}$$

Since $\widehat{\mathcal{P}}\to\mathcal{P}\times_{\mathcal{A}^\heartsuit}\mathcal{B}$ is a T-torsor, $\pi_0(\widehat{\mathcal{P}}/\mathcal{B})$ is the pullback of $\pi_0(\mathcal{P}/\mathcal{A}^\heartsuit)$, and the map $\sigma_{\mathcal{B}}$ is simply the pullback of σ from $\mathcal{A}^\heartsuit$ to $\mathcal{B}$.

When $(b,y)\in\mathcal{B}\times X^z$ is such that its image $(a,y)\in(\mathcal{A}^z\times X^z)^{rs}$ ($a\in\mathcal{A}^z$ is the image of b), we have an action of $\mathbb{X}_*(T)$ on $\widehat{\mathcal{M}}^{\mathrm{par}}_{b,y}$, see [**?**, Proof of Proposition 3.2.1, especially equation (3.4)]. On the level of points, for $\widehat{m}\in\widehat{\mathcal{M}}^{\mathrm{par}}_{b,y}$ with image $\widetilde{a}\in\widetilde{\mathcal{A}}^{z,rs}$, the action of $\lambda\in\mathbb{X}_*(T)$ on it is given by

$$\lambda(\widehat{m})=\widehat{s}(\lambda,\widetilde{a})\cdot\widehat{m} \tag{18}$$

where the right side means the action of $\widehat{\mathcal{P}}_b$ on $\widehat{\mathcal{M}}^{\mathrm{par}}_{b,y}$. By the construction of $\widehat{\mathcal{H}}_\lambda$ ($\lambda\in\mathbb{X}_*(T)$) in §3.5, $\widehat{\mathcal{H}}_\lambda$ restricted to $(\mathcal{A}^z\times X^z)^{rs}$ is given by the graph of the λ-action given above. Therefore the action of $\mathbb{X}_*(T)\subset\widetilde{W}$ on $\mathrm{H}^*_c(\widehat{\mathcal{M}}^{\mathrm{par}}_{b,y})$, as part of the action constructed in Proposition 3, is induced from the geometric action of $\mathbb{X}_*(T)$ on $\widehat{\mathcal{M}}^{\mathrm{par}}_{b,y}$ by formula (18).

4. Proof of the global main theorem

4.1. Proof of Theorem 4

We first set up some notation. Fix $S\subset T$ to be any algebraic subgroup. Then T, hence S acts on $\widehat{\mathcal{M}}^{\mathrm{Hit}}$. The complex $\widehat{f}^{\mathrm{Hit}}_!\mathbb{Q}_\ell$ (as an ind-object of $D^b(\mathcal{B})$) carries a canonical S-equivariant structure, and can be viewed as an ind-object in the S-equivariant derived category $D^b_S(\mathcal{B})=D^b([\mathcal{B}/S])$, where S acts trivially on $\mathcal{B}$. Let $p_S : [\mathcal{B}/S] \to \mathcal{B}$ be the projection. Then we have the derived functor $p_{S,*} : D^b_S(\mathcal{B}) \to D^+(\mathcal{B})$ of taking S-equivariant cohomology. We briefly recall the definition of $p_{S,*}$. An object $K \in D^b_S(\mathcal{B})$ is, by definition, a Cartesian complex of sheaves $(K_n)_{n\geq 0}$ on the simplicial scheme $(S^n \times \mathcal{B})_n \geq 0$. Using the natural projection $(p_n)_{n\geq 0}$ from $(S^n \times \mathcal{B})_{n\geq 0}$ to the constant simplicial scheme $(\mathcal{B})_{n\geq 0}$, we obtain a simplicial object $(p_{n,*}K_n)_{n\geq 0}$ in $D^b(\mathcal{B})$. We then define $p_{S,*}K \in D^+(\mathcal{B})$ as the total complex associated with the simplicial object $(p_{n,*}K_n)_{n\geq 0}$. For details, see [**?**] and [**?**]. The functor $p_{S,*}$ naturally extends to the ind-completions ind $D^b_S(\mathcal{B}) \to$ ind $D^+(\mathcal{B})$. Fix i and S, we define

$$K := \mathbf{R}^i p_{S,*} \widehat{f}^{\mathrm{Hit}}_! \mathbb{Q}_\ell.$$

For each geometric point $b \in \mathcal{B}$, the stalk of K at b is

$$K_b = \mathrm{H}^i_c([\widehat{\mathcal{M}}^{\mathrm{Hit}}_b/S]).$$

Let $p : \mathcal{B} \times X^z \to \mathcal{B}$ be the projection. We would like to show that the action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ on $p^*K = K \boxtimes \mathbb{Q}_{\ell,X^z}$ factors through the action of $\mathbb{Q}_\ell[\pi_0(\widehat{\mathcal{P}}/\mathcal{B})]$ via the homomorphism (17). This is the same as saying that the homomorphism

$$\ker(\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W \xrightarrow{\sigma_{\mathcal{B}}} \mathbb{Q}_\ell[\pi_0(\widehat{\mathcal{P}}/\mathcal{B})]) \to \mathbf{R}^0 p_* \underline{End}(p^*K)$$

is zero. Since both the source and the target are sheaves, to show it is zero it suffices to show it stalkwise, i.e., we have to show the following result.

Lemma 3. *For any geometric point $(b, x) \in \mathcal{B} \times X^z$, the action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ on the stalk $(p^*K)_{b,x} \cong \mathrm{H}^i_c([\widehat{\mathcal{M}}^{\mathrm{Hit}}_b/S])$ is independent of $x \in X^z$, and it factors through the action of $\pi_0(\widehat{\mathcal{P}}_b) \cong \pi_0(\mathcal{P}_a)$ (where a is the image of b in $\mathcal{A}^{\mathrm{Hit}}$).*

Proof. By adjunction we have

$$p_*\underline{End}(p^*K) \cong \underline{Hom}(K, p_*p^*K) = \underline{Hom}(K, \mathrm{H}^*(X^z) \otimes K).$$

Taking $\mathcal{H}^0$, using the fact that $\mathrm{H}^0(X^z) = \mathbb{Q}_\ell$, we conclude that

$$\mathbf{R}^0 p_* \underline{End}(p^*K) \cong \mathcal{H}^0 \underline{End}(K).$$

The above isomorphism can be explicitly given as the restriction to $\mathcal{B} \times \{y\}$ for *any* $y \in X^z$:

$$\mathbf{R}^0 p_* \underline{End}(p^*K) \to \mathbf{R}^0 p_* i_{y,*} \underline{End}(i^*_y p^*K) = \mathcal{H}^0 \underline{End}(K)$$

where $i_y : \mathcal{B} \times \{y\} \hookrightarrow \mathcal{B} \times X^z$ is the inclusion.

Therefore, for any given geometric point $(b,x)\in\mathcal{B}\times X^z$, in order to show that the $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$-action on $(p^*K)_{b,x}=K_b$ factors through the $\pi_0(\widehat{\mathcal{P}}_b)$-action on K_b, it suffices to show that the $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$-action on $(p^*K)_{b,y}=K_b$ factors through the $\pi_0(\widehat{\mathcal{P}}_b)$-action on K_b, for *some* $y\in X^z$. In particular we may choose y such that $(a,y)\in\widetilde{\mathcal{A}}^{rs}$. We identify $K_b=\mathrm{H}^i_c([\widehat{\mathcal{M}}^{\mathrm{Hit}}_b/S])$ as the W-invariants of $\mathrm{H}^i_c([\widehat{\mathcal{M}}^{\mathrm{par}}_{b,y}/S])$. By construction, the action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ on K_b is the restriction of its action on $\mathrm{H}^i_c([\widehat{\mathcal{M}}^{\mathrm{par}}_{b,y}/S])$. Therefore it suffices to show that the $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$-action on $\mathrm{H}^i_c([\widehat{\mathcal{M}}^{\mathrm{par}}_{b,y}/S])$ factors through $\pi_0(\widehat{\mathcal{P}}_b)$.

For $\lambda\in\mathbb{X}_*(T)$, let $|\lambda|$ be the W-orbit of λ. Let $Av(\lambda):=\sum_{\lambda'\in|\lambda|}\lambda'\in\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$. Since $(a,y)\in\widetilde{\mathcal{A}}^{rs}$, the group $\mathbb{X}_*(T)$ acts on $\widehat{\mathcal{M}}^{\mathrm{par}}_{b,y}$ by the formula (18), from which we deduce that the action of $Av(\lambda)$ on $\mathrm{H}^i_c([\widehat{\mathcal{M}}^{\mathrm{par}}_{b,y}/S])$ is the same as the action of $\sigma_{\mathcal{B}}(Av(\lambda))\in\mathbb{Q}_\ell[\pi_0(\widehat{\mathcal{P}}_b)]$ on $\mathrm{H}^i_c([\widehat{\mathcal{M}}^{\mathrm{par}}_{b,y}/S])$. This shows that the $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$-action on $(p^*K)_{b,x}=K_b$ factors through the $\pi_0(\widehat{\mathcal{P}}_b)\cong\pi_0(\mathcal{P}_a)$-action on K_b.

For the original statement of Theorem 4, we take $S=T$. In this case, we have
$$K_b=(\mathbf{R}^i_T\widehat{f}^{\mathrm{Hit}}_!\mathbb{Q}_\ell)_b=\mathrm{H}^i_c(\mathcal{M}^{\mathrm{Hit}}_a).$$
The above discussion shows that the action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ on $(p^*K)_{b,x}\cong\mathrm{H}^i_c(\mathcal{M}^{\mathrm{Hit}}_a)$ factors through $\pi_0(\widehat{\mathcal{P}}_b)=\pi_0(\mathcal{P}_a)$ for any $x\in X^z$, therefore the action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ on the stalk $(\mathbf{R}^if^{\mathrm{Hit}}_!\mathbb{Q}_\ell\boxtimes\mathbb{Q}_{\ell,X})_{a,x}=\mathrm{H}^i_c(\mathcal{M}^{\mathrm{Hit}}_a)$ factors through $\pi_0(\mathcal{P}_a)$ whenever $a\in\mathcal{A}^z$ and $x\in X^z$.

For an arbitrary geometric point $(a,x)\in\mathcal{A}^{\heartsuit}\times X$, one can find a point $z\in X-\{x\}$ such that $a(z)$ is in the regular semi-simple locus of $\mathfrak{c}$. Therefore $(a,x)\in\mathcal{A}^z\times X^z$ for some $z\in X$, and over $\mathcal{A}^z\times X^z$ we already know the factorization result from the previous paragraph. Theorem 4 is proved.

