Invent math
DOI 10.1007/s00222-013-0469-9

Motives with exceptional Galois groups and the inverse Galois problem

Zhiwei Yun

Received: 14 December 2011 / Accepted: 10 January 2013

Abstract We construct motivic ℓ-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into exceptional groups of type E_7, E_8 and G_2 whose image is Zariski dense. This answers a question of Serre. The construction is uniform for these groups and is inspired by the Langlands correspondence for function fields. As an application, we solve new cases of the inverse Galois problem: the finite simple groups $E_8(\mathbb{F}_\ell)$ are Galois groups over $\mathbb{Q}$ for large enough primes ℓ.

Mathematics Subject Classification 14D24 · 12F12 · 20G41

Contents

Z. Yun (✉)
Department of Mathematics, Stanford University, 450 Serra Mall, Bldg 380, Stanford, CA 94305, USA
e-mail: zwyun@stanford.edu

Published online: 14 May 2013

1 Introduction

1.1 Serre's question

About two decades ago, Serre raised the following question which he described as "plus hasardeuse" (English translation: more risky):

Question 1.1 (Serre [36, Sect. 8.8]) *Is there a motive M (over a number field) such that its motivic Galois group is a simple algebraic group of exceptional type G_2 or E_8?*

The purpose of this paper is to give an affirmative answer to a variant of Serre's question for E_7, E_8 and G_2, and to give applications to the inverse Galois problem.

1.1.1 Motivic Galois groups

Let us briefly recall the notion of the motivic Galois group, following [36, Sects. 1 and 2]. Let k and L be number fields. Let $\mathrm{Mot}_k(L)$ be the category of motives over k with coefficients in L (under numerical equivalences). This is an abelian category obtained by formally adjoining direct summands of smooth projective varieties over k cut out by idempotent correspondences with L-coefficients. Assuming the Standard Conjectures, the category $\mathrm{Mot}_k(L)$ becomes a semisimple L-linear Tannakian category (see Jannsen [22, Corollary 2]). Moreover, it admits a tensor structure and a fiber functor ω into Vec_L, the tensor category of L-vector spaces. For example, one may take ω to be the singular cohomology of the underlying analytic spaces (using a fixed embedding $k \hookrightarrow \mathbb{C}$) with L-coefficients. By Tannakian formalism [10], such a structure gives a group scheme G_k^{Mot} over L as the group of tensor automorphisms of ω. This is the *absolute motivic Galois group of* k.

Any motive $M \in \mathrm{Mot}_k(L)$ generates a Tannakian subcategory $\mathrm{Mot}(M)$ of $\mathrm{Mot}_k(L)$. Tannakian formalism again gives a group scheme G_M^{Mot} over L, the group of tensor automorphisms of $\omega|_{\mathrm{Mot}(M)}$. This is *the motivic Galois group of* M.

Of course Serre's question could be asked for other exceptional types. Although people hoped for an affirmative answer to Serre's question, the search within "familiar" types of varieties all failed. For example, one cannot find an abelian variety with exceptional motivic Galois groups (see [30, Corollary 1.35] for the fact that the Mumford-Tate groups of abelian varieties cannot have exceptional factors, and by [10, Theorem 6.25], the Mumford-Tate group of an abelian variety surjects onto its motivic Galois group), nor does one have Shimura varieties of type E_8, F_4 or G_2.

1.1.2 Motivic Galois representations

Let ℓ be a prime number. Fix an embedding $L \hookrightarrow \overline{\mathbb{Q}}_\ell$. For a motive $M \in \mathrm{Mot}_k(L)$, we have the *ℓ-adic realization* $\mathrm{H}(M, \overline{\mathbb{Q}}_\ell)$ which is a continuous $\mathrm{Gal}(\overline{k}/k)$-module.

Let V be a finite dimensional $\mathbb{Q}_\ell$-vector space. We call a continuous representation $\rho : \mathrm{Gal}(\overline{k}/k) \to \mathrm{GL}(V)$ *motivic* if there exists a motive $M \in \mathrm{Mot}_k(L)$ (for some number field L) such that $V \otimes \overline{\mathbb{Q}}_\ell$ is isomorphic to $\mathrm{H}(M, \overline{\mathbb{Q}}_\ell)$ as $\mathrm{Gal}(\overline{k}/k)$-modules.

Let $\widehat{G}$ be a reductive algebraic group over $\mathbb{Q}_\ell$. A continuous representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{G}(\mathbb{Q}_\ell)$ is called *motivic* if for some faithful algebraic representation V of $\widehat{G}$, the composition $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\rho} \widehat{G}(\mathbb{Q}_\ell) \to \mathrm{GL}(V)$ is motivic.

Fix an embedding $L \hookrightarrow \overline{\mathbb{Q}}_\ell$, we have an exact functor $\mathrm{H}(-, \overline{\mathbb{Q}}_\ell) : \mathrm{Mot}_k(L) \to \mathrm{Rep}(\mathrm{Gal}(\overline{k}/k), \overline{\mathbb{Q}}_\ell)$ by taking étale cohomology with $\overline{\mathbb{Q}}_\ell$-coefficients, on which the Galois group $\mathrm{Gal}(\overline{k}/k)$ acts continuously. We call this the *ℓ-adic realization functor*. For a motive $M \in \mathrm{Mot}_k(L)$, we define the *ℓ-adic motivic Galois group* of M to be the Zariski closure of the image of the representation $\rho_{M,\ell} : \mathrm{Gal}(\overline{k}/k) \to \mathrm{GL}(\mathrm{H}(M, \overline{\mathbb{Q}}_\ell))$. We denote the ℓ-adic motivic Galois group of M by $G_{M,\ell}$, which is an algebraic group over $\overline{\mathbb{Q}}_\ell$. It is expected that $G_{M,\ell} \cong G_M^{\mathrm{Mot}} \otimes_L \overline{\mathbb{Q}}_\ell$ (see [36, Sect. 3.2]).

1.2 Main results

We will answer Serre's question for ℓ-adic motivic Galois groups instead of the actual motivic Galois groups, because their existence depends on the Standard Conjectures.

Main Theorem 1.2 *Let $\widehat{G}$ be a split simple adjoint group of type A_1, E_7, E_8 or G_2. Let ℓ be a prime number. Then there exists an integer $N \geq 1$ and a continuous representation*

$$\rho : \pi_1\big(\mathbb{P}^1_{\mathbb{Z}[1/2\ell N]} - \{0, 1, \infty\}\big) \to \widehat{G}(\mathbb{Q}_\ell)$$

such that

(1) *For each geometric point* $\operatorname{Spec} k \to \operatorname{Spec} \mathbb{Z}[1/2\ell N]$, *the restriction of ρ to the geometric fiber $\mathbb{P}^1_k - \{0, 1, \infty\}$:*

$$\rho_k : \pi_1\big(\mathbb{P}^1_k - \{0, 1, \infty\}\big) \to \widehat{G}(\mathbb{Q}_\ell)$$

has Zariski dense image.

(2) *The restriction of ρ to a rational point x* : $\operatorname{Spec} \mathbb{Q} \to \mathbb{P}^1 - \{0, 1, \infty\}$:

$$\rho_x : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{G}(\mathbb{Q}_\ell)$$

is either motivic (if $\widehat{G}$ is of type E_8 or G_2) or becomes motivic when restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(i))$ *(if $\widehat{G}$ is of type A_1 or E_7).*

(3) *There exist infinitely many rational points $\{x_1, x_2, \dots\}$ of $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ such that ρ_{x_i} are mutually non-isomorphic and all have Zariski dense image.*

Corollary 1.3 *For $\widehat{G}$ a simple adjoint group of type A_1, E_7, E_8 or G_2, there exist infinitely many non-isomorphic motives over $\mathbb{Q}$ (if $\widehat{G}$ is of type E_8 or G_2) or $\mathbb{Q}(i)$ (if $\widehat{G}$ is of type E_7) whose ℓ-adic motivic Galois groups are isomorphic to $\widehat{G}$. In particular, Serre's question for ℓ-adic motivic Galois groups has an affirmative answer for A_1, E_7, E_8 and G_2.*

1.2.1 A known case

In [11], Dettweiler and Reiter constructed a rank seven *rigid* local system on $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ whose geometric monodromy is dense in G_2. The restriction of this local system to a general rational point gives a motivic Galois representation whose image is dense in G_2. We believe that our construction in the G_2 case gives the same local system as theirs (see Remark 5.12).

On the other hand, Gross and Savin [19] gave a candidate G_2-motive in the cohomology of a Siegel modular variety.

1.2.2 Application to the inverse Galois problem

The inverse Galois problem for $\mathbb{Q}$ asks whether every finite group can be realized as a Galois group of a finite Galois extension of $\mathbb{Q}$. A lot of finite simple groups are proved to be Galois groups over $\mathbb{Q}$, see [29] and [37], yet the problem is still open for many finite simple groups of Lie type. We will be concerned with finite simple groups $G_2(\mathbb{F}_\ell)$ and $E_8(\mathbb{F}_\ell)$, where ℓ is a prime. By Thompson [40] and Feit and Fong [14], $G_2(\mathbb{F}_\ell)$ is known to be a Galois group over $\mathbb{Q}$ for all primes $\ell \geq 5$. However, according to [29, Chap. II, Sect. 10], $E_8(\mathbb{F}_\ell)$ is known to be a Galois group over $\mathbb{Q}$ only for $\ell \equiv \pm 3, \pm 7, \pm 9, \pm 10, \pm 11, \pm 12, \pm 13, \pm 14 \pmod{31}$. As an application of our main construction, we solve new instances of the inverse Galois problem.

Theorem 1.4 *For sufficiently large prime ℓ, the finite simple group $E_8(\mathbb{F}_\ell)$ can be realized as Galois groups over any number field.*

1.3 The case of type A_1

To illustrate the construction in the Main Theorem, we give here an analog of the Main Theorem for $\widehat{G} = \mathrm{PGL}_2$, in which case the motives involved are more familiar. Let $k = \mathbb{F}_q$ be a finite field of characteristic not 2. The construction starts with an automorphic form of $G = \mathrm{SL}_2$. Let $T \subset B \subset G$ be the diagonal torus and the upper triangular matrices. Let $F = k(t)$ be the function field of $\mathbb{P}^1_k$. For each place v of F, let $\mathcal{O}_v$, F_v and k_v be the corresponding completed local ring, local field and residue field. For each v, we have the Iwahori subgroup $\mathbf{I}_v \subset G(\mathcal{O}_v)$ which is the preimage of $B(k_v) \subset G(k_v)$ under the reduction map $G(\mathcal{O}_v) \to G(k_v)$.

We will consider irreducible automorphic representations $\pi = \bigotimes_v \pi_v$ of $G(\mathbb{A}_F)$. We would like π to satisfy the following conditions:

- π_1 and π_∞ both contain a nonzero fixed vector under the Iwahori subgroup $\mathbf{I}_1$ and $\mathbf{I}_\infty$.
- π_0 contains a nonzero vector on which the Iwahori subgroup $\mathbf{I}_0$ acts through the quadratic character $\mu : \mathbf{I}_0 \to T(k) = k^\times \twoheadrightarrow \{\pm 1\}$.
- For $v \neq 0, 1$ or ∞, π_v is unramified.

Using similar argument as in Theorem 3.2, one can show that such an automorphic representation π exists provided $\sqrt{-1} \in k$. Langlands philosophy then predicts that there should exist a tame $\mathrm{PGL}_2(\overline{\mathbb{Q}}_\ell)$-local system on $\mathbb{P}^1_k - \{0, 1, \infty\}$ which has unipotent monodromy around the punctures 1 and ∞, and has monodromy of order two around the puncture 0. Methods from geometric Langlands theory allow us to write down this local system explicitly as follows.

