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Formule des traces et fonctorialité: le début d'un programme. (French. English, French summaries) [Trace formula and functionality: the beginnings of a program]
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The paper under review gives a proposal to solve the Langlands functoriality problem, which combines R. P. Langlands' idea of "beyond endoscopy", B. C. Ngô's proof of the Fundamental Lemma [Publ. Math. Inst. Hautes Études Sci. No. 111 (2010), 1–169; MR2653248 (2011h:22011)] and ideas coming from geometric Langlands theory.

The paper contains two major themes. One is a new trace formula approach to classifying automorphic representations of a given group G according to the groups "where they truly come from". The other is what the authors call the "adelization of the trace formula", or Poisson summation on the Steinberg-Hitchin base. The second theme is the first step towards realizing the approach outlined in the first theme.

The Langlands functoriality problem seeks to relate automorphic representations of different groups via their L-groups. Suppose G and H are reductive algebraic groups over a global field F, and we are given a homomorphism between their L-groups $\psi\colon {}^LH \to {}^LG$. Then, to any automorphic representation π_H of $H(\mathbb{A}_F)$, one expects to associate an automorphic representation π_G of $G(\mathbb{A}_F)$, such that for every place v of F at which π_H is unramified, the Satake parameter $A(\pi_{G,v})$ of $\pi_{G,v}$ (a semisimple conjugacy class of ${}^LG$) is the same as the image $\psi(A(\pi_{H,v}))$ of the Satake parameter of $\pi_{H,v}$. Thanks to the effort of many mathematicians over the past forty years or so, this expectation has become reality in several important cases, most notably in the endoscopic cases (and their twisted versions), which build upon the fundamental lemma proved by Ngô [op. cit.].

Langlands [in *Contributions to automorphic forms, geometry, and number theory*, 611–697, Johns Hopkins Univ. Press, Baltimore, MD, 2004; MR2058622 (2005f:11102); Canad. Math. Bull. **50** (2007), no. 2, 243–267; MR2317447 (2008m:11095)] proposed an approach to establish the functoriality beyond the endoscopic cases, which is also briefly recalled in Section 1 of the current paper. For an embedding $\psi\colon {}^LH \hookrightarrow {}^LG$ and an automorphic representation π_G of $G(\mathbb{A}_F)$, Langlands suggested looking at the poles of the various L-functions of π_G to detect whether π_G comes from π_H via functoriality. For each algebraic representation $\rho\colon {}^LG \to \mathrm{GL}(V)$, let $m_H(\rho)$ be the dimension of the fixed points of ${}^LH$ on V. Suppose π_G comes from some π_H; then the L-function $L(s,\pi_G,\rho)$ is expected to have a pole of order $m_H(\rho)$ at $s=1$. Conversely, by looking at the poles of the L-functions $L(s,\pi_G,\rho)$ for sufficiently many ρ's, one should be able to make a guess (although not always unique) of what ${}^LH$, or even better H, should be.

In Section 1, the authors consider a sum that assembles the partial L-functions of all automorphic representations (unramified outside a finite set of places S) together:

$$\sum_{\pi_G}\prod_{v\in S}(\pi_{G,v}(f_v))L_S(s,\pi_G,\rho); \tag{1}$$

here f_v are test functions to put at the places $v\in S$. They first observe that this sum is, at least formally, the trace of a test function on the space of all automorphic forms. In the case where F is a global function field, this test function can be written as a sum $\sum_{d\geq 0}\mathbb{K}_{\rho,d}$, and each $\mathbb{K}_{\rho,d}$ can be geometrized into a perverse sheaf $\mathcal{K}_{\rho,d}$ on the Beilinson-Drinfeld affine Grassmannian, whose pointwise Frobenius trace gives back the test function $\mathbb{K}_{\rho,d}$. The idea then is to study the poles of the sum (1) (or rather its stabilized version). The poles to the right of $\mathrm{Re}(s)=1$ should be given by those π_G that are not Ramanujan (i.e., with nontrivial SL_2-part in their Arthur parameters). For example, the rightmost pole should be given by the one-dimensional representations of G. By induction, we may assume that we understand these non-Ramanujan terms because they only involve groups smaller than G. Subtracting these non-Ramanujan terms from (1), the remaining part $\Xi(s)$ should only involve Ramanujan π_G's (i.e., with trivial SL_2-part), and should be analytic on the half plane $\mathrm{Re}(s)>1$. The poles at $s=1$ of $\Xi(s)$ should be contributed by those π_G that come via functoriality from smaller groups H satisfying $m_H(\rho)>0$. One can then hope to establish the functoriality for such pairs (G,H) by comparing the asymptotic behavior of $\Xi(s)$ at $s=1$ with a trace similar to (1) for the relevant H's (presumably via studying the geometric sides of the trace formulae).

The difficulty of realizing this approach is mainly analytic. In Sections 3–5 of the paper under review, the authors make the first step towards this end by introducing an adelic version of the trace formula.

The authors introduce the Steinberg-Hitchin base $\mathfrak{A}$ of G. When G is semisimple and simply-connected, this is the adjoint quotient of G by itself in the sense of geometric invariant theory. For the purpose of this review, we restrict to this case. Their observation is that $\mathfrak{A}$ is naturally a vector space over the global field F, so one can make sense of the Fourier transform of functions on $\mathfrak{A}(\mathbb{A}_F)$ and the Poisson summation formula holds. The authors then introduce a variant of the stable orbital integral $\theta(a;s)=\prod_v\theta_v(a;s)$ (where $a\in\mathfrak{A}(F)$, and s is a complex variable) by inserting an L-factor into the usual stable orbital integral. This function $\theta(a,s)$ then makes sense for any $a\in\mathfrak{A}(\mathbb{A}_F)$ and has good analytic properties when $\mathrm{Re}(s)$ is large. When $s=1$, $\theta(a,s)$ is equal to the usual stable orbital integral if it converges. Adélization of the trace formula means systematically working with $\theta(\cdot,s)$ instead of the usual stable orbital integrals in the trace formula.

In Section 4, the authors make sense of the Fourier transform and Poisson summation of the function $\theta(\cdot,s)$. A truncation technique of J. Getz is used. Using Poisson summation, the dominant term of the geometric side of the adelic trace formula then becomes the limit of $\widehat{\theta}(0,s)$ as $s\to 1$. In Section 5, they prove that this dominant term comes from the trivial representation of G (see Proposition 5.6), i.e.,

$$\lim_{s\searrow 1}\widehat{\theta}(0,s)=\int_{G(\mathbb{A}_F)}f(g)dg.$$

Here f is the test function used to define the orbital integral $\theta(\cdot,s)$. This gives strong support to the expectation in Section 1 that the rightmost pole of the sum (1) should come from one-dimensional automorphic representations.

The paper is not written in the usual pedagogical manner but rather has an experimental flavor. Section 3 contains a fairly detailed discussion of the Steinberg-Hitchin base and the choice of measure in orbital integrals. Other sections require more in depth background knowledge of the theory of automorphic forms. For Section 1, one may consult Langlands' paper [op. cit.; MR2058622 (2005f:11102)], where the idea of beyond endoscopy first appeared; for Section 2, some familiarity with geometric Langlands theory and the sheaf-function correspondence is needed; see for example the survey article of E. V. Frenkel [Bull. Amer. Math. Soc. (N.S.) **41** (2004), no. 2, 151–184; MR2043750 (2005e:11147)].

The paper [Bull. Math. Sci. **1** (2011), no. 1, 129–199, doi:10.1007/s13373-011-0009-0] by Frenkel and Ngô proposes a geometrization of the trace formula in the setting of geometric Langlands theory, and some expectations in Section 1 of the current paper are made precise there in certain special cases.

Reviewed by *Zhiwei Yun*