4.2. Plan of the proof of Theorem 3

The rest of the section is devoted to the proof of Theorem 3. We shall consider another action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ on $\widehat{f}^{\mathrm{par}}_!\mathbb{Q}_\ell$ given by restricting the $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$-action on $\widehat{f}^{\mathrm{par}}_!\mathbb{Q}_\ell\boxtimes\mathbb{Q}_{\ell,X^z}$ to the diagonal $\mathcal{B}\times X^z\hookrightarrow\mathcal{B}\times(X^z)^2$. This latter action will be constructed using the Hecke modification at two points, see §4.3. This new action is easier seen to factor through the action of $\pi_0(\widehat{\mathcal{P}}/\mathcal{B})$, as we shall prove in Proposition 4 (and the proof is similar to that of Theorem 4). Finally we show that the original action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ on $\widehat{f}^{\mathrm{par}}_!\mathbb{Q}_\ell$ coincides with the new one, finishing the proof.

4.3. Hecke modification at "another" point

This subsection provides preparatory tools for proving Theorem 3.

Consider the correspondence

$$\begin{tikzcd} & \widehat{\mathcal{H}}\mathrm{ecke}_2 \arrow[dl, "\overleftarrow{h_2}"'] \arrow[dr, "\overrightarrow{h_2}"] & \\ \widehat{\mathcal{M}}^{\mathrm{par}} \times X^z \arrow[r, "\widehat{f}^{\mathrm{par}}\times id"'] & \mathcal{B} \times (X^z)^2 & \widehat{\mathcal{M}}^{\mathrm{par}} \times X^z \arrow[l, "\widehat{f}^{\mathrm{par}}\times id"] \end{tikzcd} \tag{19}$$

For any scheme S, $\widehat{\mathcal{H}}\mathrm{ecke}_2(S)$ is the isomorphism classes of tuples

$$(x, y, \mathcal{E}_1, \varphi_1, t_{1,z}, \iota_{1,z}, \mathcal{E}^B_{1,x}, \mathcal{E}_2, \varphi_2, t_{2,z}, \iota_{2,z}, \mathcal{E}^B_{2,x}, \tau)$$

where

- $(x, \mathcal{E}_i, \varphi_i, t_{i,z}, \iota_{i,z}, \mathcal{E}^B_{i,x}) \in \widehat{\mathcal{M}}^{\mathrm{par}}(S)$, for $i = 1, 2$;
- $y \in X(S)$ with graph $\Gamma(y)$;
- τ is an isomorphism of objects on $S \times X^z - \Gamma(y)$:

$$\tau : (\mathcal{E}_1, \varphi_1, t_{1,z}, \iota_{1,z})|_{S\times X^z-\Gamma(y)} \xrightarrow{\sim} (\mathcal{E}_2, \varphi_2, t_{2,z}, \iota_{2,z})|_{S\times X^z-\Gamma(y)}.$$

For a point $(b, x, y) \in (\mathcal{B} \times (X^z)^2)(k)$ such that $x \neq y$, the fibers of $\overleftarrow{h_2}$ and $\overrightarrow{h_2}$ over (b, x, y) are isomorphic to the product of $\mathrm{Spr}_{\mathbf{G},\gamma_{a,y}}$ and a Springer fiber in G/B corresponding to the image of $\gamma_{a,x}$ in $\mathfrak{g}$ (here $\gamma_{a,x} \in \mathfrak{g}(\mathcal{O}_x)$ and $\gamma_{a,y} \in \mathfrak{g}(\mathcal{O}_y)$ are Kostant sections of a in the formal neighborhood of x and y; see the discussion in §**??**). If we restrict to the diagonal $\Delta_{X^z} : \mathcal{B} \times X^z \subset \mathcal{B} \times (X^z)^2$, $\widehat{\mathcal{H}}\mathrm{ecke}_2|_{\Delta_{X^z}}$ is the same as $\widehat{\mathcal{H}}\mathrm{ecke}^{\mathrm{par}}$. The reader may notice the analogy between our situation and the situation considered by Gaitsgory in [**?**], where he uses Hecke modifications at two points to deform the product $\mathcal{G}r_G \times G/B$ to Fl_G.

Let

$$\widetilde{\mathcal{B}} = \widetilde{\mathcal{A}}^{\mathrm{Hit}} \times_{\mathcal{A}^{\mathrm{Hit}}} \mathcal{B}, \qquad \widetilde{\mathcal{B}}^{rs} = \widetilde{\mathcal{A}}^{rs} \times_{\mathcal{A}^{\mathrm{Hit}}} \mathcal{B}.$$

The morphism $\widehat{f}^{\mathrm{par}}$ admits an enhancement $\widehat{\widetilde{f}} : \widehat{\mathcal{M}}^{\mathrm{par}} \to \widetilde{\mathcal{B}}$ analogous to the enhanced Hitchin vibration $\widetilde{f} : \mathcal{M}^{\mathrm{par}} \to \widetilde{\mathcal{A}}$ in [**?**, Eq.(2.2)]. Therefore we have a morphism

$$\widehat{\mathcal{H}}\mathrm{ecke}_2 \to \widehat{\mathcal{M}}^{\mathrm{par}} \times_{(\mathcal{B}\times X^z)} \widehat{\mathcal{M}}^{\mathrm{par}} \xrightarrow{\widehat{\widetilde{f}},\widehat{\widetilde{f}}} \widetilde{\mathcal{B}} \times_{(\mathcal{B}\times X^z)} \widetilde{\mathcal{B}}.$$

Let $\widehat{\mathcal{H}}\mathrm{ecke}_{2,[e]}$ be the preimage of the diagonal $\widetilde{\mathcal{B}} \subset \widetilde{\mathcal{B}} \times_{\mathcal{B}\times X^z} \widetilde{\mathcal{B}}$. One the other hand, we have the Hecke correspondence $\widehat{\mathcal{H}}\mathrm{ecke}^{\mathrm{Hit}}$ of $\widehat{\mathcal{M}}^{\mathrm{Hit}} \times X^z$ which modifies the Hitchin pair at one point. We have a commutative diagram of correspondences where the horizontal maps are given by forgetting the B-reductions.

If we restrict the left column above to $\widetilde{\mathcal{B}}^{rs} \times X^z$, all squares in the above diagram are Cartesian. Recall from [**?**, Construction 6.6.3] that for each W-orbit $|\lambda|$ in $\mathbb{X}_*(T)$, we have a graph-like closed substack $\mathcal{H}_{|\lambda|} \subset \mathcal{H}\mathrm{ecke}^{\mathrm{Hit}}$. Similarly, in the rigidified setting, we have $\widehat{\mathcal{H}}_{|\lambda|} \subset \widehat{\mathcal{H}}\mathrm{ecke}^{\mathrm{Hit}}$. Denote by $\widetilde{p}^{rs}_{\widehat{\mathcal{H}}} : \widehat{\mathcal{H}}\mathrm{ecke}_{2,[e]}|_{\widetilde{\mathcal{B}}^{rs}\times X^z} \to \widehat{\mathcal{H}}\mathrm{ecke}^{\mathrm{Hit}}$ the restriction of $\widetilde{p}_{\widehat{\mathcal{H}}}$ to $\widetilde{\mathcal{B}}^{rs} \times X^z$. Let $\widehat{\mathcal{H}}_{2,|\lambda|} \subset \widehat{\mathcal{H}}\mathrm{ecke}_{2,[e]}$ be closure of $\widetilde{p}^{rs,-1}_{\widehat{\mathcal{H}}}(\widehat{\mathcal{H}}_{|\lambda|})$. Using the formalism of cohomological correspondences in Appendix **??**, the fundamental class $[\widehat{\mathcal{H}}_{2,|\lambda|}]$ induces a map

$$[\widehat{\mathcal{H}}_{2,|\lambda|}]_{\#} : \widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell \boxtimes \mathbb{Q}_{\ell,X^z} \to \widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell \boxtimes \mathbb{Q}_{\ell,X^z}. \tag{20}$$

and an endomorphism of S-equivariant cohomology sheaves (in the notation set up in the beginning of §4.1)

$$[\widehat{\mathcal{H}}_{2,|\lambda|}]_{\#} : \mathbf{R}^i_S\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell \boxtimes \mathbb{Q}_{\ell,X^z} \to \mathbf{R}^i_S\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell \boxtimes \mathbb{Q}_{\ell,X^z}. \tag{21}$$

Proposition 4. *The endomorphism $[\widehat{\mathcal{H}}_{2,|\lambda|}]_{\#}$ on $\mathbf{R}^i_S\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell \boxtimes \mathbb{Q}_{\ell,X^z}$ in* (21) *factors through the action of $\sigma_{\mathcal{B}}(Av(\lambda)) \in \mathbb{Q}_\ell[\pi_0(\widehat{\mathcal{P}}/\mathcal{B})]$ on the first factor of $\mathbf{R}^i_S\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell \boxtimes \mathbb{Q}_{\ell,X^z}$. Here $Av(\lambda) = \sum_{\lambda'\in|\lambda|}\lambda' \in \mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$.*

Equivalently, for any geometric point $(b,x,y) \in \mathcal{B}\times(X^z)^2$, the effect of (21) *on the stalk $(\mathbf{R}^i_S\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell \boxtimes \mathbb{Q}_{\ell,X^z})_{b,x,y} \cong \mathrm{H}^i_c([\widehat{\mathcal{M}}^{\mathrm{par}}_{b,x}/S])$ is independent of y, and factors through the action of $\pi_0(\widehat{\mathcal{P}}_b) \cong \pi_0(\mathcal{P}_a)$ (where $a \in \mathcal{A}^z$ is the image of b).*

Proof. The argument is completely analogous to that of Lemma 3. We first use the adjunction for the projection onto the first two coordinates $p : (\mathcal{B}\times(X^z)^2)' \to (\mathcal{B}\times X^z)'$ to show that the action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ on the stalk $(\mathbf{R}^i_S\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell \boxtimes \mathbb{Q}_{\ell,X^z})_{b,x,y} \cong \mathrm{H}^i_c([\widehat{\mathcal{M}}^{\mathrm{par}}_{b,x}/S])$ is independent of y. Then we choose $y \in X^z$ such that $(a,y) \in \widetilde{\mathcal{A}}^{rs}$ and calculate the action of $Av(\lambda)$ on $(\mathbf{R}^i_S\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell \boxtimes \mathbb{Q}_{\ell,X^z})_{b,x,y}$ using the formula (18).

Remark 1. One can show that the assignment $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W \ni Av(\lambda) \mapsto [\widehat{\mathcal{H}}_{2,|\lambda|}]_{\#}$ extends by linearity to an *algebra* action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ on $(\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell\boxtimes\mathbb{Q}_{\ell,X^z})|_{(\mathcal{B}\times(X^z)^2)'}$, where $(\mathcal{B}\times(X^z)^2)' = (\mathcal{B}\times X^z)'\times_{\mathcal{B}}(\mathcal{B}\times X^z)'$. This can be deduced from the (G-2) property of the correspondence $\widehat{\mathcal{H}}\mathrm{ecke}_2$. We shall not use this fact.