1.3.1 The local system

We continue to assume that $\sqrt{-1} \in k$. Consider the following family of genus 3 projective smooth curves $f : C \to \mathbb{P}^1_k - \{0, 1, \infty\}$:

$$C_\lambda : y^4 = \frac{\lambda x - 1}{\lambda x(x-1)}, \quad \lambda \in \mathbb{P}^1_k - \{0, 1, \infty\}. \tag{1.1}$$

The group $\mu_4(k)$ acts on the local system $\mathbf{R}^1 f_* \overline{\mathbb{Q}}_\ell$. Let $\chi : \mu_4(k) \to \overline{\mathbb{Q}}_\ell^\times$ be a character of order four. We may decompose $\mathbf{R}^1 f_* \overline{\mathbb{Q}}_\ell$ according to the action of $\mu_4(k)$:

$$\mathbf{R}^1 f_* \overline{\mathbb{Q}}_\ell = L_{\mathrm{sgn}} \oplus L_\chi \oplus L_{\overline{\chi}}.$$

Here L_{sgn}, L_χ and $L_{\overline{\chi}}$ are rank two local system defined on $\mathbb{P}^1 - \{0, 1, \infty\}$. In fact L_{sgn} is the H^1 of the Legendre family of elliptic curves. By Katz's results on the local monodromy of middle convolutions, one can show [1] that the local geometric monodromy of both L_χ and $L_{\overline{\chi}}$ at 0, 1 and ∞ are conjugate to

$$\mu_0 \sim \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}; \qquad \mu_1 \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \qquad \mu_\infty \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The $\mathrm{PGL}_2(\overline{\mathbb{Q}}_\ell) = \mathrm{SO}_3(\overline{\mathbb{Q}}_\ell)$-local systems $\mathrm{Sym}^2(L_\chi)(1)$ and $\mathrm{Sym}^2(L_{\overline{\chi}})(1)$ are canonically isomorphic, and this is the local system predicted by the Langlands correspondence. Moreover, it can be shown that this $\mathrm{PGL}_2(\overline{\mathbb{Q}}_\ell)$ local system comes from a $\mathrm{PGL}_2(\mathbb{Q}_\ell)$-local system via extension of coefficient fields, and this construction can in fact be modified to work even when $\sqrt{-1} \notin k$.

The family of curves (1.1) naturally shows up when one preforms the geometric Hecke operators (the analog of the operators T_p on modular forms) to the geometric analog (called an *automorphic sheaf*) of a particular automorphic form in the representation π considered above.

1.3.2 Switching to number fields

The construction in Sect. 1.3.1 makes perfect sense if we replace the finite field k by any field of characteristic not equal to 2, and in particular $\mathbb{Q}$. The resulting PGL_2-local system $\mathrm{Sym}^2(L_\chi)(1)$ over $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ is the output of our Main Theorem in the case of type A_1. This local system is visibly motivic (at least when base changed to $\mathbb{Q}(i)$) because it is part of the H^1 of the family of curves (1.1) cut out by the μ_4-action.

[1] This is communicated to the author by N.Katz. It can also be deduced from our more general results in Propositions 5.4, 5.3 and Remark 5.6.

1.4 Description of the motives

Now we give an explicit description of the local systems that appear in the Main Theorem as direct summands of cohomology of smooth varieties, which explains why they are motivic. Let G be a split simply-connected group of type A_1, D_{2n}, E_7, E_8 or G_2, defined over $\mathbb{Q}$. Let $\widehat{G}$ be its Langlands dual group defined over $\mathbb{Q}_\ell$.

Let $\theta^\vee$ be the coroot of G corresponding to the highest root θ. Let $V_{\theta^\vee}$ be the irreducible representation of $\widehat{G}$ with highest weight $\theta^\vee$, which we call the *quasi-minuscule representation* of $\widehat{G}$. This is either the adjoint representation (if G is simply-laced) or the seven dimensional representation of G_2. For $x \in \mathbb{Q} - \{0, 1\}$, let ρ_x^{qm} be the composition

$$\rho_x^{\mathrm{qm}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\rho_x} \widehat{G}(\mathbb{Q}_\ell) \to \mathrm{GL}(V_{\theta^\vee})$$

where ρ_x is as in Main Theorem (2). We will describe ρ_x^{qm} motivically.

Let P be the "Heisenberg parabolic" subgroup of G containing T with roots $\{\beta \in \Phi_G | \langle \beta, \theta^\vee \rangle \geq 0\}$. The unipotent radical of P is a Heisenberg group whose center has Lie algebra $\mathfrak{g}_\theta$, the highest root space. The contracted product gives a line bundle over the partial flag variety G/P:

$$Y = G \overset{P}{\times} \mathfrak{g}_\theta^*$$

which is a smooth variety over $\mathbb{Q}$ of dimension $2h^\vee - 2$. Here $h^\vee$ is the dual Coxeter number of G, i.e., $h^\vee = 2, 4n - 2, 18, 30$ and 4 for $G = A_1, D_{2n}, E_7, E_8$ and G_2 respectively. So the corresponding variety Y has dimension $2, 8n - 6, 34, 58$ and 6 in the cases A_1, D_{2n}, E_7, E_8 and G_2 respectively. There is a divisor $D_x \subset Y$ depending algebraically on x which is cut out by a modular interpretation of Y. Unfortunately we have not yet seen a direct way of describing the divisors D_x. There is also a finite group scheme $\widetilde{A}$ over $\mathbb{Q}$ and an $\widetilde{A}$-torsor

$$\widetilde{Y}_x \to Y - D_x. \tag{1.2}$$

The group scheme $\widetilde{A}$ is well-known to experts in real Lie groups, and will be defined in Sect. 2.6.1. It is a central extension of $T[2] \cong \mu_2^{\mathrm{rank}\ G}$ by μ_2. Let $\mathbb{Q}_\ell[\widetilde{A}(\overline{\mathbb{Q}})]$ be group algebra of $\widetilde{A}(\overline{\mathbb{Q}})$, and let $\mathbb{Q}_\ell[\widetilde{A}(\overline{\mathbb{Q}})]_{\mathrm{odd}}$ be the subspace where the central μ_2 acts via the sign representation.

Consider the middle dimensional cohomology $\mathrm{H}_c^{2h^\vee-2}(\widetilde{Y}_x, \mathbb{Q}_\ell)$. We take its direct summand $\mathrm{H}_c^{2h^\vee-2}(\widetilde{Y}_x, \mathbb{Q}_\ell)_{\mathrm{odd}}$ on which the central μ_2 of $\widetilde{A}$ acts via the sign representation. Then there is an $\widetilde{A}(\overline{\mathbb{Q}}) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant isomorphism

$$\mathrm{H}_c^{2h^\vee-2}(\widetilde{Y}_x, \overline{\mathbb{Q}}_\ell)_{\mathrm{odd}}\big(h^\vee - 1\big) \cong \rho_x^{\mathrm{qm}} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell\big[\widetilde{A}(\overline{\mathbb{Q}})\big]_{\mathrm{odd}}.$$

Here $(h^\vee - 1)$ means Tate twist. This realizes ρ_x^{qm} as a direct summand of the cohomology of a smooth variety (possibly after base change to $\mathbb{Q}(i)$), showing it is motivic.

1.5 The general construction

In the main body of the paper, we work with a simply-connected almost simple split group G over a field k with $\mathrm{char}(k) \neq 2$. We assume G to satisfy two conditions

(1) The longest element in the Weyl group of G acts by -1 on the Cartan subalgebra.
(2) G is oddly-laced: i.e., the ratio between the square lengths of long roots and short roots of G is odd.

By the Dynkin diagram classification (see [7, Planche I–IX]), the above conditions are equivalent to that G is of type A_1, D_{2n}, E_7, E_8 or G_2. Our goal is to construct $\widehat{G}(\mathbb{Q}_\ell)$-local systems over $\mathbb{P}^1_k - \{0, 1, \infty\}$, where $\widehat{G}$ is the Langlands dual group of G. Motivated by the work of Dettweiler and Reiter [11] on G_2, we would also like the local monodromy of these local systems around the punctures 0, 1 and ∞ to lie in specific conjugacy classes in $\widehat{G}$. These local monodromy conditions can be translated into local conditions on automorphic representations of $G(\mathbb{A}_F)$ via the hypothetical local Langlands correspondence. So our construction again starts with an automorphic representation of $G(\mathbb{A}_F)$ with prescribed local behavior.

Step I Let $F = k(t)$ be the function field of $\mathbb{P}^1_k$, where k is a finite field. We consider automorphic representations $\pi = \bigotimes'_{v\in|\mathbb{P}^1|} \pi_v$ satisfying the following conditions

- π_1 has a nonzero fixed vector under the Iwahori subgroup $\mathbf{I}_1 \subset G(F_1)$;
- π_∞ has a nonzero fixed vector under the parahoric $\mathbf{P}_\infty \subset G(F_\infty)$;
- π_0 has an eigenvector on which $\mathbf{P}_0 \subset G(F_0)$ acts through a nontrivial quadratic character $\mu : \mathbf{P}_0 \to \{\pm 1\}$;
- π_v is unramified for $v \neq 0, 1$ or ∞.

Gross, Reeder and Yu [33] establishes a bijection between regular elliptic conjugacy classes in W and certain class of parahoric subgroups of a p-adic group. Under this bijection, the parahoric subgroup $\mathbf{P}_0$ corresponds to the element $-1 \in W$. Its reductive quotient admits a unique nontrivial quadratic character μ. The parahoric $\mathbf{P}_\infty$ has the same type as $\mathbf{P}_0$. For details, we refer to Sect. 2.3.

Step II Show that such automorphic representations do exist and are very limited in number. The argument for this relies on a detailed study of the structure of the double coset

$$G(F)\backslash G(\mathbb{A}_F)/\left(\mathbf{P}_0 \times \mathbf{I}_1 \times \mathbf{P}_\infty \times \prod_{v\neq 0,1,\infty} G(\mathcal{O}_v)\right). \tag{1.3}$$

Up to making a finite extension of k, we have

$$\sum_{\pi \text{ as above}} m(\pi)\dim \pi_0^{(\mathbf{P}_0,\mu)} \dim \pi_1^{\mathbf{I}_1} \dim \pi_\infty^{\mathbf{P}_\infty} = \#ZG. \tag{1.4}$$

Here, $m(\pi)$ is the multiplicity of π in the automorphic spectrum, $\pi_0^{(\mathbf{P}_0,\mu)}$ is the μ-eigenspace under $\mathbf{P}_0$, and ZG is the center of G. Note that the central character of π has to be trivial, so the multiplicity $\#ZG$ in the above formula is *not* the contribution from different central characters.

Neither Step I nor Step II actually appear in the main body of the paper. We start directly with a geometric reinterpretation of the previous two steps, which makes sense for any field k with $\mathrm{char}(k) \neq 2$.

Step III We interpret the double coset (1.3) as the k-points of a moduli stack $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{I}_1, \mathbf{P}_\infty)$: the moduli stack of principal G-bundles over $\mathbb{P}^1$ with three level structures at $0, 1$ and ∞ as specified by the parahoric subgroups. In fact we will consider a variant $\mathrm{Bun} = \mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{I}_1, \mathbf{P}_\infty)$ of this moduli stack on which the geometric analog of the quadratic character μ can be defined. These moduli stacks are defined in Sect. 3.2. Automorphic functions on the double coset are upgraded to "odd" sheaves on Bun, which are studied in Sect. 3.3. Theorem 3.2 is crucial in understanding the structure of such odd sheaves: they correspond to odd representations of the finite group $\widetilde{A}$ we mentioned before.

Step IV We take an irreducible odd sheaf $\mathcal{F}$ on Bun, and apply geometric Hecke operators to it. A geometric Hecke operator $\mathbb{T}(\mathcal{K}, -)$ is a geometric analog of an integral transformation, which depends on a "kernel sheaf" $\mathcal{K}$. The kernel $\mathcal{K}$ is an object in the Satake category, which is equivalent to the category of algebraic representations of $\widehat{G}$. The resulting sheaf $\mathbb{T}(\mathcal{K}, \mathcal{F})$ is over $\mathrm{Bun} \times (\mathbb{P}^1 - \{0, 1, \infty\})$. In Theorem 4.2(1), we prove that $\mathcal{F}$ is an eigen object under geometric Hecke operators: every Hecke operator $\mathbb{T}(\mathcal{K}, -)$ transforms $\mathcal{F}$ to a sheaf of the form $\mathcal{F} \boxtimes \mathcal{E}(\mathcal{K})$ on $\mathrm{Bun} \times (\mathbb{P}^1 - \{0, 1, \infty\})$, where $\mathcal{E}(\mathcal{K})$ is a local system on $\mathbb{P}^1 - \{0, 1, \infty\}$. The collection $\{\mathcal{E}(\mathcal{K})\}_\mathcal{K}$ forms a tensor functor from the Satake category (which is equivalent to $\mathrm{Rep}(\widehat{G})$) to the category of local systems on $\mathbb{P}^1 - \{0, 1, \infty\}$, and gives the desired $\widehat{G}$-local system $\mathcal{E}$ on $\mathbb{P}^1 - \{0, 1, \infty\}$.

We see that the local system $\mathcal{E}$ depends on the choice of an odd sheaf $\mathcal{F}$. In fact there are exactly $\#ZG$ odd central characters χ of $\widetilde{A}$, each giving a unique irreducible odd representation V_χ of $\widetilde{A}$. Each V_χ in turn gives an irreducible odd sheaf $\mathcal{F}_\chi$ on Bun, and hence a $\widehat{G}$-local system $\mathcal{E}_\chi$ by the above procedure.

Step V Rationality issue. The irreducible odd representations of $\widetilde{A}$ are only defined over $\mathbb{Q}(i)$. Therefore the automorphic sheaf $\mathcal{F}_\chi$ has $\mathbb{Q}'_\ell = \mathbb{Q}_\ell(i)$-coefficients. Moreover, $\mathcal{F}_\chi$ may not be defined over Bun_k when the central character χ is not fixed by $\mathrm{Gal}(\overline{k}/k)$: it is sometimes only defined over $\mathrm{Bun}_{k'}$ where $k' = k(\sqrt{-1})$. The previous step gives us a $\widehat{G}(\mathbb{Q}'_\ell)$-local system on $\mathbb{P}^1_{k'} - \{0, 1, \infty\}$. We need to apply two descent arguments, first for the ground field (from k' to k) and then for the coefficient field (from $\mathbb{Q}'_\ell$ to $\mathbb{Q}_\ell$). These are the contents of Theorem 4.2(2) and (3).

Step VI Finally we extend the local system $\mathcal{E}_\chi$ from $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ to $\mathbb{P}^1_{\mathbb{Z}[1/2\ell N]} - \{0, 1, \infty\}$, such that its restriction along $\mathbb{P}^1_{\mathbb{F}_p} - \{0, 1, \infty\}$ is the same as $\mathcal{E}_{\chi,\mathbb{F}_p}$ constructed using the base field $\mathbb{F}_p$. This is done in Proposition 4.5.

1.5.1 Proofs of other main results

The above steps finish the constructive part of the Main Theorem. We indicate where the other main results are proved.

- For k an algebraically closed field, the density of the image of ρ_k is proved in Theorem 5.7, whose proof depends on detailed analysis of local monodromy in Sect. 5.2.
- The fact that ρ_x is motivic is proved in Proposition 4.6.
- To see there exists a rational number x such that ρ_x has dense image, we only need to use a variant of Hilbert irreducibility [39, Theorem 2]. In Corollary 5.9, we give an effective criterion for ρ_x to have dense image, and also prove that there are infinitely many non-isomorphic ρ_x's.
- The application to the inverse Galois problem (Theorem 1.4) is given in Sect. 5.6.

1.6 Conjectures and generalizations

In Sect. 5.5, we list some conjectural properties of the local and global monodromy of the local systems we construct. In Sect. 5.6.3, we also suggest an approach to proving Theorem 1.4 via the rigidity method.

In this paper we impose two conditions on the group G, namely $-1 \in W$ and odd-lacedness. We plan to remove these conditions and work with all types of almost simple groups in future work. For groups without -1 in their Weyl groups, we shall replace the constant group scheme $G \times \mathbb{P}^1$ by a quasisplit form of it. For doubly-laced groups, the analog of Theorem 3.2 is no longer true, but this can be circumvented by considering automorphic sheaves in a certain quotient category of sheaves. We hope these extensions will solve more instances of the inverse Galois problem.

2 Group-theoretic preliminaries

Our construction of motives with exceptional Galois groups will involve some known group-theoretic results. We collect them in this section for future reference.

Throughout the paper, we fix k to be a field with $\mathrm{char}(k) \neq 2$.

2.1 The group G

Let G be a split almost simple simply-connected group over k. Let $\mathfrak{g}$ be its Lie algebra. We fix a maximal torus T of G and a Borel subgroup B containing it. These data give a based root system $\Delta_G \subset \Phi_G \subset \mathbb{X}^*(T)$ (where Φ_G stands for roots and Δ_G for simple roots) and a Weyl group W. Throughout this paper, except in Sect. 2.4, we make the following assumption

$$\textit{The longest element in } W \textit{ acts as } -1 \textit{ on } \mathbb{X}^*(T). \tag{2.1}$$

Examining the classification of G [7, Planche I–IX], this means that G is of type $A_1, B_n, C_n, D_{2n}, E_7, E_8, F_4$ or G_2.

We call G *oddly-laced* if the ratio between the square lengths of long roots and short roots of G is odd. The classification shows that G is oddly-laced if and only if it is of type A_n, D_n, E_n or G_2.

We denote the center of G by ZG. The adjoint form $G^{\mathrm{ad}} := G/ZG$ has maximal torus $T^{\mathrm{ad}} := T/ZG$ and standard Borel subgroup $B^{\mathrm{ad}} := B/ZG$.

The highest root of G is denoted by θ. We denote by ρ (resp. $\rho^\vee$) half the sum of positive roots (coroots) of G. Let h and $h^\vee$ denote the Coxeter number and dual Coxeter number of G. We recall that $h - 1 = \langle \theta, \rho^\vee \rangle$ and $h^\vee - 1 = \langle \rho, \theta^\vee \rangle$.

The flag variety of G will be denoted by $f\ell_G$.

2.2 Loop groups

We review the definition of loop groups as functors following [13, Definition 1]. The fact that these functors are (ind-)representable is proved by Faltings in [13, Sect. 2]. The loop group LG is the functor assigning every k-algebra R the group $G(R((t)))$, where $R((t))$ is the ring of formal Laurent series in one variable t with coefficients in R. It is representable by an ind-scheme. Similarly we define the positive loops L^+G to be the functor $R \mapsto G(R[[t]])$, which is representable by a pro-algebraic group. In practice, we have a smooth curve and we denote its completion at a k-point x by $\mathcal{O}_x$, which is isomorphic to $k[[t]]$ but not canonically so. Let F_x be the field of fractions of $\mathcal{O}_x$. We define L_x^+G (resp. L_xG) to be the group (ind-)scheme over the residue field $k(x)$ representing the functor $R \mapsto G(R\widehat{\otimes}_{k(x)}\mathcal{O}_x)$ (resp. $R \mapsto G(R\widehat{\otimes}_{k(x)}F_x)$).

By Bruhat-Tits theory, for each facet $\underline{a}$ in the Bruhat-Tits building of $G(k((t)))$, there is a smooth group scheme $\mathcal{P}_{\underline{a}}$ over $k[[t]]$ with connected fibers whose generic fiber is $G \times_{\operatorname{Spec} k} \operatorname{Spec} k((t))$. We call such $\mathcal{P}_{\underline{a}}$ a *Bruhat-Tits group scheme*. Let $\mathbf{P}_{\underline{a}}$ be the functor $R \mapsto \mathcal{P}_{\underline{a}}(R[[t]])$, which is representable by a pro-algebraic group over k. We call $\mathbf{P}_{\underline{a}}$ a *parahoric subgroup* of LG. The conjugacy classes of parahoric subgroups of LG are classified by proper subsets of the nodes of the extended Dynkin diagram of G.

The group L^+G is a particular parahoric subgroup of LG, corresponding to the Bruhat-Tits group scheme $\mathcal{G} = G \times \operatorname{Spec} k[[t]]$. The Borel subgroup $B \subset G$ gives another parahoric subgroup called an Iwahori subgroup $\mathbf{I} \subset L^+G \subset LG$: $\mathbf{I}$ represents the functor $R \mapsto \{g \in G(R[[t]]); g \mod t \in B(R)\}$.

2.3 A class of parahoric subgroups

In [33], Gross, Reeder and Yu define a bijection between regular elliptic conjugacy classes of W and certain conjugacy classes of parahoric subgroups of LG. In particular, the longest element $-1 \in W$ corresponds to a conjugacy class of parahoric subgroups of LG. Here is an explicit description of a particular parahoric subgroup in this conjugacy class. Recall that the maximal torus T gives an apartment $\mathfrak{A}(T)$ in the building of LG. The apartment $\mathfrak{A}(T)$ is a torsor under $\mathbb{X}_*(T) \otimes \mathbb{R}$, and parahoric subgroups containing L^+T correspond to facets of $\mathfrak{A}(T)$. The parahoric subgroup L^+G corresponds to a point in $\mathfrak{A}(T)$. Using this point as the origin, we may identify $\mathfrak{A}(T)$ with $\mathbb{X}_*(T) \otimes \mathbb{R}$. In particular, $\frac{1}{2}\rho^\vee$, viewed as a point in $\mathfrak{A}(T)$, lies in a unique facet, and hence determines a parahoric subgroup $\mathbf{P}_{\frac{1}{2}\rho^\vee}$. The conjugacy class of $\mathbf{P}_{\frac{1}{2}\rho^\vee}$ is the image of $-1 \in W$ under Gross-Reeder-Yu's map.

Let K be the maximal reductive quotient of $\mathbf{P}_{\frac{1}{2}\rho^\vee}$. This is a connected split reductive group over k. Since $\mathbf{P}_{\frac{1}{2}\rho^\vee}$ is defined using a facet in the apartment $\mathfrak{A}(T)$, K contains T as a maximal torus. A root $\alpha \in \Phi_G$ belongs to the root system of K if and only if there is an affine root $\alpha + n\delta$ ($n \in \mathbb{Z}$, δ is the imaginary root) vanishing at $\frac{1}{2}\rho^\vee$, i.e., if and only if $\langle \rho^\vee, \alpha\rangle$ is an even integer. Therefore K can be identified with the fixed point subgroup $G^{\rho^\vee(-1)} \subset G$ of the inner involution $\rho^\vee(-1) \in G^{\mathrm{ad}}$ (note that $\rho^\vee$ is cocharacter of T^{ad}). We will study the group K in more detail in Sect. 2.5.

2.4 Results on involutions

In this subsection, we assume k is algebraically closed, and we do not distinguish notationwise a variety over k and its set of k-points. We shall first recall two fundamental results on involutions on G established by Springer and E. Cartan, which are valid for any connected reductive group G.

Proposition 2.1 (Springer) *Let G be a connected reductive group over k. Let* $\mathrm{Inv}(G^{\mathrm{ad}})$ *be the set of involutions in G^{ad}. We have*

(1) *Any involution $\tau \in \mathrm{Inv}(G^{\mathrm{ad}})$ is B^{ad}-conjugate to an element in $N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$;*
(2) *There are only finitely many B-conjugacy classes on* $\mathrm{Inv}(G^{\mathrm{ad}})$;
(3) *For any $\tau \in \mathrm{Inv}(G^{\mathrm{ad}})$, there are only finitely many G^{τ}-orbits on $f\ell_G$;*
(4) *Let $\tau \in \mathrm{Inv}(G^{\mathrm{ad}}) \cap N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$, then $G^{\tau} \cap B = T^{\tau} \cdot N_{\tau}$ where N_{τ} is a unipotent group.*

Proof (1) Fix $\tau_0 \in \mathrm{Inv}(G^{\mathrm{ad}}) \cap T^{\mathrm{ad}}$. Let $S = \{x \in G^{\mathrm{ad}} | x\mathrm{Ad}(\tau_0)(x) = 1\}$. Then G^{ad} acts on S via $g * x = gx\mathrm{Ad}(\tau_0)(g^{-1})$. Right multiplication by τ_0 gives an isomorphism $S \xrightarrow{\sim} \mathrm{Inv}(G^{\mathrm{ad}})$, which intertwines the $*$-action and the conjugation action of G^{ad}. Springer [38, Lemma 4.1(i)] shows that every B^{ad}-orbit on S (through the $*$-action) intersects $N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$. Therefore, every B^{ad}-conjugacy class in $\mathrm{Inv}(G^{\mathrm{ad}})$ also intersects $N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})\tau_0 = N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$.