4.4. Proof of Theorem 3

We consider two actions of $Av(\lambda)$ on $\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell|_{(\mathcal{B}\times X^z)'}$:

- Action α_1: this is given by the restriction of the $\widetilde{W}$-action constructed in §3.4. This is the action involved in the statement of Theorem 3.
- Action α_Δ: this is given by restricting the action $[\widehat{\mathcal{H}}_{2,|\lambda|}]_\#$ in (20) to the diagonal $\Delta_{X^z} : (\mathcal{B}\times X^z)' \hookrightarrow (\mathcal{B}\times (X^z)^2)'$. Note that $\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell = \Delta^*_{X^z}(\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell \boxtimes \mathbb{Q}_{\ell,X^z})$.

We claim that the actions α_1 and α_Δ are the same. Recall from [**?**, Appendix A.4] the notion of the pullback of a cohomological correspondence, which extends to the situation of cohomological correspondences over algebraic spaces locally of finite type. See the discussion at the end of Appendix **??**. By [**?**, Lemma A.4.1], the action α_Δ of $Av(\lambda)$ is given by the pullback class $\Delta^*_{X^z}[\widehat{\mathcal{H}}_{2,|\lambda|}] \in \mathrm{Corr}(\widehat{\mathcal{H}}';\mathbb{Q}_\ell,\mathbb{Q}_\ell)$, where $\widehat{\mathcal{H}}' \subset \widehat{\mathcal{H}}\mathrm{ecke}^{\mathrm{par}}$ is a large enough closed sub-correspondence with both maps to $\widehat{\mathcal{M}}^{\mathrm{par}}$ proper. For the notation $\mathrm{Corr}(-;-,-)$, see Appendix **??**. Let $\Delta^{rs}_{X^z} : (\mathcal{B}\times X^z)^{rs} \hookrightarrow (\mathcal{B}\times (X^z)^2)^{rs}$ be the restriction of Δ_{X^z}. Over $(\mathcal{B}\times (X^z)^2)^{rs}$, $\widehat{\mathcal{H}}^{rs}_{2,|\lambda|}$ is finite étale over $\widehat{\mathcal{M}}^{\mathrm{par}}\times X^z$ via both projections, therefore $\Delta^{rs,*}_{X^z}[\widehat{\mathcal{H}}^{rs}_{2,|\lambda|}] = [\widehat{\mathcal{H}}^{rs}_{2,|\lambda|}|_{\Delta^{rs}_{X^z}}]$, the fundamental class of the base change of $\widehat{\mathcal{H}}^{rs}_{2,|\lambda|}$ via $\Delta^{rs}_{X^z}$. However, restricting the left column of the diagram (**??**) to $\widetilde{\mathcal{B}}^{rs} \hookrightarrow \widetilde{\mathcal{B}}\times X^z$ (as the graph of the projection $\widetilde{\mathcal{B}}^{rs}\to X^z$), we get

$$\Delta^{rs,-1}_{X^z}(\widehat{\mathcal{H}}^{rs}_{2,|\lambda|}) = \bigsqcup_{\lambda'\in|\lambda|}\widehat{\mathcal{H}}^{rs}_{\lambda'}. \tag{22}$$

On the other hand, the action α_1 of $Av(\lambda)$ is given by the class $\sum_{\lambda'\in|\lambda|}[\widehat{\mathcal{H}}_{\lambda'}] \in \mathrm{Corr}(\widehat{\mathcal{H}}';\mathbb{Q}_\ell,\mathbb{Q}_\ell)$ (enlarge the correspondence $\widehat{\mathcal{H}}'$ defined before to contain all the $\widehat{\mathcal{H}}_{\lambda'}$ with $\lambda'\in|\lambda|$). Over $(\mathcal{B}\times X^z)'$, both classes $\Delta^*_{X^z}[\widehat{\mathcal{H}}_{2,|\lambda|}]$ and $\sum_{\lambda'\in|\lambda|}[\widehat{\mathcal{H}}_{\lambda'}]$ are supported on $\widehat{\mathcal{H}}'$, which has property (G-2) by Lemma 2. Applying Lemma **??**, the equality (22) implies

$$(\Delta^*_{X^z}[\widehat{\mathcal{H}}_{2,|\lambda|}])_\# = \sum_{\lambda'\in|\lambda|}[\widehat{\mathcal{H}}_{\lambda'}]_\#$$

as endomorphisms of $\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell|_{(\mathcal{B}\times X^z)'}$. This proves the claim.

By Proposition 4, the action α_Δ of $Av(\lambda)$ on $\mathbf{R}^i_S\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell|_{(\mathcal{B}\times X^z)'} = \Delta^*_{X^z}(\mathbf{R}^i_S\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell\boxtimes\mathbb{Q}_{\ell,X^z}|_{(\mathcal{B}\times(X^z)^2)'})$ factors through (17). Since the action α_1 on $\mathbf{R}^i_S\widehat{f}^{\mathrm{par}}_{!}\mathbb{Q}_\ell|_{(\mathcal{B}\times X^z)'}$ is the same as α_Δ, it also factors through (17).

To finish the proof of Theorem 3, we take $S=T$ and argue as in the final part of §4.1. Taking the stalk at a point of $(\mathcal{B}\times X^z)'$, we conclude:

Corollary 2. *Let $S \subset T$ be an algebraic subgroup and $(b, x) \in (\mathcal{B} \times X^z)'(k)$. Then for each i the action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ on $\mathrm{H}_c^i([\widehat{\mathcal{M}}_{b,x}^{\mathrm{par}}/S])$ factors through the action of $\pi_0(\widehat{\mathcal{P}}_b) = \pi_0(\mathcal{P}_a)$ via σ.*

5. From global to local

In this section, we prove the local main theorem (Theorem 1) and Theorem 2.

5.1. Initial reduction

We use the notations from §2.7. Suppose $G \to G'$ is a central isogeny with *finite* kernel. Let $\gamma \in \mathfrak{g}(F)$ be a regular semisimple element and γ' its image in $\mathfrak{g}'(F)$. We claim that the validity of Theorem 1 for γ' implies its validity for γ. For a notation $(-)$ attached to G, we use the notation $(-)'$ to denote its counterpart for G'. The induced morphism between affine flag varieties $\mathrm{Fl}_G \to \mathrm{Fl}_{G'}$ identifies Fl_G homeomorphically with a union of connected components of $\mathrm{Fl}_{G'}$. This morphism restricts to a morphism $\theta : \mathrm{Spr}_\gamma \to \mathrm{Spr}_{\gamma'}$, which also identifies Spr_γ with a union of connected components of $\mathrm{Spr}_{\gamma'}$. Note that $\widetilde{W}$ can be identified with a subgroup of $\widetilde{W}'$, under which the surjective map $\theta^* : \mathrm{H}^*(\mathrm{Spr}_{\gamma'}) \twoheadrightarrow \mathrm{H}^*(\mathrm{Spr}_\gamma)$, and its H_c^* analog, are both $\widetilde{W}$-equivariant. Finally, the construction in §2.7 gives a commutative diagram

$$\begin{array}{ccc} \mathbb{Q}_\ell[\mathbb{X}_*(T)]^W & \longrightarrow & \mathbb{Q}_\ell[\pi_0(LG_\gamma)] \\ \downarrow & & \downarrow \\ \mathbb{Q}_\ell[\mathbb{X}_*(T')]^{W'} & \longrightarrow & \mathbb{Q}_\ell[\pi_0(LG'_{\gamma'})] \end{array}$$

Therefore, if the action of $\mathbb{Q}_\ell[\mathbb{X}_*(T')]^{W'}$ on $\mathrm{H}^*(\mathrm{Spr}_{\gamma'})$ or $\mathrm{H}_c^*(\mathrm{Spr}_{\gamma'})$ factors through $\mathbb{Q}_\ell[\pi_0(LG'_{\gamma'})]$, so does the action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$. Hence the action of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$ on the quotient space $\mathrm{H}^*(\mathrm{Spr}_\gamma)$ or $\mathrm{H}_c^*(\mathrm{Spr}_\gamma)$ also factors through $\mathbb{Q}_\ell[\pi_0(LG'_{\gamma'})]$, and hence through $\mathbb{Q}_\ell[\pi_0(LG_\gamma)]$.

We apply the above discussion to the central isogeny $G \to G' = G^{\mathrm{ad}} \times G^{\mathrm{ab}}$, where G^{ad} is the adjoint form of G and $G^{\mathrm{ab}} = G/[G, G]$ is the abelianization of G. We then reduce to the case where G is a product of an adjoint group and a torus. Since Theorem 1 trivially holds for tori, we reduce to the case where G is of adjoint type. In the sequel we will assume that G is of adjoint type.

5.2. The local data

Assume $a(\gamma) \in \mathfrak{c}(\mathcal{O}_F)$, otherwise the affine Springer fiber is empty. Recall the local cameral cover diagram (6). As before, we choose a component $Spec\widetilde{\mathcal{O}}^!_F$ of $Spec\widetilde{\mathcal{O}}_F$ with stabilizer $W^!$. By our assumption on $\mathrm{char}(k)$, $F^! = Frac(\widetilde{\mathcal{O}}^!_F)$ is a tamely ramified extension of F, hence $W^! = Gal(F^!/F)$ is a cyclic subgroup of W. Fix a generator w of $W^!$ and a primitive m-th root of unity $\zeta \in \mu_m(k)$, where $m = \#W^!$.

5.3. The global data

Let $X = \mathbb{P}^1$. Fix $z \in X(k)\backslash\{0,\infty\}$. Let $\widetilde{X}$ be the contracted product $\widetilde{X} = W \overset{W^!}{\times} \mathbb{P}^1$, where w acts on $\mathbb{P}^1$ by $t \mapsto \zeta t$. We take $\widetilde{X}^!$ to be the component $\{1\} \times \mathbb{P}^1 \subset \widetilde{X}$. The morphism $\pi : \widetilde{X} \to \widetilde{X}^! /\!\!/ W^! = X$ is a branched W-cover. The point 0 has a unique preimage in $\widetilde{X}^!$, which we denote by $0^!$. We fix a $W^!$-equivariant isomorphism

$$\widetilde{\mathcal{O}}^!_F \cong \mathcal{O}_{\widetilde{X}^!,0^!}$$

This induces a W-equivariant isomorphism $\widetilde{\mathcal{O}}_F \cong \mathcal{O}_{\widetilde{X},0}$ (the latter being the completion of $\widetilde{X}$ along $\pi^{-1}(0)$), and also an isomorphism $\mathcal{O}_F \cong \mathcal{O}_{X,0}$.