(2) follows from (1), see [38, Corollary 4.3(i)].

(3) We have an embedding $G^{\tau}\backslash G \hookrightarrow \mathrm{Inv}(G^{\mathrm{ad}})$ given by $g \mapsto g^{-1}\tau g$, which is equivariant under the right translation by G and the conjugation action of G on $\mathrm{Inv}(G^{\mathrm{ad}})$. By (2), there are finitely many right B-orbits on $G^{\tau}\backslash G$. Therefore, there are only finitely many left G^{τ}-orbits on $G/B = f\ell_G$.

(4) See [38, Proposition 4.8]. □

Proposition 2.2 (E. Cartan) *Let G be a connected reductive group over k.*

(1) *For any involution $\tau \in \mathrm{Aut}(G)$, we have*

$$\dim \mathfrak{g}^{\tau} \geq \#\Phi_G/2. \tag{2.2}$$

When equality holds, we call τ a split Cartan involution.
(2) *All split Cartan involutions are conjugate under G^{ad}.*

Proof (1) Let $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ be the decomposition of $\mathfrak{g}$ into $+1$ and -1 eigenspaces of τ. Let $\mathfrak{s} \subset \mathfrak{g}^-$ be a maximal abelian subalgebra of $\mathfrak{g}^-$ consisting of semisimple elements. The centralizer $\mathfrak{l} = Z_{\mathfrak{g}}(\mathfrak{s})$ is a Levi subalgebra (with center $\mathfrak{s}$) on which τ acts. We have a similar eigenspace decomposition $\mathfrak{l}^{\mathrm{der}} = \mathfrak{l}^{\mathrm{der},+} \oplus \mathfrak{l}^{\mathrm{der},-}$ under the τ-action. If $\mathfrak{l}^{\mathrm{der},-} \neq 0$, it must contain a semisimple element Y, and hence $\mathrm{Span}\{\mathfrak{s}, Y\} \subset \mathfrak{g}^-$ is a larger abelian subalgebra than $\mathfrak{s}$ which also consists of semisimple elements. Contradiction.

Therefore $\mathfrak{l}^{\mathrm{der},-}=0$ and $\mathfrak{l}^-=\mathfrak{s}$. Let $X\in\mathfrak{s}$ be a generic element. Then the map $\mathrm{ad}(X):\mathfrak{g}^-/\mathfrak{s}\to\mathfrak{g}^+$ is injective. In fact, by the genericity of X, $\ker(\mathrm{ad}(X))$ commutes with the whole $\mathfrak{s}$ and hence lies in $\mathfrak{l}^-/\mathfrak{s}=0$. From the injectivity of $\mathrm{ad}(X)$, we get the desired estimate

$$\dim\mathfrak{g}^\tau\geq\frac{1}{2}(\dim\mathfrak{g}-\dim\mathfrak{s})\geq\frac{1}{2}(\dim\mathfrak{g}-\dim\mathfrak{t})=\frac{1}{2}\#\Phi_G. \tag{2.3}$$

(2) When equality in (2.3) holds, $\mathfrak{s}$ must be a Cartan subalgebra of $\mathfrak{g}$. Let τ and τ' be two split Cartan involutions with corresponding Cartan subalgebras $\mathfrak{s}$ and $\mathfrak{s}'$ in the (-1)-eigenspaces. Since $\mathfrak{s}=\mathrm{Ad}(x)(\mathfrak{s}')$ for some $x\in G^{\mathrm{ad}}$, by changing τ' to $\mathrm{Ad}(x)\tau'$, we may assume $\mathfrak{s}=\mathfrak{s}'$. Let S^{ad} be the maximal torus in G^{ad} with Lie algebra $\mathfrak{s}$. Since $\tau^{-1}\tau'$ acts by identity on $\mathfrak{s}$, $\tau^{-1}\tau'\in S^{\mathrm{ad}}$. Write $\tau'=\tau s$ for some $s\in S^{\mathrm{ad}}$. Since the square map $[2]:S^{\mathrm{ad}}\to S^{\mathrm{ad}}$ is surjective, we may write $s=\sigma^2$ for $\sigma\in S^{\mathrm{ad}}$. Then, in the group $\mathrm{Ad}(G)$, we have $\tau'=\tau\sigma^2=\tau(\sigma)\tau\sigma=\sigma^{-1}\tau\sigma$, hence τ' is conjugate to τ via $\sigma\in G^{\mathrm{ad}}$. □

The following lemma gives two explicit ways of constructing split Cartan involutions.

Lemma 2.3 *Suppose* $-1\in W$. *Then*

(1) *Any lifting of* $-1\in W$ *to* $N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$ *is a split Cartan involution.*
(2) *The element* $\rho^\vee(-1)\in G^{\mathrm{ad}}$ *is also a split Cartan involution.*

Proof (1) Let τ_0 be a lifting of $-1\in W$ to $N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$ of order two. Any other lifting has the form $\tau=t\tau_0$ for some $t\in T^{\mathrm{ad}}$. Since $(t\tau_0)^2=t\mathrm{Ad}(\tau_0)t=tt^{-1}=1$, τ is also an involution. Moreover, τ acts by -1 on the Cartan $\mathfrak{t}=\mathrm{Lie}\,T$ and interchanges root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. Therefore $\dim\mathfrak{g}^\tau=\#\Phi_G/2$.

(2) It suffice to consider the case G is simple. The root space $\mathfrak{g}_\alpha$ belongs to the (-1)-eigenspace of $\rho^\vee(-1)$ if and only if $\langle\alpha,\rho^\vee\rangle$ is odd, i.e., α has odd height. A case-by-case analysis of simple groups G with $-1\in W$ shows that we always have $\#\Phi_G/2+\mathrm{rank}\,G$ roots with odd height. Hence $\dim\mathfrak{g}^{\rho^\vee(-1)}=\dim\mathfrak{g}-(\#\Phi_G/2+\mathrm{rank}\,G)=\#\Phi_G/2$. □

Let $\tau\in N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$ be an involution. Then τ acts on $\mathfrak{t}$ and permutes the roots of G. Let $\Phi_G^\tau\subset\Phi_G$ be those roots which are fixed by τ. Let L be the Levi subgroup of G containing T with root system Φ_G^τ. Let L^{der} be the derived group of L, which is a semisimple group. Let $\mathfrak{l}$ (resp. $\mathfrak{l}^{\mathrm{der}}$) be the Lie algebra of L (resp. L^{der}).

Lemma 2.4 *In the above situation, suppose further that* τ *is a split Cartan involution. Then*

(1) $\mathfrak{t}^\tau$ *is a Cartan subalgebra of* $\mathfrak{l}^{\mathrm{der}}$, *equivalently* Φ_G^τ *span* $\mathfrak{t}^{*,\tau}$;

(2) *τ restricts to a split Cartan involution on L^{der}. In particular, it is nontrivial on each simple factor L^{der}.*

Proof We calculate the dimension of $\mathfrak{g}^\tau$. For those roots α which are not fixed by τ, $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\tau(\alpha)}$ contains a τ-fixed line. Therefore

$$\begin{aligned}\dim \mathfrak{g}^\tau &= \frac{\#\Phi_G - \#\Phi_G^\tau}{2} + \dim \mathfrak{l}^{\mathrm{der},\tau} + \dim \mathfrak{t}^\tau - \mathrm{rank}\ \mathfrak{l}^{\mathrm{der}} \\ &\geq \frac{\#\Phi_G - \#\Phi_G^\tau}{2} + \frac{\#\Phi_G^\tau}{2} = \frac{\#\Phi_G}{2}.\end{aligned}$$

This calculation gives another proof of Cartan's inequality (2.2) for τ.

When τ is a split Cartan involution, the above inequality is an equality. In particular, we have $\dim \mathfrak{t}^\tau = \mathrm{rank}\ \mathfrak{l}^{\mathrm{der}}$ and $\dim \mathfrak{l}^{\mathrm{der},\tau} = \#\Phi_G^\tau/2$. These two equalities imply the two parts of the lemma respectively. □

2.5 Minimal symmetric subgroups of G

By Cartan's inequality, a symmetric subgroup H of G (fixed point under some involution) has dimension at least $\#\Phi_G/2$. When $\dim H = \#\Phi_G/2$, we call H a *minimal symmetric subgroup* of G. By Proposition 2.2, all minimal symmetric subgroups are conjugate under $G^{\mathrm{ad}}(\overline{k})$.

Let $K = G^{\rho^\vee(-1)}$, keeping in mind it is also the reductive quotient K of the parahoric $\mathbf{P}_{\frac{1}{2}\rho^\vee}$ defined in Sect. 2.3. This is a minimal symmetric subgroup by Lemma 2.3. It is connected since G is assumed to be simply-connected. We will use K as a model to study properties of minimal symmetric subgroups.

2.5.1 The root system of K

Recall that the root system Φ_K is a subsystem of Φ_G consisting of roots α with even height (i.e., $\langle \rho^\vee, \alpha\rangle$ is even). On the other hand, being the reductive quotient of the parahoric subgroup $\mathbf{P}_{\frac{1}{2}\rho^\vee}$ of LG, the Dynkin diagram of K is also a subset of the extended Dynkin diagram of G. A case-by-case examination shows that the Dynkin diagram of K is obtained from the extended Dynkin diagram of G by removing either one node (if G is not of type A_1 or C_n), or two extremal nodes (if G is of type A_1 or C_n), and all the adjacent edges. We tabulate the types of K in each case:

Type of G	Type of K
A_1	$\mathbb{G}_m$
B_{2n}	$B_n \times D_n$
B_{2n+1}	$B_n \times D_{n+1}$
C_n	$A_{n-1} \times \mathbb{G}_m$
D_{2n}	$D_n \times D_n$
E_7	A_7
E_8	D_8
F_4	$A_1 \times C_3$
G_2	$A_1 \times A_1$

Next we analyze how far K is from being simply-connected.

Lemma 2.5 *Let $\mathbb{Z}\Phi_K^\vee \subset \mathbb{X}_*(T)$ be the coroot lattice of K. Then*

$$\mathbb{X}_*(T)/\mathbb{Z}\Phi_K^\vee \cong \begin{cases} \mathbb{Z} & G \text{ is of type } A_1 \text{ or } C_n \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise} \end{cases}$$

Proof When G is not of type A_1 or C_n, the simple roots of K are $\Delta_K = (\Delta - \{\alpha'\}) \cup \{-\theta\}$, where θ is the highest root of G and α' is the node we remove from the affine Dynkin diagram of G. Hence the coroot lattice of K is spanned by the simple coroots $\alpha^\vee$ of G with the only exception $\alpha'^\vee$, which is replaced by $\theta^\vee$. We write $\theta^\vee = \sum_{\alpha\in\Delta_G} c(\alpha)\alpha^\vee$, then we have a canonical isomorphism

$$\mathbb{X}_*(T)/\mathbb{Z}\Phi_K^\vee \cong \mathbb{Z}/c\bigl(\alpha'\bigr)\mathbb{Z}.$$

By examining all the Dynkin diagrams, we find $c(\alpha')$ is always equal to 2.

If G is of type A_1 or C_n, then $K \cong \mathrm{GL}_n$ has fundamental group $\mathbb{Z}$. □

2.5.2 Canonical double covering

By Lemma 2.5, we observe that in all cases, there is a canonical double covering

$$1 \to \mu_2 \to \widetilde{K} \to K \to 1$$

with $\widetilde{K}$ a connected reductive group. To emphasize the particular μ_2 as $\ker(\widetilde{K} \to K)$, we denote it by $\mu_2^{\ker}$. When G is *not* of type A_1 or C_n, $\widetilde{K}$ is the simply-connected form of K.

2.6 A remarkable finite 2-group

This subsection collects several facts about a remarkable finite 2-group $\widetilde{A}$, which will appear as the stabilizer of a base point in the moduli stack of G-bundles we consider.

2.6.1 *Definition of* $\widetilde{A}(B', K')$

We start with a minimal symmetric subgroup K' of G. The group K' acts on the flag variety $f\ell_G$ by conjugating Borel subgroups. According to Proposition 2.1(3), K' acts on $f\ell_G$ with finitely many orbits, therefore there is a unique open K'-orbit $U' \subset f\ell_G$. A Borel subgroup $B' \subset G$ is *in general position* with K' if the corresponding point in $f\ell_G$ lies in U'. Since $\dim K' = \dim f\ell_G = \dim U' = \#\Phi_G/2$, we conclude that (B', K') are in general position if and only if $B' \cap K'$ is finite.