Lemma 4. *For any $N > 0$, there exists an integer $d = d(N)$ such that for any divisor D on X, disjoint from $\{0, z, \infty\}$ and $\deg(D) \geq d$, there exists a section $a : X \to \mathfrak{c}_D$ such that*

1. *There is a W-equivariant birational morphism $\widetilde{X} \to X_a$ (i.e., $\widetilde{X}$ is the normalization of the cameral curve X_a);*
2. *Let a_0 be the restriction of a to $Spec\mathcal{O}_{X,0}$. Then $a_0 \equiv a(\gamma) \mod \varpi^N$ (since D is disjoint from 0, a_0 is an element in $\mathfrak{c}(\mathcal{O}_{X,0}) = \mathfrak{c}(\mathcal{O}_F)$);*
3. *$a(z) \in \mathfrak{c}^{rs}$, i.e., $a \in \mathcal{A}^z$.*

Proof. To give such a section a is equivalent to giving a W-equivariant morphism $\widetilde{a} : \widetilde{X} \to \mathfrak{t}_D$ whose induced map between W-quotients satisfies local conditions (2) and (3) (which is then automatically birational because a is regular semisimple at z).

Let $\mathcal{L}$ be the following coherent sheaf on X

$$\mathcal{L} = (\mathfrak{t} \otimes \pi_*\mathcal{O}_{\widetilde{X}})^W.$$

Since $\mathrm{char}(k)$ does not divide $\#W$, this is a vector bundle over X. First fix any divisor D on X not containing $\{0, z, \infty\}$. Giving a W-equivariant map $\widetilde{X} \to \mathfrak{t}_D$ is equivalent to giving a section $\widetilde{a} \in \mathrm{H}^0(X, \mathcal{L}(D))$. The conditions (2)(3) will be satisfied if we can find such a section $\widetilde{a}$ which approximates given sections at 0 and z to high order. This can obviously be achieved provided $\deg(D)$ is large.

Let N be the integer in Proposition 2: whenever $a(\gamma') \equiv a(\gamma) \mod \varpi^N$, then Theorem 1 is true for $a(\gamma')$ if and only if it is true for $a(\gamma)$. Applying Lemma 4 to this N, we obtain a divisor $D = 2D'$ on X and $a : X \to \mathfrak{c}_D$ satisfying the properties therein. By choosing $\deg(D)$ large enough, we can make sure that $(\mathcal{A}^\heartsuit \times X)' \supset (\mathcal{A}^\heartsuit \times X)_\delta$ for $\delta = \delta(a;0)$, hence $(a,0) \in (\mathcal{A}^\heartsuit \times X)'$ (see the discussion in §3.6). We use this D to define the rigidified parabolic Hitchin moduli $\widehat{\mathcal{M}}^{\mathrm{par}}$. With this a we have the cameral curve $q_a : X_a \to X$. The image of $\widetilde{X}^!$ in X_a is a component $X^!_a$. Let $\mathrm{Sing}(a) \subset X$ be a subset which contains $0, \infty$ and the locus where $q_a : X_a \to X$ is not étale.

5.4. *Relation between local and global Picard*

Let $x \in X(k)$. The discussion in §2.6 can be applied to $\gamma(a,x) \in \mathfrak{g}(F_x)$, the Kostant section of the restriction of a to the formal disk $Spec\mathcal{O}_x$. In particular, we obtain $\mathcal{P}_{a,x}, P_{a,x}, \Lambda_{a,x}$ and $R_{a,x}$, and an exact sequence

$$1 \to R_{a,x} \to P_{a,x} \to \Lambda_{a,x} \to 1. \tag{23}$$

Note that $R_{a,x}$ is the same group scheme which appeared in (12). For each $x \in X(k)\backslash\{z\}$, we have a commutative diagram

$$\begin{tikzcd} P_{a,x} \arrow[r] \arrow[d] & \widehat{\mathcal{P}}_a \arrow[d] \\ \Lambda_{a,x} \arrow[r] & P^\flat_a \end{tikzcd} \tag{24}$$

Let $b = (a, t_z) \in \mathcal{B}(k)$ lifting a such that $t_z \in X^!_a$. The choice of t_z gives an identification $J_{a,z} \xrightarrow{\sim} T$. Recall the exact sequence (12). By [**?**, §4.11] we have $\mathrm{H}^0(X, J^\flat_a) = T^w$, hence the first arrow in (12) gives a map $T^w \to J_{a,z} = T$, which is easily checked to be the canonical inclusion $T^w \hookrightarrow T$. Let $S \subset T$ be a torus which is complementary to the neutral component of T^w, so that $S \cap T^w$ is a finite diagonalizable group over k. By the assumption on $\mathrm{char}(k)$, $S \cap T^w$ is in fact discrete.

Combining the exact sequences (12), (23) and the diagram (24), we get a map between exact sequences

$$\begin{tikzcd} 1 \arrow[r] & S \times \prod_{x\in\mathrm{Sing}(a)} R_{a,x} \arrow[r] \arrow[d, "\alpha"] & S \times \prod_{x\in\mathrm{Sing}(a)} P_{a,x} \arrow[r] \arrow[d, "\beta"] & \prod_{x\in\mathrm{Sing}(a)} \Lambda_{a,x} \arrow[r] \arrow[d, "\gamma"] & 1 \\ 1 \arrow[r] & (J_{a,z} \times \prod_{x\in\mathrm{Sing}(a)} R_{a,x})/T^w \arrow[r] & \widehat{\mathcal{P}}_a \arrow[r] & P^\flat_a \arrow[r] & 1 \end{tikzcd} \tag{25}$$

By the choice of S, α is surjective. Let $P^{\ker}_a = \ker(\beta)$ and $\Lambda^{\ker}_a = \ker(\gamma)$. The snake lemma gives an exact sequence between the kernels

$$1 \to S \cap T^w \to P^{\ker}_a \to \Lambda^{\ker}_a \to 1. \tag{26}$$

Here $S \cap T^w$ is viewed as a subgroup of $T^w = \mathrm{H}^0(X, J^\flat_a)$, hence maps diagonally into $S \times \prod_x R_{a,x}$. Note that $S \cap T^w$ is a discrete group scheme, so is $P^{\ker}_a$, hence we may identify $P^{\ker}_a$ with its k-points.

By Lemma **??** below, the map γ in (25) is also surjective. Hence so is β and we have an exact sequence

$$1 \to P^{\ker}_a \to S \times \prod_{x\in\mathrm{Sing}(a)} P_{a,x} \to \widehat{\mathcal{P}}_a \to 1 \tag{27}$$

Lemma 5. *In our situation, the coarse moduli space $P^\flat_a$ of $\mathcal{P}^\flat_a$ is a discrete group. The natural map $\iota_0 : \Lambda_{a,0} \to P^\flat_a$ is an isomorphism, and both groups are canonically isomorphic to $\mathbb{X}_*(T)_w/$torsion.*

Proof. By [**?**, Corollaire 4.8.1], the finite type Néron model $J^\flat_a$ of J_a is $(Res_{\widetilde{X}^!/X}(T\times\widetilde{X}^!))^w$. The Lie algebra of $P^\flat_a$ is $\mathrm{H}^1(X, LieJ^\flat_a) = \mathrm{H}^1(X, (\mathfrak{t}\otimes\mathcal{O}_{\widetilde{X}^!})^w) \subset \mathfrak{t}\otimes\mathrm{H}^1(\widetilde{X}^!, \mathcal{O}_{\widetilde{X}^!}) = 0$ since $\widetilde{X}^! \cong \mathbb{P}^1$. Therefore $P^\flat_a$ is discrete.

The restriction of $J^\flat_a$ to $Spec\mathcal{O}_F$ is $(Res_{\widetilde{\mathcal{O}}^!_F/\mathcal{O}_F} T)^w$. Therefore $\Lambda_{a,0} = J_a(F)/J^\flat_a(\mathcal{O}_F) = T(F^!)^w/T(\widetilde{\mathcal{O}}^!_F)^w$. Taking the w-invariants of the exact sequence

$$1 \to T(\widetilde{\mathcal{O}}^!_F) \to T(F^!) \xrightarrow{ev} \mathbb{X}_*(T) \to 0$$

we get that $\Lambda_{a,0}$ is the image of the map

$$ev : J_a(F) = T(F^!)^w \to \mathbb{X}_*(T)^w.$$

Similarly, from the injection $J^\flat_a \hookrightarrow Res_{\widetilde{X}^!/X}(T\times\widetilde{X}^!)$ we deduce a natural map

$$\deg : P^\flat_a \to \mathrm{H}^1(\widetilde{X}^!, T)^w = \mathbb{X}_*(T)^w.$$

Consider the diagram

$$\begin{tikzcd} \mathbb{X}_*(T)_w \arrow[r, "\alpha"] \arrow[d, equal] & \Lambda_{a,0} \arrow[r, "ev"] \arrow[d, "\iota_0"] & \mathbb{X}_*(T)^w \arrow[d, equal] \\ \mathbb{X}_*(T)_w \arrow[r, "\beta"] & P_a^\flat \arrow[r, "\deg"] & \mathbb{X}_*(T)^w \end{tikzcd} \tag{28}$$

Here α is a surjection induced from the map $J_a^{\flat,0} \to J_a^\flat$ (here $(-)^0$ denotes fiber-wise neutral component), and the fact that $J_a(F)/J_a^{\flat,0}(\mathcal{O}_F) \cong \mathbb{X}_*(T)_w$ [**?**, Lemme 3.9.4]. The map β is a surjection induced from the same map $J_a^{\flat,0} \to J_a^\flat$, and the fact that $\pi_0(Pic(X, J_a^{\flat,0})) \cong \mathbb{X}_*(T)_w$ [**?**, Proposition 6.4, Corollaire 6.7]. The diagram (**??**) is easily seen to be commutative.

The fact that β is surjective shows that ι_0 is also surjective. The fact that ev is injective shows that ι_0 is also injective. Therefore ι_0 is an isomorphism. Now $\ker(\alpha) \cong J_a^\flat(\mathcal{O}_F)/J_a^{\flat,0}(\mathcal{O}_F)$ is torsion and $\Lambda_{a,0}$ is torsion-free since ev is injective, hence the first row of (**??**) identifies $\Lambda_{a,0}$ with the torsion-free quotient of $\mathbb{X}_*(T)_w$.

Lemma 6. *The natural map*

$$P_a^{\ker} \to \prod_{x \in \mathrm{Sing}(a)\backslash\{0\}} \pi_0(P_{a,x})$$

is injective with finite cokernel.

Proof. Consider the following map between short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & S \cap T^w \arrow[r] \arrow[d, "\alpha"] & P_a^{\ker} \arrow[r] \arrow[d, "\beta"] & \Lambda_a^{\ker} \arrow[r] \arrow[d, "\gamma"] & 1 \\ 1 \arrow[r] & \prod_{x \in \mathrm{Sing}(a)\backslash\{0\}} \pi_0(R_{a,x}) \arrow[r] & \prod_{x \in \mathrm{Sing}(a)\backslash\{0\}} \pi_0(P_{a,x}) \arrow[r] & \prod_{x \in \mathrm{Sing}(a)\backslash\{0\}} \Lambda_{a,x} \arrow[r] & 1 \end{tikzcd} \tag{29}$$

By Lemma **??**, $\Lambda_{a,0} \xrightarrow{\sim} P_a^\flat$, hence γ is an isomorphism. Applying the snake lemma to (**??**), we get $\ker(\alpha) \cong \ker(\beta)$ and $coker(\beta) \cong coker(\alpha)$, which is finite.