Now let (B', K') be in general position. Let $\widetilde{K}'$ be the canonical double cover of K' as in Sect. 2.5.2. Let $\widetilde{A}(B', K')$ be the preimage of $B' \cap K'$ in $\widetilde{K}'$. This is a finite group scheme over k. Because all such pairs (B', K') are simultaneously conjugate under $G(\overline{k})$, the isomorphism class of the finite group $\widetilde{A}(B', K')(\overline{k})$ is independent of the choice of (B', K'). For this reason, when talking about $\widetilde{A}(B', K')(\overline{k})$ as an abstract group, use denote it simply by $\widetilde{A}(\overline{k})$.

2.6.2 *Structure of* $\widetilde{A}(\overline{k})$

Below we will use a particular pair (B', K') in general position to study the structure of $\widetilde{A}(\overline{k})$. By Lemma 2.3(1), any lift $\tau \in N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})(k)$ of $-1 \in W$ is a split Cartan involution. Let $K' = G^{\tau}$, then K' is a minimal symmetric subgroup. The intersection $B \cap K' = B^{\tau}$ is the 2-torsion subgroup $T[2] \subset T$. In particular, the standard Borel subgroup B is in general position with K'. The structure of $\widetilde{A}(B, K')(\overline{k})$ is worked out in [2], as we shall now recall. Below we denote $\widetilde{A}(B, K')$ simply by $\widetilde{A}$.

Associated to the central extension

$$1 \to \mu_2^{\mathrm{ker}} \to \widetilde{A} \to T[2] \to 1 \tag{2.4}$$

is a quadratic form

$$q : T[2] \to \mu_2^{\mathrm{ker}}$$

which assigns to $a \in T[2]$ the element $\widetilde{a}^2 \in \mu_2^{\mathrm{ker}}$, here $\widetilde{a} \in \widetilde{A}(\overline{k})$ is any lifting of a. Associated to this quadratic form there is a symplectic form (or, which amounts to the same in characteristic 2, a symmetric bilinear form)

$$\langle \cdot, \cdot \rangle : T[2] \times T[2] \to \mu_2^{\mathrm{ker}}$$

defined by

$$\langle a, b \rangle = q(ab)q(a)q(b).$$

The symplectic form may be computed by the commutator pairing

$$\langle a, b \rangle = \widetilde{a}\widetilde{b}\widetilde{a}^{-1}\widetilde{b}^{-1}$$

where $\widetilde{a}, \widetilde{b} \in \widetilde{A}(\overline{k})$ are liftings of a and b.

On the coroot lattice $R^\vee = \mathbb{Z}\Phi_G^\vee$, we have a unique W-invariant symmetric bilinear form

$$(\cdot,\cdot) : R^\vee \times R^\vee \to \mathbb{Z}$$

such that $(\alpha^\vee, \alpha^\vee) = 2$ if $\alpha^\vee$ is a short coroot. Note that for types B_n, C_n and F_4, $(\alpha^\vee, \alpha^\vee) = 4$ if $\alpha^\vee$ is a long coroot; for type G_2, $(\alpha^\vee, \alpha^\vee) = 6$ if $\alpha^\vee$ is a long coroot.

Lemma 2.6 *The following hold.*

(1) *Identifying* $T[2]$ *with* $R^\vee/2R^\vee$, *we have*

$$q(a) = (-1)^{(a,a)/2} \in \mu_2^{\mathrm{ker}}, \quad \text{for } a \in R^\vee/2R^\vee \cong T[2]. \tag{2.5}$$

(2) *Let* A_0 *be the kernel of the symplectic form* $\langle \cdot,\cdot \rangle$. *If* G *is oddly-laced*, *then* $A_0 = ZG[2]$.

Proof (1) Our pairing $(\cdot,\cdot)$ on $R^\vee$ is the same one as in [2, Sect. 3]. The symplectic form $\langle \cdot,\cdot \rangle$ is determined by Matsumoto, see [2, Eq. (3.3)]:

$$\left\langle \alpha^\vee(-1), \beta^\vee(-1) \right\rangle = (-1)^{(\alpha^\vee,\beta^\vee)}, \quad \text{for two coroots } \alpha^\vee, \beta^\vee. \tag{2.6}$$

Also, [1, Theorem 1.6] says that a root α of G is metaplectic (which is equivalent to saying $q(\alpha^\vee(-1)) = -1 \in \mu_2^{\mathrm{ker}}$) if and only if $\alpha^\vee$ is *not* a long coroot in type B_n, C_n and F_4. Therefore, checking the coroot lengths under the pairing $(\cdot,\cdot)$, we find

$$q\left(\alpha^\vee(-1)\right) = (-1)^{(\alpha^\vee,\alpha^\vee)/2} \quad \text{for any coroot } \alpha^\vee. \tag{2.7}$$

The two identities (2.6) and (2.7) together imply (2.5).

(2) follows from [2, Lemma 3.12] in which the base field was $\mathbb{R}$, but $ZG(\mathbb{R})$ can be canonically identified with $ZG[2]$. □

2.6.3 *Odd representations of* $\widetilde{A}(\overline{k})$

A representation V of $\widetilde{A}(\overline{k})$ is called *odd*, if μ_2^{ker} acts via the sign representation on V. Let $\mathrm{Irr}(\widetilde{A}(\overline{k}))_{\mathrm{odd}}$ be the set of irreducible odd representations of $\widetilde{A}(\overline{k})$.

Recall that $A_0 \subset T[2]$ is the kernel of the pairing $\langle \cdot,\cdot \rangle$. The pairing descends to a nondegenerate symplectic form on $T[2]/A_0$. Let $\widetilde{A}_0 \subset \widetilde{A}$ be the preimage of A_0. This is the center of $\widetilde{A}$. We have a central extension

$$1 \to \widetilde{A}_0(\overline{k}) \to \widetilde{A}(\overline{k}) \to T[2]/A_0 \to 1$$

whose commutator pairing $T[2]/A_0 \times T[2]/A_0 \to \mu_2^{\ker}$ is nondegenerate. Let $\widetilde{A}_0(\bar{k})^*_{\text{odd}}$ be the set of characters $\chi : \widetilde{A}_0(\bar{k}) \to \overline{\mathbb{Q}}^\times$ such that $\chi|_{\mu_2^{\ker}}$ is the sign representation. By the Stone-von Neumann Theorem, for every odd central character $\chi \in \widetilde{A}_0(\bar{k})^*_{\text{odd}}$, there is up to isomorphism a unique irreducible $\overline{\mathbb{Q}}$-representation V_χ of $\widetilde{A}(\bar{k})$ with central character χ. Therefore we have a canonical a bijection

$$\operatorname{Irr}\big(\widetilde{A}(\bar{k})\big)_{\text{odd}} \xrightarrow{\sim} \widetilde{A}_0(\bar{k})^*_{\text{odd}}.$$

In particular, the number of irreducible objects in $\operatorname{Rep}(\widetilde{A}, \overline{\mathbb{Q}})_{\text{odd}}$ is $\#\widetilde{A}_0(\bar{k})^*_{\text{odd}} = \#A_0$.

We tabulate the structure of $\widetilde{A}_0$ for oddly-laced groups with $-1 \in W$:

Type of G	$\widetilde{A}_0$	$\#\operatorname{Irr}(\widetilde{A}(\bar{k}))_{\text{odd}}$
E_8, G_2	μ_2	1
A_1, E_7	μ_4	2
D_{4n}	μ_2^3	4
D_{4n+2}	$\mu_4 \times \mu_2$	4

2.6.4 Rationality issues for $\widetilde{A}$

Above we only considered $\widetilde{A}(\bar{k})$ as an abstract group. Now we would like to consider the group scheme structure on $\widetilde{A}$ which complicates its representation theory. We will study odd representations of the semi-direct product $\Gamma = \widetilde{A}(\bar{k}) \rtimes \operatorname{Gal}(\bar{k}/k)$.

Let (B', K') be any pair in general position as in Sect. 2.6.1 and $\widetilde{A} = \widetilde{A}(B', K')$ be the finite group scheme over k. We have an exact sequence $1 \to \mu_2^{\ker} \to \widetilde{A} \to T'[2] \to 1$ of group schemes over k, in which both $\mu_2^{\ker}$ and $T'[2]$ are discrete group schemes. Therefore the only way $\operatorname{Gal}(\bar{k}/k)$ can act on $\widetilde{A}(\bar{k})$ is through a homomorphism $T'[2] \to \mu_2^{\ker}$. Therefore $\operatorname{Gal}(\bar{k}/k)$ acts on $\widetilde{A}(\bar{k})$ through a finite quotient $\overline{\Gamma} \hookrightarrow \operatorname{Hom}(T'[2], \mu_2^{\ker})$. In particular, $\overline{\Gamma}$ is isomorphic to a direct sum of $\mathbb{Z}/2\mathbb{Z}$.

When G is of type D_{4n}, E_8 or G_2, the group scheme $\widetilde{A}_0$ is a product of μ_2's, hence $\operatorname{Gal}(\bar{k}/k)$ acts trivially on $\widetilde{A}_0(\bar{k})$ and its group of characters.

When G is of type A_1, D_{4n+2}, E_7, $\mu_2^{\ker}$ is contained in some $\mu_4 \subset \widetilde{A}_0$. Therefore $\operatorname{Gal}(\bar{k}/k)$ acts trivially on the odd central characters if and only if $\sqrt{-1} \in k$. We henceforth make the assumption:

$$\textit{When } G \textit{ is of type } A_1, D_{4n+2} \textit{ or } E_7, \textit{we assume that } \sqrt{-1} \in k. \tag{2.8}$$

Lemma 2.7 *Assume* (2.8) *holds. For each odd central character χ of $\widetilde{A}_0(\overline{k})$, the irreducible $\widetilde{A}(\overline{k})$-module V_χ can be given a Γ-module structure extending the $\widetilde{A}(\overline{k})$-action. Moreover, such a Γ-module V_χ can be defined over $\mathbb{Q}(i)$.*

Proof We first show that V_χ extends to a Γ-module over $\overline{\mathbb{Q}}$. By our assumption on k, the central character χ is fixed by $\overline{\Gamma}$. For any $\overline{\gamma} \in \overline{\Gamma}$, let ${}^{\overline{\gamma}}V_\chi$ be the same space V_χ with $a \in \widetilde{A}(\overline{k})$ acting as $\overline{\gamma}(a)$ in the original action. Since the central character of ${}^{\overline{\gamma}}V_\chi$ is still χ, there is an $\widetilde{A}(\overline{k})$-isomorphism $\phi_{\overline{\gamma}} : V_\chi \xrightarrow{\sim} {}^{\overline{\gamma}}V_\chi$, unique up to a scalar in $\overline{\mathbb{Q}}^\times$. The obstruction for making the collection $\{\phi_{\overline{\gamma}}\}$ into an action of $\overline{\Gamma}$ on V_χ is a class in $\mathrm{H}^2(\overline{\Gamma}, \overline{\mathbb{Q}}^\times)$. Since $\overline{\Gamma}$ is a direct sum of $\mathbb{Z}/2\mathbb{Z}$, and $\mathrm{H}^2(\mathbb{Z}/2\mathbb{Z}, \overline{\mathbb{Q}}_\ell^\times) = \overline{\mathbb{Q}}^\times/\overline{\mathbb{Q}}^{\times 2} = 1$, there is no obstruction, and V_χ extends to an $\widetilde{A}(\overline{k}) \rtimes \overline{\Gamma}$-module over $\overline{\mathbb{Q}}$, hence a Γ-module over $\overline{\mathbb{Q}}$.