In order to show that β is injective, it suffices to show that α is injective. By definition, $\infty \in \mathrm{Sing}(a)$. By [**?**, Proposition 3.9.7], we have an isomorphism $\pi_0(P_{a,\infty}) \cong \mathbb{X}_*(T)_w$ because G is of adjoint type. Also by [**?**, Proposition 3.9.7], the map $\pi_0(T^w) \to \mathbb{X}_*(T)_w = \pi_0(P_{a,\infty})$ is an embedding whose image is the torsion part of $\mathbb{X}_*(T)_w$. Since $T^w \to P_{a,\infty}$ factors through $R_{a,\infty}$, we see that $\pi_0(T^w) \to \pi_0(R_{0,\infty})$ is injective. By the choice of S (complementary to the neutral component of T^w), $S \cap T^w$ injects into $\pi_0(T^w)$, therefore $S \cap T^w \hookrightarrow \pi_0(R_{a,\infty})$, proving that α is injective. This finishes the proof.

Proposition 5. *For any $x \in X(k)$, the following diagram is commutative:*

$$\begin{tikzcd} \mathbb{Q}_\ell[\mathbb{X}_*(T)]^W \arrow[r, "\sigma_{a,x}"] \arrow[rr, bend right, "\sigma_a"'] & \mathbb{Q}_\ell[\pi_0(P_{a,x})] \arrow[r] & \mathbb{Q}_\ell[\pi_0(\mathcal{P}_a)] \end{tikzcd} \tag{30}$$

Here $\sigma_{a,x}$ is the homomorphism (1) *applied to $\gamma(a,x)$, and σ_a is the stalk of* (2) *at a.*

Proof. Recall the choice t_z gives a point $z^! \in X_a^{!,rs}$ over z. Also we have $x^! \in X_a^!$ over x. Consider the diagram

$$\begin{tikzcd} & \pi_0(P_{a,x}) \arrow[dr, "\iota_x"] & \\ \mathbb{X}_*(T) \arrow[ur, "\widetilde{\sigma}_{a,x^!}"] \arrow[rr, "\tau"] \arrow[dr, "s(-,z^!)"'] & & \pi_0(\mathcal{P}_a) \\ & \pi_0(\mathcal{G}r_{J_a,z}) = \pi_0(P_{a,z}) \arrow[ur, "\iota_z"'] & \end{tikzcd} \tag{31}$$

Here $s(-,z^!)$ is the map in (13), and the composition $\iota_z \cdot s(-,z^!)$ is used to define σ_a, see §3.8. The homomorphism $\widetilde{\sigma}_{a,x^!}$ is defined in (7) (quoted from [**?**, Proposition 3.9.2]), which was used to define $\sigma_{a,x}$. Therefore, in order to show that (**??**) is commutative, if suffices to show that the outer square of (**??**) is commutative.

The arrow τ in (**??**) is defined in [**?**, Proposition 6.8] and [**?**, Proposition 4.10.3]. We shall show that the two triangles in (**??**) are both commutative. The commutativity of the upper triangle follows from the compatibility of Ngô's constructions, see the proof of [**?**, Proposition 4.10.3]. On the other hand, tracing through the definition of $s(-,z^!)$ in [**?**, Lemma 3.2.5], we see that $s(-,z^!)$ is the same as Ngô's map $\widetilde{\sigma}_{a,z^!}$, therefore the lower triangle of (**??**) is also commutative. This finishes the proof.

5.5. Product formula

For each $x \in \mathrm{Sing}(a)$, choosing a trivialization of $\mathcal{O}_X(D)$ at y, we may view the restriction of a on $Spec\mathcal{O}_x$ as an element $a_x \in \mathfrak{c}(\mathcal{O}_x)$. Let $\gamma(a,x) = \epsilon(a_x) \in \mathfrak{g}(\mathcal{O}_x)$ be the Kostant section. Define

$$\mathrm{Spr}_{\mathbf{P},a,x} := \mathrm{Spr}_{\mathbf{P},\gamma(a,x)}.$$

Recall from [**?**, §4.2.4] that we have a global Kostant section $\epsilon(a) = (\mathcal{E},\varphi) \in \mathcal{M}_a^{\mathrm{Hit}}$. Picking any isomorphism $\iota_z : (\mathcal{E},\varphi)_z \xrightarrow{\sim} (G,t_z)$ gives a point $\widehat{\epsilon}(b) \in \widehat{\mathcal{M}}_b^{\mathrm{Hit}}$. As in [**?**, §4.15], by gluing the local Hitchin pairs at $x \in \mathrm{Sing}(a)$ with $\widehat{\epsilon}(b)$ we get a morphism

$$\mathrm{Spr}_{a,0} \times \prod_{x\in\mathrm{Sing}(a)\backslash\{0\}} \mathrm{Spr}_{\mathbf{G},a,x} \to \widehat{\mathcal{M}}_{b,0}^{\mathrm{par}} \tag{32}$$

which intertwines the $\prod_{x\in\mathrm{Sing}(a)} P_{a,x}$-action on the LHS and the $\widehat{\mathcal{P}}_a$-action on the right. To alleviate notions, let

$$Z = \prod_{x\in\mathrm{Sing}(a)\backslash\{0\}} \mathrm{Spr}_{\mathbf{G},a,x}.$$

A rigidified version of the product formula ([**?**, Proposition 4.15.1], [**?**, Proposition 2.4.1]) gives a homeomorphism

$$(\mathrm{Spr}_{a,0} \times Z) \overset{\prod_{x\in\mathrm{Sing}(a)} P_{a,x}}{\times} \widehat{\mathcal{P}}_a \to \widehat{\mathcal{M}}_{b,0}^{\mathrm{par}}. \tag{33}$$

Dividing both sides of (**??**) by S, using (**??**), we get a homeomorphism of stacks

$$\mathrm{Spr}_{a,0} \overset{P_a^{\mathrm{ker}}}{\times} Z \xrightarrow{\text{homeo.}} [\widehat{\mathcal{M}}_{b,0}^{\mathrm{par}}/S]. \tag{34}$$

5.6. Pulling apart components

[1] By [**?**, §3.10.2], the irreducible components of $\mathrm{Spr}_{\mathbf{G},a,x}$ are in bijections with $\pi_0(P_{a,x})$. Hence, by Lemma **??**, P_a^{ker} permutes the irreducible components of $Z = \prod_{x\in\mathrm{Sing}(a)\backslash\{0\}} \mathrm{Spr}_{\mathbf{G},a,x}$ freely with finitely many orbits. We pick one irreducible component $Y_\alpha \subset Z$ from each P_a^{ker}-orbit. Let $Y = \cup_\alpha Y_\alpha \subset Z$ be the union of these orbit-representatives of irreducible components. Let $Y^{reg} \subset Y$ be the intersection of Y with the regular locus $Z^{reg} = \prod \mathrm{Spr}^{reg}_{\mathbf{G},a,x}$ (cf. [**?**, Lemma 3.3.1]). By construction, the action map $P_a^{\mathrm{ker}} \times Y \to Z$ induces a bijection between irreducible components, and restricts to an isomorphism on the regular loci. Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Spr}_{a,0} \times Y^{reg} \arrow[r, hook, "j_Y"] \arrow[d, "\wr"] & \mathrm{Spr}_{a,0} \times Y \arrow[d, "\nu"] & \\
\mathrm{Spr}_{a,0} \overset{P_a^{\mathrm{ker}}}{\times} Z^{reg} \arrow[r, hook, "j"] & \mathrm{Spr}_{a,0} \overset{P_a^{\mathrm{ker}}}{\times} Z \arrow[r, "(??)"] & [\widehat{\mathcal{M}}_{b,0}^{\mathrm{par}}/S]
\end{tikzcd}$$

[1] This part of the argument was suggested by Y.Varshavsky.

From this we get a commutative diagram on compactly supported cohomology

$$\begin{tikzcd}
\mathrm{H}^*_c(\mathrm{Spr}_{a,0}\times Y^{reg}) \arrow[r, "j_{Y,!}"] & \mathrm{H}^*_c(\mathrm{Spr}_{a,0}\times Y) & \\
\mathrm{H}^*_c(\mathrm{Spr}_{a,0}\overset{P_a^{\ker}}{\times} Z^{reg}) \arrow[u, "\wr"] \arrow[r, "j_!"] & \mathrm{H}^*_c(\mathrm{Spr}_{a,0}\overset{P_a^{\ker}}{\times} Z) \arrow[u, "\nu^*"'] \arrow[r, "\sim"] & \mathrm{H}^*_c([\widehat{\mathcal{M}}^{\mathrm{par}}_{b,0}/S])
\end{tikzcd}$$

Here ν^* makes sense because it is proper. In other words, the map $j_{Y,!}$ on compactly supported cohomology can be factored as

$$j_{Y,!}:\mathrm{H}^*_c(\mathrm{Spr}_{a,0}\times Y^{reg})\xrightarrow{j_!}\mathrm{H}^*_c([\widehat{\mathcal{M}}^{\mathrm{par}}_{b,0}/S])\xrightarrow{\nu^*}\mathrm{H}^*_c(\mathrm{Spr}_{a,0}\times Y). \tag{35}$$

Let $d=\dim Y$. Using the Künneth formula, and taking only the top cohomology of Y^{reg} and Y, we get

$$\mathrm{H}^i_c(\mathrm{Spr}_{a,0})\otimes\mathrm{H}^{2d}_c(Y^{reg})\longrightarrow\mathrm{H}^{i+2d}_c([\widehat{\mathcal{M}}^{\mathrm{par}}_{b,0}/S])\longrightarrow\mathrm{H}^i_c(\mathrm{Spr}_{a,0})\otimes\mathrm{H}^{2d}(Y)$$

The composition of the above maps is an isomorphism because $\mathrm{H}^{2d}_c(Y^{reg})\xrightarrow{\sim}\mathrm{H}^{2d}(Y)=\mathbb{Q}_\ell[Irr(Y)]$. Therefore the map $j_!$ in (**??**) gives an injection

$$\mathrm{H}^i_c(\mathrm{Spr}_{a,0})\hookrightarrow\mathrm{H}^i_c(\mathrm{Spr}_{a,0})\otimes\mathrm{H}^{2d}_c(Y^{reg})\hookrightarrow\mathrm{H}^{i+2d}_c([\widehat{\mathcal{M}}^{\mathrm{par}}_{b,0}/S]). \tag{36}$$

where the first map is given by the inclusion of the fundamental class $[Y^{reg}]\in\mathrm{H}^{2d}_c(Y^{reg})$.

Proposition 6. *The map* (**??**) *intertwines the* $\widetilde{W}\times\pi_0(P_{a,0})$*-action on the first factor of the LHS and the* $\widetilde{W}\times\pi_0(\mathcal{P}_a)=\widetilde{W}\times\pi_0(\widehat{\mathcal{P}}_a/S)$*-action on the RHS via the natural map* $P_{a,0}\to\widehat{\mathcal{P}}_a$.

Proof. The fact that this map intertwines the $\pi_0(P_{a,0})$-action on the LHS and the $\pi_0(\mathcal{P}_a)$-action on the RHS is clear from the equivariance property of (**??**).

It remains to prove the $\widetilde{W}$-equivariance of (**??**). For each standard parahoric $\mathbf{P}$, we have a diagram consisting of two Cartesian squares

$$\begin{tikzcd}
\mathcal{S}\mathrm{pr}_{a,0}\times Y^{reg} \arrow[r, hook] \arrow[d, "\nu_{\mathbf{P}}"] & \widehat{\mathcal{M}}^{\mathrm{par}}_{b,0} \arrow[r, "ev_0"] \arrow[d, "\widehat{\mathrm{For}}^{\mathbf{P}}_{\mathbf{I}}"] & {[\widetilde{\mathfrak{l}}_{\mathbf{P}}/L_{\mathbf{P}}]} \arrow[d, "\pi_{\mathbf{P}}"] \\
\mathcal{S}\mathrm{pr}_{\mathbf{P},a,0}\times Y^{reg} \arrow[r, hook] & \widehat{\mathcal{M}}_{\mathbf{P},b,0} \arrow[r, "ev_0"] & {[\mathfrak{l}_{\mathbf{P}}/L_{\mathbf{P}}]}
\end{tikzcd} \tag{37}$$

such that the outer square is the product of Y^{reg} with the diagram (4) and the right square is the restriction of the diagram (10) at $(b,0)\in\mathcal{B}\times X^z$.

The $W_{\mathbf{P}}$-action on $\mathrm{H}_c^*(\mathrm{Spr}_{a,0} \times Y^{reg})$ and $\mathrm{H}_c^*(\widehat{\mathcal{M}}_{b,0}^{\mathrm{par}})$ are constructed from these diagrams using the parahorics $\mathbf{P}$ and the classical Springer theory for $L_{\mathbf{P}}$. The diagram (**??**) implies that (**??**) is $W_{\mathbf{P}}$-equivariant for every $\mathbf{P}$, hence W_{aff}-equivariant. Similarly, using a diagram connecting (5) and its global analogue, one shows that (**??**) is also $\Omega_{\mathbf{I}}$-equivariant. Putting together, we conclude that (**??**) is $\widetilde{W}$-equivariant.

5.7. Conclusion of the proof of Theorem 1

Consider the diagram

$$
\begin{tikzcd}
\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W \otimes \arrow[d, equal] & \mathrm{H}_c^i(\mathrm{Spr}_{a,0}) \arrow[r, "\sigma_{a,0}\otimes id"] \arrow[d, "(??)"] \arrow[rrr, bend left, "\alpha_{\mathrm{loc}}"] & \mathbb{Q}_\ell[\pi_0(P_{a,0})] \otimes \arrow[d, "\iota_0"] & \mathrm{H}_c^i(\mathrm{Spr}_{a,0}) \arrow[r, "act_{\mathrm{loc}}"] \arrow[d, "(??)"] & \mathrm{H}_c^i(\mathrm{Spr}_{a,0}) \arrow[d, "(??)"] \\
\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W \otimes & \mathrm{H}_c^{i+2d}([\widehat{\mathcal{M}}_{b,0}^{\mathrm{par}}/S]) \arrow[r, "\sigma_a\otimes id"] \arrow[rrr, bend right, "\alpha_{\mathrm{glob}}"'] & \mathbb{Q}_\ell[\pi_0(\mathcal{P}_a)] \otimes & \mathrm{H}_c^{i+2d}([\widehat{\mathcal{M}}_{b,0}^{\mathrm{par}}/S]) \arrow[r, "act_{\mathrm{glob}}"] & \mathrm{H}_c^{i+2d}([\widehat{\mathcal{M}}_{b,0}^{\mathrm{par}}/S])
\end{tikzcd}
$$

where the upper α_{loc} and α_{glob} are actions maps of $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$; act_{loc} and act_{glob} are action maps of the π_0's. By Proposition **??**, the left side square is commutative. By Proposition **??**, the right side square and the outer square are commutative. By Corollary 2, the lower triangle is commutative. Our goal is to prove that the upper triangle is commutative. From the known commutativity, we conclude that $act_{\mathrm{loc}} \circ (\sigma_{a,0} \otimes id)$ and α_{loc} are the same if we further compose them with (**??**). Since (**??**) is injective, they must be equal before composition, i.e., the upper triangle is commutative. This proves Theorem 1 for $\mathrm{Spr}_{a,0}$. Since $a_0 \equiv a(\gamma) \in \varpi^N$ by construction, the theorem also holds for Spr_γ by Proposition 2. This finishes the proof of Theorem 1.

5.8. Duality between homology and compactly supported cohomology

The next goal is to prove Theorem 2. We consider the following general setting. Let Λ be a group. Let X be a scheme, locally of finite type over k with a free Λ-action such that X/Λ is representable by a proper scheme. The main examples we have in mind are $X = \mathrm{Spr}_\gamma$ with the action of a lattice $\Lambda \subset LG_\gamma$.

A *Λ-covering* of X is a Λ-equivariant morphism

$$f : Y \times \Lambda \to X$$

where Y is a scheme, Λ acts trivially on Y and acts as translations on Λ itself, such that the induced morphism $\bar{f}: Y \to X/\Lambda$ is proper and surjective. For example, we may take $Y \subset X$ to be the union of representatives of the Λ-orbits on the irreducible components of X, and take f to be the action map.

Let $X_0 = \Lambda \times Y$ and $X_n = X_0 \times_X \times \cdots \times_X X_0$ ($n+1$ terms). The collection $\mathfrak{X} = (X_n)_{n\geq 0}$ together with the natural face maps (projections) and degeneration maps (diagonal maps) forms a simplicial resolution of X. Each X_n carries a diagonal Λ-action which is also free. Let $Y_n = X_n/\Lambda$. Then Y_n is naturally identified with the $(n+1)$-fold Cartesian product $Y \times_{X/\Lambda} Y \times \cdots \times_{X/\Lambda} Y$. We form the simplicial scheme $\mathfrak{Y} = (Y_n)_{n\geq 0}$ again using the obvious face and degeneration maps. Then $\mathfrak{Y}$ is a simplicial resolution of X/Λ, and each Y_n is a proper scheme.

The natural projection $\xi_n : X_n \to Y_n$ is in fact a trivial Λ-torsor. The trivialization is given by $\eta_n : \Lambda \times Y_n \to X_n$ defined as $(\lambda, y_0, \cdots, y_n) \mapsto (\lambda, y_0; \lambda\lambda_1, y_1; \cdots; \lambda\lambda_n, y_n)$, where $\lambda_i \in \Lambda$ is the unique element such that $\lambda_i y_i = y_0$. It is easy to see that η_n is a Λ-equivariant isomorphism. However the simplicial structures of $\mathfrak{X}$ and $\mathfrak{Y}$ are not preserved by the maps η_n. In other words, the map $\xi : \mathfrak{X} \to \mathfrak{Y}$ is a Λ-torsor in the category of simplicial schemes, which is trivializable over each Y_n but not necessarily trivializable as simplicial schemes.

By cohomological descent for proper surjective morphisms (see [**?**, §5.3] and [**?**, Prop. 4.3.2]), $\mathrm{H}^*_c(X)$ is canonically isomorphic to the compactly supported cohomology of the simplicial scheme $\mathfrak{X}$. We would like to calculate $\mathrm{H}^*_c(\mathfrak{X})$ using $\mathrm{H}^*_c(\mathfrak{Y})$. When working with a finite coefficient ring R, we may resolve the constant sheaf R on $\mathfrak{Y}$ by a complex $K^0 \to K^1 \to \cdots$ on $\mathfrak{Y}$ with injective terms. We form the double complex $D^{i,j}(f)_R = \Gamma_c(Y_i, K^j|_{Y_i})$ (see [**?**, §5.2.3]) with differentials in the i-index induced from the simplicial structure and in the j-index induced from the differentials on K^*. We may view the double complex $D^{*,*}(f)_R$ as an object $D(f)_R$ in the filtered derived category $D^bF(R\text{-mod})$, by taking the stupid filtration in the i-index, i.e., $Gr^i D(f)_R \cong D^{i,*}(f)_R[-i]$, which is quasi-isomorphic to $\mathbf{R}\Gamma_c(Y_i, R)$. Let $\omega : D^bF(R\text{-mod}) \to D^b(R\text{-mod})$ be the functor of forgetting the filtration (which is the same as taking the simple complex associated with a double complex, when applied to a filtered complex given by the stupid filtration of a double complex). Then $\mathbf{R}\Gamma_c(\mathfrak{Y}, R)$ is quasi-isomorphic to $\omega D(f)_R$. Similarly, $\mathbf{R}\Gamma_c(\mathfrak{X}, R)$ is quasi-isomorphic to $\omega C(f)_R$, where $C(f)_R$ is a filtered complex of R-modules obtained via the stupid filtration in the i-index of the double complex $C^{i,j}(f)_R = \Gamma_c(X_i, \xi_i^*(K^j|_{Y_i})) \cong R[\Lambda] \otimes D^{i,j}(f)_R$. Note that the differentials in the i-direction on $C^{i,j}(f)_R$ are *not* simply the identity map on $R[\Lambda]$ tensored with the differentials on $D^{i,j}(f)_R$, but are induced from the face maps of $\mathfrak{X}$. But in any case the differentials on $C^{*,*}(f)_R$ are $R[\Lambda]$-linear, and hence we may view $C^{*,*}(f)_R$ as a double complex of $R[\Lambda]$-modules, which defines a filtered complex $C(f)_R \in D^bF(R[\Lambda]\text{-mod})$ with $Gr^iC(f)_R \cong R[\Lambda] \otimes Gr^iD(f)_R$. Passing to inverse limits for $R = \mathbb{Z}/\ell^n\mathbb{Z}$ and then inverting ℓ, we get an object $C(f) \in D^bF(\mathbb{Q}_\ell[\Lambda]\text{-mod})$ such that $\omega C(f)$ is quasi-isomorphic to $\mathbf{R}\Gamma_c(\mathfrak{X})$. In this way, we have upgraded the complex $\mathrm{H}^*_c(X)$ to an object $\mathrm{H}^*_c(X)^\sharp := \omega C(f)$ in the derived category $D^b(\mathbb{Q}_\ell[\Lambda]\text{-mod})$.