Next we show that every $\widetilde{A}(\overline{k}) \rtimes \overline{\Gamma}$-module is defined over $\mathbb{Q}(i)$. In fact, for every element $(a, \overline{\gamma}) \in \widetilde{A}(\overline{k}) \rtimes \overline{\Gamma}$, $(a, \overline{\gamma})^2 = (a \cdot \overline{\gamma}(a), \overline{\gamma}^2) = (a^2 c, 1) \in \mu_2^{\mathrm{ker}}$ because $\overline{\gamma}(a) = ac$ for some $c \in \mu_2^{\mathrm{ker}}$ and $a^2 \in \mu_2^{\mathrm{ker}}$ as well. Therefore all elements of $\widetilde{A}(\overline{k}) \rtimes \overline{\Gamma}$ have order divisible by four. By Brauer's theorem [35, 12.3, Theorem 24], all irreducible representations of $\widetilde{A}(\overline{k}) \rtimes \overline{\Gamma}$ can be defined over $\mathbb{Q}(i)$. Since the Γ-module V_χ is inflated from an $\widetilde{A}(\overline{k}) \rtimes \overline{\Gamma}$-module, it is also defined over $\mathbb{Q}(i)$. □

Remark 2.8 The Γ-module structure on V_χ constructed in Lemma 2.7 is not unique. Different Γ-actions on V_χ extending the $\widetilde{A}(\overline{k})$-action differ by twisting by a character of $\mathrm{Gal}(\overline{k}/k)$.

3 The automorphic sheaves

In this section, we will analyze the geometry of a moduli stack of G-bundles over $\mathbb{P}^1$ with level structures. We will exhibit some "automorphic" sheaves on this moduli stack, which will be used to construct the local systems stated in the Main Theorem in later sections.

3.1 Convention

We work under the setting of Sect. 2.1 and in particular under Assumption (2.1). All stacks without subscripts are defined over k.

Let ℓ be a prime number different from $\mathrm{char}(k)$. For technical reasons, let $\mathbb{Q}'_\ell = \mathbb{Q}_\ell(\sqrt{-1})$ be an extension of $\mathbb{Q}_\ell$ of degree at most two. All sheaves we consider are either $\mathbb{Q}_\ell$ or $\mathbb{Q}'_\ell$-sheaves under the étale topology on algebraic stacks defined over k. For an algebraic stack X over k, we use $D^b(X, \mathbb{Q}'_\ell)$ or simply $D^b(X)$ to denote the derived category of constructible $\mathbb{Q}'_\ell$-complexes on X, as defined in [23, 24]. When X is a global quotient $X = [H\backslash Y]$, $D^b(X)$ can be identified with the equivariant derived category $D^b_H(Y)$.

3.2 Moduli stacks of G-bundles

In this subsection, we introduce moduli stacks of G-bundles on $\mathbb{P}^1$ with certain level structures. This is the starting point of our geometric construction, and automorphic sheaves will live on these moduli stacks.

3.2.1 Moduli of bundles with parahoric level structures

Fix a set of k-points $S \subset \mathbb{P}^1(k)$. For each $x \in S$, fix a parahoric subgroup $\mathbf{P}_x \subset L_xG$. Generalizing the definition in [41, Sect. 4.2], we can define the an algebraic stack $\mathrm{Bun}_G(\mathbf{P}_x; x \in S)$ classifying G-torsors on $\mathbb{P}^1$ with $\mathbf{P}_x$-level structures at x (in [41, Sect. 4.2] we only considered the case S is a singleton, but the same construction generalizes to the case of multiple points). Note that when the parahoric $\mathbf{P}_x$ is not contained in $G(\mathcal{O}_x)$ for some $x \in S$, points in $\mathrm{Bun}_G(\mathbf{P}_x; x \in S)$ do not give well-defined G-bundles: they only give G-bundles over $X - S$.

In Sect. 2.3 we defined a particular parahoric subgroup $\mathbf{P}_{\frac{1}{2}\rho^\vee}$ of the loop group LG. Let $\mathbf{P}_0 \subset L_0G$ be the parahoric subgroup in the conjugacy class of $\mathbf{P}_{\frac{1}{2}\rho^\vee}$ which contains the standard Iwahori subgroup $\mathbf{I}_0$ (defined using B). At $\infty \in \mathbb{P}^1$, let $\mathbf{I}^{\mathrm{opp}}_\infty \subset L_\infty G$ be the Iwahori subgroup defined by the Borel $B^{\mathrm{opp}} \supset T$ opposite to B. Let $\mathbf{P}_\infty \subset L_\infty G$ be the parahoric subgroup in the conjugacy class of $\mathbf{P}_{\frac{1}{2}\rho^\vee}$ which contains $\mathbf{I}^{\mathrm{opp}}_\infty$.

The maximal reductive quotients of $\mathbf{P}_0$ and $\mathbf{P}_\infty$ are denoted by K_0 and K_∞ respectively, which are both isomorphic to the symmetric subgroup K of G studied in Sect. 2.5. The Weyl groups of K_0 and K_∞ can be identified with the subgroup of the affine Weyl group $\widetilde{W} = \mathbb{X}_*(T) \rtimes W$ generated by simple reflections which fix the alcove of $\mathbf{P}_0$ or $\mathbf{P}_\infty$. We denote this Weyl group by W_K, and understand it as a subgroup of $\widetilde{W}$ as above when needed.

The moduli stack $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$ will be our main interest till Sect. 3.2.3. Let us briefly recall its definition in a form convenient for our purposes. Let $\mathbf{P}^+_0$ (resp. $\mathbf{P}^+_\infty$) be the pro-unipotent radical of $\mathbf{P}_0$ (resp. $\mathbf{P}_\infty$). Since $\mathbf{P}^+_0$ and $\mathbf{P}^+_\infty$ are normal subgroups of $\mathbf{I}_0$ and $\mathbf{I}^{\mathrm{opp}}_\infty$, the moduli stack $\mathrm{Bun}_G(\mathbf{P}^+_0, \mathbf{P}^+_\infty)$ of G-bundles over $\mathbb{P}^1$ with $\mathbf{P}^+_0$ and $\mathbf{P}^+_\infty$ level structures at 0 and ∞ can be defined: it is an $(\mathbf{I}_0/\mathbf{P}^+_0) \times (\mathbf{I}^{\mathrm{opp}}_\infty/\mathbf{P}^+_\infty)$-torsor over the more familiar moduli stack $\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{I}^{\mathrm{opp}}_\infty)$. The groups K_0 and K_∞ act on $\mathrm{Bun}_G(\mathbf{P}^+_0, \mathbf{P}^+_\infty)$ by changing the level structures, and we define $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$ as the quotient stack $[(K_0 \times K_\infty)\backslash \mathrm{Bun}_G(\mathbf{P}^+_0, \mathbf{P}^+_\infty)]$.

3.2.2 Birkhoff decomposition

Let $\mathcal{P}^{\mathrm{triv}}_{\mathbb{A}^1}$ be the trivial G-bundle over $\mathbb{A}^1$ together with the tautological $\mathbf{P}_0$-level structure at 0. Let Γ_0 be the group ind-scheme of automorphisms of $\mathcal{P}^{\mathrm{triv}}_{\mathbb{A}^1}$: for any k-algebra R, $\Gamma_0(R) = \mathrm{Aut}_{\mathbb{A}^1_R}(\mathcal{P}^{\mathrm{triv}}_{\mathbb{A}^1_R})$. Recall from [21, Proposition 1.1] (where the Iwahori version was stated; our parahoric situation follows from the Iwahori version) that we have an isomorphism of stacks

$$[\Gamma_0\backslash L_\infty G/\mathbf{P}_\infty] \xrightarrow{\sim} \mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty),$$

and we have the Birkhoff decomposition

$$L_\infty G(\bar{k}) = \bigsqcup_{W_K\backslash\widetilde{W}/W_K} \Gamma_0(\bar{k})\widetilde{w}\mathbf{P}_\infty(\bar{k}). \tag{3.1}$$

In (3.1), every double coset $\widetilde{w} \in W_K\backslash\widetilde{W}/W_K$ represents a k-point $\mathcal{P}_{\widetilde{w}}$ of $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$, which is obtained by gluing $\mathcal{P}^{\mathrm{triv}}_{\mathbb{A}^1}$ with the parahoric subgroup $\mathrm{Ad}(\widetilde{w})\mathbf{P}_\infty$ of $L_\infty G$.

We sometimes will also use the stack $\mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_\infty) = [K_0\backslash \mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_\infty^+)]$. The trivial G-bundle with its tautological $\mathbf{P}_0^+$ and $\mathbf{P}_\infty$ level structures give a k-point $\star \in \mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_\infty)$, whose automorphism group $\Gamma_0 \cap \mathbf{P}_0^+ \cap \mathbf{P}_\infty$ is trivial. The image of $\star$ in $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$ corresponds to the unit coset $1 \in W_K\backslash\widetilde{W}/W_K$ in the decomposition (3.1), which has automorphism group $\Gamma_0 \cap \mathbf{P}_\infty \xrightarrow{\sim} K_0$ (the map here is given by the projection $\Gamma_0 \hookrightarrow \mathbf{P}_0 \to K_0$). Thus we get an embedding $[\{\star\}/K_0] \hookrightarrow \mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$. Since $\mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_\infty) \to \mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$ is a K_0-torsor, the preimage of $[\{\star\}/K_0]$ in $\mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_\infty)$ is precisely the point $\{\star\}$.

Lemma 3.1 *The substack* $[\{\star\}/K_0]$ *of* $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$ *is the non-vanishing locus of a section of a line bundle on* $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$*. In particular, the embedding* $j_0 : \mathbb{B}K_0 = [\{\star\}/K_0] \hookrightarrow \mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$ *is open and affine.*

Proof Let $d = \dim G$ and let Bun_d denote the moduli stack of rank d vector bundles on $\mathbb{P}^1$. We shall define a morphism

$$\mathrm{Ad}^+ : \mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty) \to \mathrm{Bun}_d \tag{3.2}$$

For this, first consider the morphism $\mathrm{Ad} : \mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_\infty) \to \mathrm{Bun}_d$ obtained by taking the adjoint bundle. Let us recall its construction. Let R be a locally noetherian k-algebra. Fix an affine coordinate t on $\mathbb{A}^1$ and identify the formal neighborhood of $\{0\}_R$ and $\{\infty\}_R$ in $\mathbb{P}^1_R$ with $\mathrm{Spf}\ R[[t]]$ and $\mathrm{Spf}\ R[[t^{-1}]]$ respectively. An object $\mathcal{P} \in \mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_\infty)(R)$ defined a G-bundle $\mathcal{P}_{\mathbb{G}_m}$ over $\mathbb{G}_{m,R} = \mathbb{P}^1_R - \{0, \infty\}$, hence the adjoint bundle $\mathrm{Ad}(\mathcal{P}_{\mathbb{G}_m})$ over $\mathbb{G}_{m,R}$ of rank d. The infinitesimal automorphisms of $\mathcal{P}|_{\mathrm{Spec}R((t))}$ preserving the

$\mathbf{P}_0^+$-level structure give an $R[[t]]$-submodule $\Lambda_0^+(\mathcal{P})$ of $\mathrm{Ad}(\mathcal{P}_{\mathbb{G}_m})|_{\mathrm{Spec}\, R((t))}$. The infinitesimal automorphisms of $\mathcal{P}|_{\mathrm{Spec}\, R((t^{-1}))}$ preserving the $\mathbf{P}_0$-level structure gives an $R[[t^{-1}]]$-submodule $\Lambda_\infty(\mathcal{P})$ of $\mathrm{Ad}(\mathcal{P}_{\mathbb{G}_m})|_{\mathrm{Spec}\, R((t^{-1}))}$. The adjoint bundle $\mathrm{Ad}(\mathcal{P})$ is obtained by glueing $\mathrm{Ad}(\mathcal{P}|_{\mathbb{G}_m})$ with $\Lambda_0^+(\mathcal{P})$ and $\Lambda_\infty(\mathcal{P})$.

Since $\mathbf{P}_0^+$ is normal in $\mathbf{P}_0$, changing the $\mathbf{P}_0^+$-level structure of $\mathcal{P}$ within a fixed $\mathbf{P}_0$-level structure does not change the $R[[t]]$-module $\Lambda_0^+(\mathcal{P})$. Hence the construction of Λ_0^+ descends to $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$, and so does the adjoint bundle construction. This finishes the definition of the morphism (3.2).