Similarly, for a finite ring R, we have a double complex $K_{i,j}(f)_R$ such that for fixed i, $K_{i,*}(f)_R$ is quasi-isomorphic to $\mathbf{R}\Gamma_c(Y_i, \mathbb{D}_{Y_i,R})$ (whose cohomology calculates the R-homology of Y_i). The double complex $H_{i,j}(f)_R = R[\Lambda] \otimes K_{i,j}(f)_R$ (again the differentials in the i-index is not simply obtained from the tensor product) then calculates the homology $\mathrm{H}_*(\mathfrak{X}, R) \cong \mathrm{H}_*(X, R)$: first view $H_{*,*}(f)_R$ as a filtered complex $H(f)_R$ with $Gr^{-i}H(f)_R = R[\Lambda]\otimes\mathbf{R}\Gamma_c(Y_i, \mathbb{D}_{Y_i,R})[i]$, then $\mathbf{R}\Gamma_c(\mathfrak{X}, \mathbb{D}_{\mathfrak{X},R})$ is quasi-isomorphic to $\omega H(f)_R$. For each fixed i, we have a canonical quasi-isomorphism of complexes of $R[\Lambda]$-modules

$$\begin{aligned} &Gr^{-i}H(f)_R \cong R[\Lambda] \otimes \mathbf{R}\Gamma_c(Y_i, \mathbb{D}_{Y_i,R}) \\ \cong\ &\mathbf{R}Hom_{R[\Lambda]}(R[\Lambda] \otimes \mathbf{R}\Gamma_c(Y_i, R), R[\Lambda]) \cong \mathbf{R}Hom_{R[\Lambda]}(Gr^iC(f)_R, R[\Lambda]). \end{aligned} \tag{38}$$

Here we used the fact that each Y_i is proper. Moreover, the isomorphism (**??**) is compatible with the simplicial structure as i varies. Passing to inverse limits for $R = \mathbb{Z}/\ell^n\mathbb{Z}$ and then inverting ℓ, we have upgraded the homology complex $\mathrm{H}_*(X)$ to an object $\mathrm{H}_*(X)^\sharp := \omega H(f) \in D^b(\mathbb{Q}_\ell[\Lambda]\text{-mod})$, and obtained an isomorphism in $D^b(\mathbb{Q}_\ell[\Lambda]\text{-mod})$ from (**??**):

$$\mathrm{H}_*(X)^\sharp \cong \mathbf{R}Hom_{\mathbb{Q}_\ell[\Lambda]}(\mathrm{H}^*_c(X)^\sharp, \mathbb{Q}_\ell[\Lambda]). \tag{39}$$

Finally, neither the isomorphism classes of the objects $\mathrm{H}^*_c(X)^\sharp, \mathrm{H}_*(X)^\sharp \in D^b(\mathbb{Q}_\ell[\Lambda]\text{-mod})$ nor the isomorphism (**??**) depend on the choice of the Λ-covering. In fact, for any two Λ-coverings $f : Y \times \Lambda \to X$ and $f' : Y' \times \Lambda \to X$ are both dominated by the third $f'' : Y'' \times \Lambda \to X$, where $Y'' = (Y \times \{0\}) \times_X (Y' \times \Lambda)$. To emphasize the *a priori* dependence on the Λ-covering, we write $\mathrm{H}^*_c(X)^\sharp_f, \mathrm{H}_*(X)^\sharp_f$ etc to denote the upgraded objects constructed using f. Let $g : Y'' \to Y$ and $g' : Y'' \to Y'$ be projections. They induce maps on simplicial schemes $g_* : Y''_* \to Y_*$ and $g'_* : Y''_* \to Y'_*$, which then induce isomorphisms in the category $D^b(\mathbb{Q}_\ell[\Lambda]\text{-mod})$

$$\mathrm{H}^*_c(X)^\sharp_f \xrightarrow{g^*} \mathrm{H}^*_c(X)^\sharp_{f''} \xleftarrow{g'^*} \mathrm{H}^*_c(X)^\sharp_{f'}, \tag{40}$$

$$\mathrm{H}_*(X)^\sharp_f \xleftarrow{g_*} \mathrm{H}_*(X)^\sharp_{f''} \xrightarrow{g'_*} \mathrm{H}_*(X)^\sharp_{f'}. \tag{41}$$

Therefore the isomorphism classes of $\mathrm{H}_c^*(X)^\sharp, \mathrm{H}_*(X)^\sharp \in D^b(\mathbb{Q}_\ell[\Lambda]\text{-mod})$ are independent of the choice of the Λ-covering. Moreover, the quasi-isomorphisms in (**??**) and (**??**) intertwine the dualities of the type (**??**). Therefore, f, f' and f'' all give the same isomorphism (**??**).

5.9. Proof of Theorem 2

[2] Now let $X = \mathrm{Spr}_\gamma$ and $\Lambda \subset LG_\gamma$ be a free abelian subgroup considered in [**?**, Proposition 2.1], which acts freely on Spr_γ with proper quotient [**?**, Proposition 3.1(b), Corollary 3.1]. In fact, the proof in [**?**] also applies to $\mathrm{Spr}_{\mathbf{P},\gamma}$ for any parahoric $\mathbf{P} \subset LG$. Hence Λ also acts freely on $\mathrm{Spr}_{\mathbf{P},\gamma}$ with proper quotient.

The discussion in §**??** gives the upgraded objects

$$\mathrm{H}_c^*(\mathrm{Spr}_\gamma)^\sharp, \mathrm{H}_*(\mathrm{Spr}_\gamma)^\sharp \in D^b(\mathbb{Q}_\ell[\Lambda]\text{-mod}) \tag{42}$$

and the canonical isomorphism (**??**) now reads

$$\mathrm{H}_*(\mathrm{Spr}_\gamma)^\sharp \cong \mathbf{R}Hom_{\mathbb{Q}_\ell[\Lambda]}(\mathrm{H}_c^*(\mathrm{Spr}_\gamma)^\sharp, \mathbb{Q}_\ell[\Lambda]). \tag{43}$$

Lemma 7. *The upgraded objects in* (**??**) *carry* $\widetilde{W}\times\pi_0(P_{a(\gamma)})$*-actions (lifting the actions on the plain vector spaces), and the isomorphism* (**??**) *is* $\widetilde{W} \times \pi_0(P_{a(\gamma)})$*-equivariant.*

Proof. For each parahoric $\mathbf{P}$, we pick a Λ-covering $f_{\mathbf{P}} : Y_{\mathbf{P}} \times \Lambda \to \mathrm{Spr}_{\mathbf{P},\gamma}$ and define a Λ-covering $f : Y \times \Lambda \to \mathrm{Spr}_\gamma$ by requiring the left square of the following diagram to be Cartesian

$$\begin{tikzcd} Y \times \Lambda \arrow[r, "f"] \arrow[d] & \mathrm{Spr}_\gamma \arrow[r] \arrow[d] & {[\widetilde{\mathfrak{l}}_{\mathbf{P}}/L_{\mathbf{P}}]} \arrow[d] \\ Y_{\mathbf{P}} \times \Lambda \arrow[r, "f_{\mathbf{P}}"] & \mathrm{Spr}_{\mathbf{P},\gamma} \arrow[r] & {[\mathfrak{l}_{\mathbf{P}}/L_{\mathbf{P}}]} \end{tikzcd} \tag{44}$$

where the right square is topologically Cartesian by (4). We have shown in §**??** that the upgraded objects (**??**) are independent of the choice of Λ-coverings, and here we shall use this particular Λ-covering to define them. The construction of the $W_{\mathbf{P}}$-action on $\mathrm{H}_c^*(\mathrm{Spr}_\gamma)$ (resp. $\mathrm{H}_*(\mathrm{Spr}_\gamma)$) in §2.3 then gives a $W_{\mathbf{P}}$-action on the filtered complexes $C(f)$ (resp. $H(f)$) calculating $\mathrm{H}_c^*(\mathrm{Spr}_\gamma)^\sharp$ (resp. $\mathrm{H}_*(\mathrm{Spr}_\gamma)^\sharp$), and these $W_{\mathbf{P}}$-actions are compatible with the duality between $Gr^iC(f)$ and $Gr^{-i}H(f)$ as complexes

[2] The idea of proving Theorem 2 by duality of the type (**??**) was suggested by R.Bezrukavnikov.

of $\mathbb{Q}_\ell[\Lambda]$-modules as in (**??**). This gives the $W_{\mathbf{P}}$-action on $\mathrm{H}_c^*(\mathrm{Spr}_\gamma)^\sharp$ and $\mathrm{H}_*(\mathrm{Spr}_\gamma)^\sharp$, and proves that (**??**) is $W_{\mathbf{P}}$-equivariant. The actions of $\Omega_{\mathbf{I}}$ and $\pi_0(P_{a(\gamma)})$ are given by the actions of $\Omega_{\mathbf{I}}$ and $P_{a(\gamma)}$ on Spr_γ itself, which clearly lift to the objects in (**??**) and are intertwined by (**??**). One easily checks that the $\pi_0(P_{a(\gamma)})$-action commutes with the $W_{\mathbf{P}}$ and $\Omega_{\mathbf{I}}$ actions. Using a variant of the diagram (5) incorporating the Λ-coverings as in (**??**), one checks that the commutation relation between $W_{\mathbf{P}}$ and $\Omega_{\mathbf{I}}$ continues to hold after upgrading. This finishes the proof.

By (**??**) and the above lemma, we have a $\widetilde{W} \times \pi_0(P_{a(\gamma)})$-equivariant spectral sequence

$$E_2^{-p,-q} = Ext^{-p}_{\mathbb{Q}_\ell[\Lambda]}(\mathrm{H}_c^q(\mathrm{Spr}_\gamma), \mathbb{Q}_\ell[\Lambda]) \Rightarrow \mathrm{H}_{p+q}(\mathrm{Spr}_\gamma),$$

which necessarily converges because $\mathbb{Q}_\ell[\Lambda]$ has cohomological dimension $rk(\Lambda)$. Therefore, this gives a finite decreasing filtration Fil^p on $\mathrm{H}_i(\mathrm{Spr}_\gamma)$ such that

$$Gr^p_{\mathrm{Fil}}\mathrm{H}_i(\mathrm{Spr}_\gamma) = E_\infty^{p,-i-p}.$$

Since the $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$-action on the E_2 page factors through $\pi_0(LG_\gamma)$ by Theorem 1, so does the $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$-action on E_∞. Therefore, the $\mathbb{Q}_\ell[\mathbb{X}_*(T)]^W$-action on $Gr^p_{\mathrm{Fil}}\mathrm{H}_i(\mathrm{Spr}_\gamma)$ also factors through $\pi_0(LG_\gamma)$. Since $E_2^{p,q} = 0$ unless $0 \leq p \leq rk(\Lambda)$, the same is true for $E_\infty^{p,*} = Gr^p_{\mathrm{Fil}}\mathrm{H}_*(\mathrm{Spr}_\gamma)$. Note that $rk(\Lambda)$ is the same as the F-rank of G_γ. This proves Theorem 2.

A. Sheaves and correspondences on spaces locally of finite type

In this appendix, all algebraic spaces are locally of finite type over k.