For an object $\mathcal{P} \in \mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_\infty)(R)$ as above, the cohomology $\mathrm{H}^i(\mathbb{P}^1_R, \mathrm{Ad}(\mathcal{P}))$ may be calculated using a Čech complex as follows. The $R[[t]]$-module $\Lambda_0^+(\mathcal{P})$ and $\mathrm{Ad}(\mathcal{P}|_{\mathbb{G}_m})$ glue to give a vector bundle on $\mathbb{A}^1_R$, whose global sections form an $R[t]$-submodule M_0^+ of the $R[t, t^{-1}]$-module $\Gamma(\mathbb{G}_{m,R}, \mathrm{Ad}(\mathcal{P}|_{\mathbb{G}_m}))$. Similarly, the $R[[t^{-1}]]$-module $\Lambda_\infty(\mathcal{P})$ and $\mathrm{Ad}(\mathcal{P}|_{\mathbb{G}_{m,R}})$ glue to give a vector bundle on $\mathbb{P}^1_R - \{\infty\}$, whose global sections form an $R[t^{-1}]$ submodule M_∞ of $\Gamma(\mathbb{G}_{m,R}, \mathrm{Ad}(\mathcal{P}|_{\mathbb{G}_m}))$. The two term complex placed at degrees zero and one:

$$M_0^+ \oplus M_\infty \to \Gamma\big(\mathbb{G}_{m,R}, \mathrm{Ad}(\mathcal{P})\big). \tag{3.3}$$

then computes the cohomology $\mathrm{H}^*(\mathbb{P}^1_R, \mathrm{Ad}(\mathcal{P}))$. As we discussed above, this complex only depends on the image of $\mathcal{P}$ in $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$.

We claim that the image of $\mathcal{P}$ in $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$ is the base point $\star$ if and only if $\mathrm{H}^*(\mathbb{P}^1_R, \mathrm{Ad}(\mathcal{P})) = 0$, i.e., the map (3.3) is an isomorphism. Since the terms in (3.3) are flat R-modules, we may reduce to the case R is an algebraically closed field, which we may assume to be $\overline{k}$. Suppose the image of $\mathcal{P}$ in $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)(\overline{k})$ corresponds to a double coset $\widetilde{w} \in W_K \backslash \widetilde{W} / W_K$ under the Birkhoff decomposition (3.1). Since $\mathcal{P}$ is glued from the trivial G-bundle with the tautological $\mathbf{P}_0$-level structure at 0 and $\mathrm{Ad}(\widetilde{w})\mathbf{P}_\infty$ at ∞, we may identify the complex (3.3) in this case with

$$\big(\mathfrak{g}[t] \cap \mathrm{Lie}\, \mathbf{P}_0^+\big) \oplus \mathrm{Ad}(\widetilde{w})\big(\mathfrak{g}\big[t^{-1}\big] \cap \mathrm{Lie}\, \mathbf{P}_\infty\big) \to \mathfrak{g}\big[t, t^{-1}\big]. \tag{3.4}$$

For $\widetilde{w}$ not equal to the trivial double coset, the two summands of the left side both contain the affine root space (inside $\mathfrak{g}[t, t^{-1}]$) of some affine root, hence (3.4) has a nontrivial kernel. If $\widetilde{w}$ is the trivial double coset, every affine root space appears either in $\mathrm{Lie}\, \mathbf{P}_0^+$ or in $\mathrm{Lie}\, \mathbf{P}_\infty$ but not both, so (3.3) is an isomorphism. This proves the claim.

By Grothendieck's classification of vector bundles on $\mathbb{P}^1$, the only vector bundle on $\mathbb{P}^1$ of rank d with trivial cohomology is the bundle $\mathcal{O}(-1)^{\oplus d}$, which is cut out by the non-vanishing of a section of the inverse of the determinant line bundle $\mathcal{L}_{\det}$ on Bun_d. Therefore, $\{\star\}/K_0 \subset \mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$ is the non-vanishing locus of a section of the line bundle $\mathrm{Ad}^{+,*}\mathcal{L}_{\det}^{-1}$. $\square$

3.2.3 A $\mu_2^{\ker}$-gerbe over the moduli stack

Let $\widetilde{\mathbf{P}}_0 = \mathbf{P}_0 \times_{K_0} \widetilde{K}_0$, where $\widetilde{K}_0$ is the canonical double cover of K_0 as in Sect. 2.5.2. We would like to define the moduli stack $\mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{P}_\infty)$, which will be a $\mu_2^{\ker}$-gerbe over $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$. Recall that $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$ is defined as the quotient of $\mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_\infty^+)$ by the $K_0 \times K_\infty$-action. Now instead we take quotient by $\widetilde{K}_0 \times K_\infty$, whose action on $\mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_\infty^+)$ still factors through $K_0 \times K_\infty$; i.e., we define

$$\mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{P}_\infty) := \big[(\widetilde{K}_0 \times K_\infty)\backslash \mathrm{Bun}_G\big(\mathbf{P}_0^+, \mathbf{P}_\infty^+\big)\big].$$

This is a $\mu_2^{\ker} = \ker(\widetilde{K}_0 \to K_0)$-gerbe over $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$.

3.2.4 Level structure at three points

Let $\mathbf{I}_1 \subset L_1 G$ be the Iwahori subgroup defined by B. We let

$$\mathrm{Bun} := \mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{I}_1, \mathbf{P}_\infty).$$

As above, this is defined as the quotient of $\mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{I}_1, \mathbf{P}_\infty^+)$ by $\widetilde{K}_0 \times K_\infty$; alternatively, it may be defined as the quotient of $\mathrm{Bun}^+ := \mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{I}_1, \mathbf{P}_\infty)$ by $\widetilde{K}_0$.

Consider the diagram

$$\begin{array}{ccccc} \mathrm{Bun}^+ & \longrightarrow & \mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_\infty) & & \\ \downarrow & & \downarrow & \searrow & \\ \mathrm{Bun} & \longrightarrow & \mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{P}_\infty) & \longrightarrow & \mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty) \end{array} \tag{3.5}$$

where the vertical maps are $\widetilde{K}_0$-torsors and the square is 2-Cartesian. The preimage of the base point $\star \in \mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_\infty)$ in Bun^+ corresponds to all Borel reductions of *the* trivial G-bundle at $t = 1$, and hence can be identified with the flag variety $f\ell_G$. Taking the preimage of the open substack $[\{\star\}/K_0] \subset \mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$ in various stacks in diagram (3.5), we get

$$\begin{array}{ccccc} f\ell_G & \longrightarrow & \{\star\} & & \\ \downarrow & & \downarrow & \searrow & \\ [\widetilde{K}_0\backslash f\ell_G] & \longrightarrow & [\{\star\}/\widetilde{K}_0] & \longrightarrow & [\{\star\}/K_0] \end{array}$$

each term being an open substack of the corresponding term in (3.5). In particular, we get an affine open embedding $j_1 : [\widetilde{K}_0\backslash f\ell_G] \hookrightarrow \mathrm{Bun}$.

We spell out the action of $\widetilde{K}_0$ on $f\ell_G$. By construction, the action is given by the homomorphism

$$\widetilde{K}_0 \to K_0 \cong \Gamma_0 \cap \mathbf{P}_\infty \subset G\big[t, t^{-1}\big] \xrightarrow{t=1} G \tag{3.6}$$

and the natural action of G on $f\ell_G$. Recall that K_0 is the reductive quotient of the parahoric subgroup $\mathbf{P}_0$, hence the root system Φ_{K_0} of K_0 is identified with a subset of the affine roots Φ_G^{aff} of G. Under the homomorphism (3.6), K_0 get mapped isomorphically onto the subgroup of G containing T whose root system is the image of the projection $\Phi_{K_0} \subset \Phi_G^{\mathrm{aff}} \to \Phi_G$. By the discussion in Sect. 2.5.1, the image of (3.6) is a symmetric subgroup of G corresponding to a split Cartan involution.

By Proposition 2.1(3), K_0 acts on $f\ell_G$ with finitely many orbits. Therefore there is a unique open K_0-orbit $U \subset f\ell_G$. We thus get an open embedding

$$j : [\widetilde{K}_0 \backslash U] \subset [\widetilde{K}_0 \backslash f\ell_G] \xrightarrow{j_1} \mathrm{Bun}.$$

3.2.5 A base point of U

We would like to fix a point $u_0 \in U(k)$, and henceforth assume

$$\textit{The base field } k \textit{ is such that } U(k) \neq \varnothing. \tag{3.7}$$

We argue that this assumption is fulfilled whenever $\mathrm{char}(k)$ is zero or sufficiently large. First we observe that the groups K_0 and G can be extended to split reductive group schemes $\underline{K}_0$ and $\underline{G}$ over $\mathbb{Z}[1/2N_0]$ for some positive integer N_0. Let $\underline{U}$ be the canonical model of U over $\mathbb{Z}[1/2N_0]$ obtained as the open $\underline{K}$-orbit on $f\ell_{\underline{G}}$. Since $U_{\mathbb{Q}}$ is a rational variety, its $\mathbb{Q}$-points are Zariski dense. Therefore $\underline{U}(\mathbb{Q}) \neq \varnothing$. We fix $u_0 \in \underline{U}(\mathbb{Q})$, which then extends to a point $\underline{u}_0 \in \underline{U}(\mathbb{Z}[1/2N_0N_1])$ for some other positive integer N_1. Therefore, whenever $p = \mathrm{char}(k)$ does not divide $2N_0N_1$, the point $\underline{u}_0$ induces a point of $U(\mathbb{F}_p)$, which we view as a k-point of U. We denote this k-point also by u_0, and fix this choice for the rest of the paper.

The point $u_0 \in U(k) \subset f\ell_G(k)$ fixed above corresponds to a Borel subgroup $B_0 \subset G$ (defined over k) which is in general position with K_0 (see Sect. 2.6.1). In Sect. 2.6.1 we made a definition of a finite group scheme $\widetilde{A}(B_0, K_0)$ whenever B_0 and K_0 are in general position. We denote $\widetilde{A}(B_0, K_0)$ simply by $\widetilde{A}$ and let $A = K_0 \cap B_0$. Projecting A to the Cartan quotient $B_0 \to T_0$ induces an isomorphism $A \cong T_0[2]$ (see Sect. 2.6.2), and $\widetilde{A}$ is a central extension of A by μ_2^{ker}. Since $\mathrm{char}(k) \neq 2$, A is a discrete group over k, and we may identify A with $A(k) = A(\bar{k})$. However, $\widetilde{A}$ may not be a discrete group over k, so it is important to distinguish among $\widetilde{A}$, $\widetilde{A}(k)$ and $\widetilde{A}(\bar{k})$. The structure and representation theory of the finite 2-group $\widetilde{A}(\bar{k})$ is worked out in Sect. 2.6.

3.3 Sheaves on the moduli stack of G-bundles

In this subsection, we will define certain irreducible perverse sheaves on Bun which are geometric incarnations of automorphic forms of a particular kind. We assume (2.8) and (3.7) hold in this subsection.

3.3.1 *The category of odd sheaves*

Since $\mu_2^{\ker} = \ker(\widetilde{K}_0 \to K_0)$ acts trivially on Bun^+, $\mu_2^{\ker}$ is in the automorphism group of every point of Bun. An object $\mathcal{F} \in D^b(\mathrm{Bun}) = D^b_{\widetilde{K}_0}(\mathrm{Bun}^+)$ thus carries an action of $\mu_2^{\ker}$. Therefore we have a decomposition

$$D^b_{\widetilde{K}_0}\big(\mathrm{Bun}^+\big) = D^b_{\widetilde{K}_0}\big(\mathrm{Bun}^+\big)_{\mathrm{even}} \oplus D^b_{\widetilde{K}_0}\big(\mathrm{Bun}^+\big)_{\mathrm{odd}}$$

according to whether $\mu_2^{\ker}$ acts trivially (the even part) or through the sign representation (the odd part). We then define

$$D^b(\mathrm{Bun})_{\mathrm{odd}} := D^b_{\widetilde{K}_0}\big(\mathrm{Bun}^+\big)_{\mathrm{odd}}.$$

Similarly, we define $D^b_{\widetilde{K}_0}(U)_{\mathrm{odd}}$ as the full subcategory of $D^b_{\widetilde{K}_0}(U)$ on which $\mu_2^{\ker}$ acts by the sign representation.