A.1. The category of sheaves

Let X be an algebraic space over k which is locally of finite type. Let $\mathrm{Ft}(X)$ be the set of open subsets $U \subset X$ which are of finite type over k. We define

$$\underleftarrow{D}^b(X) := \varprojlim_{U \in \mathrm{Ft}(X)} D^b(U)$$

When X itself is of finite type over k, $\mathrm{Ft}(X)$ has a final object X, so obviously $\underleftarrow{D}^b(X) = D^b(X)$.

Concretely, an object in $\underleftarrow{D}^b(X)$ is a system of complexes $\mathcal{F}_U \in D^b(U)$ for each open subset $U \subset X$ of finite type over k, together with isomorphisms $\varphi^U_V : j^*\mathcal{F}_U \xrightarrow{\sim} \mathcal{F}_V$ for each open embedding $j : V \hookrightarrow U$ satisfying obvious transitivity conditions. A morphism $\alpha : \{\mathcal{F}_U\} \to \{\mathcal{G}_U\}$ is a system of maps $\alpha_U : \mathcal{F}_U \to \mathcal{G}_U$ in $D^b(U)$ such that α_U restricts to α_V on V.

Examples of objects in $\underleftarrow{D}^b(X)$ include the constant sheaf $\mathbb{Q}_{\ell,X} := \{\mathbb{Q}_{\ell,U}\}$ and the dualizing complex $\mathbb{D}_X := \{\mathbb{D}_U\}$.

A.2. Functors

Let $f : X \to Y$ be a morphism which is locally of finite type. We have the following functors

1. $f^* : \underleftarrow{D}^b(X) \to \underleftarrow{D}^b(Y)$. For $U \in \mathrm{Ft}(X)$, $f(U)$ is contained in some $V \in \mathrm{Ft}(Y)$. Denote by $f_{U,V} : U \to V$ the restriction of f. We define $(f^*\mathcal{G})_U = f^*_{U,V}\mathcal{G}_V$.
2. $f^! : \underleftarrow{D}^b(X) \to \underleftarrow{D}^b(Y)$, defined in a similarly way as f^*: $(f^!\mathcal{G})_U := f^!_{U,V}\mathcal{G}_V$.
3. If f is of finite type, we have
$$f_! : \underleftarrow{D}^b(X) \to \underleftarrow{D}^b(Y)$$
For $V \in \mathrm{Ft}(Y)$, $f^{-1}(V) \in \mathrm{Ft}(X)$. Let $f_V : f^{-1}(V) \to V$ be the restriction of f. We define $(f_!\mathcal{F})_V := f_{V,!}\mathcal{F}_{f^{-1}(V)}$.
In general, if f is only locally of finite type, we have
$$f_! : \underleftarrow{D}^b(X) \to \mathrm{ind}\ \underleftarrow{D}^b(Y)$$
where ind $\underleftarrow{D}^b(Y)$ denotes the category of ind-objects in $\underleftarrow{D}^b(Y)$. We define $f_!\mathcal{F}$ as the ind-object $\varinjlim_{U \in \mathrm{Ft}(X)} f_{U,!}\mathcal{F}_U$, where $f_U : U \to Y$, the restriction of f, is of finite type, and $f_{U,!}$ is defined above.
4. If f is of finite type, we have
$$f_* : \underleftarrow{D}^b(X) \to \underleftarrow{D}^b(Y)$$
defined in a similar way as $f_!$: $(f_*\mathcal{F})_V := f_{V,*}\mathcal{F}_{f^{-1}(V)}$.
In general, if f is only locally of finite type, we have
$$f_* : \underleftarrow{D}^b(X) \to \mathrm{pro}\ \underleftarrow{D}^b(Y)$$
where pro $\underleftarrow{D}^b(Y)$ denotes the category of pro-objects in $\underleftarrow{D}^b(Y)$. We define $f_*\mathcal{F}$ as the pro-object $\varprojlim_{U \in \mathrm{Ft}(X)} f_{U,*}\mathcal{F}_U$.

In particular, we can still define

$$\mathbf{H}_*(X/Y) := f_!\mathbb{D}_X \in \mathrm{ind}\ \underleftarrow{D}^b(Y).$$

When $Y = Speck$, we have

$$\mathrm{H}_c^*(X) = f_!\mathbb{Q}_{\ell,X}, \quad \mathrm{H}_*(X) = f_!\mathbb{D}_X \in \mathrm{ind}\ D^b(\mathbb{Q}_\ell\text{-vector spaces});$$
$$\mathrm{H}^*(X) = f_*\mathbb{Q}_{\ell,X}, \quad \mathrm{H}_*^{\mathrm{BM}}(X) = f_*\mathbb{D}_X \in \mathrm{pro}\ D^b(\mathbb{Q}_\ell\text{-vector spaces}).$$

A.3. Cohomological correspondences

In this appendix, we extend the formalism of cohomological correspondences (see [**?**] and [**?**, Appendix A]) to situations where the relevant algebraic spaces are locally of finite type.

Consider a correspondence diagram

$$\begin{tikzcd} & C \arrow[dl, "\overleftarrow{c}"'] \arrow[dr, "\overrightarrow{c}"] & \\ X \arrow[r, "f"'] & S & Y \arrow[l, "g"] \end{tikzcd} \tag{45}$$

where

- S is locally of finite type over a field k;
- f, g are locally of finite type;
- $\overrightarrow{c}$ is proper and $\overleftarrow{c}$ is of finite type.

For $\mathcal{F} \in D^b(X)$ and $\mathcal{G} \in D^b(Y)$, we define as in [**?**, Definition A.1.1]

$$\mathrm{Corr}(C; \mathcal{F}, \mathcal{G}) := Hom_{\underleftarrow{D}^b(C)}(\overrightarrow{c}^*\mathcal{G}, \overleftarrow{c}^!\mathcal{F}).$$

We call an element $\zeta \in \mathrm{Corr}(C; \mathcal{F}, \mathcal{G})$ a *cohomological correspondence* between $\mathcal{F}$ and $\mathcal{G}$ with support on C.

Given $\zeta \in \mathrm{Corr}(C; \mathcal{F}, \mathcal{G})$, we define

$$\zeta_\# : g_!\mathcal{G} \xrightarrow{g_!(\mathrm{ad.})} g_!\overrightarrow{c}_*\overrightarrow{c}^*\mathcal{G} \xrightarrow{g_!\overrightarrow{c}_*\zeta} g_!\overrightarrow{c}_*\overleftarrow{c}^!\mathcal{F} = g_!\overrightarrow{c}_!\overleftarrow{c}^!\mathcal{F} = f_!\overleftarrow{c}_!\overleftarrow{c}^!\mathcal{F} \xrightarrow{f_!(\mathrm{ad.})} f_!\mathcal{F}.$$

In the equality above, we used $\overrightarrow{c}_! = \overrightarrow{c}_*$ since it is proper. Arrows indexed by "ad." all come from the relevant adjunction for the morphisms $\overleftarrow{c}$ and $\overrightarrow{c}$, which are of finite type. Note that $\zeta_\#$ is a morphism in ind $D^b(S)$.

Most of the results in [**?**, Appendix A] are still valid in this extended situation. In particular, the results on pull-backs of cohomological correspondences in [**?**, Appendix A.4] extends *verbatim*.

A.4. Composition

Suppose we have the following diagram

$$
\begin{tikzcd}
& & C \arrow[dl, "\overleftarrow{d}"] \arrow[dr, "\overrightarrow{d}"'] \arrow[ddll, bend right, "\overleftarrow{c}"'] \arrow[ddrr, bend left, "\overrightarrow{c}"] & & \\
& C_1 \arrow[dl, "\overleftarrow{c_1}"'] \arrow[dr, "\overrightarrow{c_1}"] & & C_2 \arrow[dl, "\overleftarrow{c_2}"] \arrow[dr, "\overrightarrow{c_2}"'] & \\
X \arrow[drr, "f"'] & & Y \arrow[d, "g"] & & Z \arrow[dll, "h"] \\
& & S & &
\end{tikzcd}
\tag{46}
$$

where $C = C_1 \times_Y C_2$ and C_1 and C_2 satisfy the conditions in beginning of §**??**. Since $\overrightarrow{c_1}, \overrightarrow{c_2}$ are proper, so are $\overrightarrow{d}$ and $\overrightarrow{c}$. Similarly, $\overleftarrow{c}$ is of finite type. Hence C, as a correspondence between X and Z, also satisfies the conditions in the beginning of §**??**.

Let $\mathcal{F} \in \underleftarrow{D}^b(X), \mathcal{G} \in \underleftarrow{D}^b(Y)$ and $\mathcal{H} \in \underleftarrow{D}^b(Z)$. The convolution product defined in [**?**, Appendix A.2] extends to the current situation, giving a bilinear map

$$\circ : \mathrm{Corr}(C_1; \mathcal{F}, \mathcal{G}) \otimes \mathrm{Corr}(C_2, \mathcal{G}, \mathcal{H}) \to \mathrm{Corr}(C; \mathcal{F}, \mathcal{H}).$$

The following statement is a variant of [**?**, Lemma A.2.1], and is proved by a diagram-chasing:

Lemma 8. *Let* $\zeta_1 \in \mathrm{Corr}(C_1; \mathcal{F}, \mathcal{G})$ *and* $\zeta_2 \in \mathrm{Corr}(C_2; \mathcal{G}, \mathcal{H})$*. Then*

$$(\zeta_1 \circ \zeta_2)_\# = \zeta_{1,\#} \circ \zeta_{2,\#} : h_! \mathcal{H} \to f_! \mathcal{F}.$$

The associativity of the convolution $\circ$ also holds, see [**?**, Lemma A.2.2].

A.5. Property (G-2)

From now on we assume both X and Y are smooth of equidimension d. Recall from [**?**, Appendix A.6] that we say C has Property (G-2) with respect to an open subset $U \subset S$ if $\dim C_U \leq d$ and the image of $C - C_U \to X \times_S Y$ has dimension $< d$.

[**?**, Lemma A.6.2] now reads

Lemma 9. *Suppose C satisfies (G-2) with respect to $U \subset S$. Let $\zeta, \zeta' \in \mathrm{Corr}(C; \mathbb{Q}_{\ell,X}, \mathbb{Q}_{\ell,Y})$. If $\zeta|_U = \zeta'|_U \in \mathrm{Corr}(C_U; \mathbb{Q}_{\ell,X_U}, \mathbb{Q}_{\ell,Y_U})$, then $\zeta_\# = \zeta'_\# \in Hom_S(g_! \mathbb{Q}_{\ell,Y}, f_! \mathbb{Q}_{\ell,X})$.*

Acknowledgements. The author would like to thank R.Bezrukavnikov, B-C.Ngô and Y.Varshavsky for helpful discussions. He also appreciates the suggestions of the referee which improve the presentation of the paper. The author's research is supported by the NSF grants DMS-0969470, DMS-1261660, DMS-1302071 and the Packard Foundation.

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