Theorem 3.2 *Assume that G is oddly-laced and $-1 \in W$ (i.e., G is of type A_1, D_{2n}, E_7, E_8 or G_2). Then the restriction functor induced by the open embedding $j : [\widetilde{K}_0 \backslash U] \hookrightarrow \mathrm{Bun}$*

$$j^* : D^b(\mathrm{Bun})_{\mathrm{odd}} \to D^b_{\widetilde{K}_0}(U)_{\mathrm{odd}}$$

is an equivalence of categories with inverse equal to both $j_!$ and j_*.

The proof of the theorem will occupy Sect. 3.4. If G is of type B_n, C_n or F_4, the statement in the above theorem does not hold.

For the rest of the paper, we assume G is of type A_1, D_{2n}, E_7, E_8 or G_2.

Our sought-for automorphic sheaf is an object of $D^b(\mathrm{Bun})_{\mathrm{odd}}$, which, by the above theorem, is determined by its restriction to U. Let $\mathrm{Loc}_{\widetilde{K}_0}(U)_{\mathrm{odd}}$ denote the abelian category of $\mathbb{Q}'_\ell$-local systems on U which are equivariant under $\widetilde{K}_0$ and on which $\mu_2^{\ker}$ acts by the sign representation. In order to construct automorphic sheaves, we would like to understand the category $\mathrm{Loc}_{\widetilde{K}_0}(U)_{\mathrm{odd}}$ in more concrete terms.

Lemma 3.3 *Consider the pro-finite group $\Gamma = \widetilde{A}(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/k)$ introduced in Sect. 2.6.4. Restriction to the point u_0 (which was fixed in Sect. 3.2.5) gives* *an equivalence of tensor categories*

$$u_0^* : \mathrm{Loc}_{\widetilde{K}_0}(U)_{\mathrm{odd}} \xrightarrow{\sim} \mathrm{Rep}_{\mathrm{cont}}\big(\Gamma, \mathbb{Q}'_\ell\big)_{\mathrm{odd}}$$

where the right side denotes continuous finite dimensional $\mathbb{Q}'_\ell$-representations of Γ on which $\mu_2^{\ker}$ acts by the sign representation.

Proof Since $\widetilde{K}_0$ acts on U transitively with stabilizer $\widetilde{A}$ at u_0, pullback to $\{u_0\}$ gives an equivalence of tensor categories

$$u_0^* : \mathrm{Loc}_{\widetilde{K}_0}(U)_{\mathrm{odd}} \xrightarrow{\sim} \mathrm{Loc}_{\widetilde{A}}(\mathrm{Spec}\ k)_{\mathrm{odd}}$$

where the subscript "odd" on the right side refers to the action of $\mu_2^{\ker} \subset \widetilde{A}$.

An $\widetilde{A}$-equivariant local system on Spec k is first of all a lisse $\mathbb{Q}'_\ell$-sheaf on Spec k, which is the same as a continuous representation V of $\mathrm{Gal}(\overline{k}/k)$ over E. The $\widetilde{A}$-equivariant structure gives an $\widetilde{A}(\overline{k})$-action on V. The two actions are compatible in the sense that

$$\gamma \cdot \widetilde{a} \cdot v = \gamma(\widetilde{a}) \cdot \gamma \cdot v, \quad \text{for } \gamma \in \mathrm{Gal}(\overline{k}/k), \widetilde{a} \in \widetilde{A}(\overline{k}), v \in V.$$

Therefore these two actions together give a continuous action of $\Gamma = \widetilde{A}(\overline{k}) \rtimes \mathrm{Gal}(\overline{k}/k)$ on V. The oddness condition on the local system is equivalent to that the action of $\mu_2^{\ker} \subset \Gamma$ is via the sign representation. □

3.3.2 The local system $\mathcal{F}_\chi$

In Lemma 2.7 we showed that the irreducible odd representation V_χ of $\widetilde{A}(\overline{k})$ can be extended to a Γ-module. Moreover, it is defined over $\mathbb{Q}(i)$. By Lemma 3.3, $V_\chi \otimes_{\mathbb{Q}(i)} \mathbb{Q}'_\ell$ gives rise to a geometrically irreducible local system $\mathcal{F}_\chi \in \mathrm{Loc}_{\widetilde{K}_0}(U, \mathbb{Q}'_\ell)_{\mathrm{odd}}$. By Remark 2.8, the construction of $\mathcal{F}_\chi$ is canonical only up to twisting by a continuous character of $\mathrm{Gal}(\overline{k}/k)$.

3.3.3 Variant of cleanness

Let S be a scheme over k and we may similarly consider $D^b(\mathrm{Bun} \times S)_{\mathrm{odd}}$. The cleanness theorem 3.2 implies that the functor $(u_0 \times \mathrm{id}_S)^*$ also induces an equivalence of categories

$$(u_0 \times \mathrm{id}_S)^* : D^b(\mathrm{Bun} \times S)_{\mathrm{odd}} \xrightarrow{\sim} D^b_{\widetilde{K}_0}(U \times S)_{\mathrm{odd}} \xrightarrow{\sim} D^b_{\widetilde{A}}(S)_{\mathrm{odd}} \tag{3.8}$$

where $\widetilde{A}$ acts trivially on S in the last expression. With the help of $j_!\mathcal{F}_\chi$, we may decompose objects in $D^b(\mathrm{Bun} \times S)_{\mathrm{odd}}$ in terms of the "basis" $j_!\mathcal{F}_\chi$ as χ

ranges over odd characters of $\widetilde{A}_0$. This decomposition will be made precise in Lemma 3.4 below, after some preparatory remarks.

We may decompose an object in the $D^b_{\widetilde{A}}(S)_{\mathrm{odd}}$ according to the action of $\widetilde{A}_0$. Let $\widetilde{A}^*_{0,\mathrm{odd}}$ be the set of odd characters of $\widetilde{A}_0$. This way we get a decomposition for every object $\mathcal{H}\in D^b(\mathrm{Bun}\times S)_{\mathrm{odd}}$:

$$(u_0\times\mathrm{id})^*\mathcal{H}=\bigoplus_{\chi\in\widetilde{A}^*_{0,\mathrm{odd}}}(u_0\times\mathrm{id})^*\mathcal{H}_\chi. \tag{3.9}$$

3.3.4 Invariants under a finite group scheme action

Let S be a scheme on which a finite group scheme H acts (both defined over k). We digress a bit to define what H-invariants mean for an H-equivariant perverse sheaf $\mathcal{F}$ on S. Let k' be a finite Galois extension of k over which H becomes a discrete group scheme. Let $\mathcal{F}_{k'}$ be the pull back of $\mathcal{F}$ to $S_{k'}=S\otimes_k k'$. We first take the perverse subsheaf of $H(k')$-invariants $\mathcal{F}^{H(k')}_{k'}\subset\mathcal{F}_{k'}$. The descent datum for $\mathcal{F}_{k'}$ (descending from $S_{k'}$ to S) restricts to a descent datum of $\mathcal{F}^{H(k')}_{k'}$. Since perverse sheaves satisfy étale descent, $\mathcal{F}^{H(k')}_{k'}$ descends to a perverse sheaf $\mathcal{F}^H$ on S. The construction of $\mathcal{F}^H$ is canonically independent of the choice of k', and we call $\mathcal{F}^H\in\mathrm{Perv}(S)$ the H-invariants of $\mathcal{F}$.

Lemma 3.4 *Let $\mathcal{H}\in D^b(\mathrm{Bun}\times S)_{\mathrm{odd}}$ such that $(u_0\times\mathrm{id}_S)^*\mathcal{H}$ is concentrated in a single perverse degree, then we have a canonical decomposition*

$$\mathcal{H}\cong\bigoplus_{\chi\in\widetilde{A}^*_{0,\mathrm{odd}}}(j_!\mathcal{F}_\chi)\boxtimes\left(V^*_\chi\otimes(u_0\times\mathrm{id}_S)^*\mathcal{H}_\chi\right)^{\widetilde{A}}. \tag{3.10}$$

Proof By applying a shift, we may assume that $\mathcal{K}=(u_0\times\mathrm{id}_S)^*\mathcal{H}$ is an $\widetilde{A}$-equivariant *perverse* sheaf. By the variant of cleanness (3.8), it suffice to show that the pullback of the two sides of (3.10) along $u_0\times\mathrm{id}$ are canonically isomorphic as $\widetilde{A}$-equivariant perverse sheaves, i.e., we have to show

$$\mathcal{K}\cong\bigoplus_{\chi\in\widetilde{A}^*_{0,\mathrm{odd}}}V_\chi\otimes\left(V^*_\chi\otimes\mathcal{K}_\chi\right)^{\widetilde{A}} \tag{3.11}$$

Base changing to k' where $\widetilde{A}$ becomes discrete, we have a canonical decomposition $\mathcal{F}\cong\bigoplus_{\chi\in\widetilde{A}^*_{0,\mathrm{odd}}}V_\chi\otimes\mathrm{Hom}_{\widetilde{A}(k')}(V_\chi,\mathcal{K})$. Writing $\mathcal{K}=\bigoplus_\chi\mathcal{K}_\chi$ according to the $\widetilde{A}_0$-action. Since $\mathrm{Hom}_{\widetilde{A}(k')}(V_\chi,\mathcal{K}_{\chi'})=0$ for $\chi\neq\chi'$, we may rewrite the decomposition as $\mathcal{F}\cong\bigoplus_{\chi\in\widetilde{A}^*_{0,\mathrm{odd}}}V_\chi\otimes\mathrm{Hom}_{\widetilde{A}(k')}(V_\chi,\mathcal{K}_\chi)=\bigoplus_{\chi\in\widetilde{A}^*_{0,\mathrm{odd}}}V_\chi\otimes(V^*_\chi\otimes\mathcal{K}_\chi)^{\widetilde{A}(k')}$. Canonicity guarantees descent from k' to k, which then gives (3.11). □

3.4 Proof of Theorem 3.2

To prove the theorem, it suffices to show that for any $\mathcal{F} \in D^b(\mathrm{Bun})_{\mathrm{odd}}$, the stalks of $\mathcal{F}$ outside the open substack $[\widetilde{K}_0\backslash U]$ are zero. This statement being geometric, we may assume k is algebraically closed. The rest of this subsection is devoted to the proof of this vanishing statement.

3.4.1 $\mathcal{F}$ is zero on $[\widetilde{K}_0\backslash(f\ell_G - U)]$

We first restrict $\mathcal{F}$ to the open subset $[\widetilde{K}_0\backslash f\ell_G] \subset \mathrm{Bun}$. Let $O(x) = K_0xB/B$ be a K_0-orbit on $f\ell_G$ which is not open. The goal is to show that $\mathcal{F}|_{O(x)} = 0$.

Write $K_0 = G^{\tau_0}$ for some split Cartan involution $\tau_0 \in G^{\mathrm{ad}}$. Consider the involution $\tau = x^{-1}\tau x \in G^{\mathrm{ad}}$. By Proposition 2.1(1), up to right-multiplying x by an element in B, we may assume that $\tau \in N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$. Let $[\tau] \in W$ be the image of τ, which is an involution in W. Then $O(x)$ is the open orbit if $[\tau] = -1 \in W$, so we assume $[\tau] \neq -1$.

Conjugating by x^{-1} gives an isomorphism

$$O(x) = K_0/\big(K_0 \cap xBx^{-1}\big) \xrightarrow{\sim} xK_0x^{-1}/\big(xK_0x^{-1} \cap B\big) = G^\tau/\big(G^\tau \cap B\big) \tag{3.12}$$

which intertwines the K_0-action on $O(x)$ and the G^τ-action on $G^\tau/(G^\tau \cap B)$. Since G^τ is also a minimal symmetric subgroup of G, we have the canonical double cover $\nu' : \widetilde{G^\tau} \to G^\tau$ as in Sect. 2.5.2. Under the isomorphism (3.12), $\mathcal{F}|_{O(x)}$ can be viewed as an object in $D^b_{\widetilde{G^\tau}}(G^\tau/G^\tau \cap B)_{\mathrm{odd}}$, where oddness refers to the action of $\ker(\nu')$, which we still denote by $\mu_2^{\ker}$ in the sequel. Since $\widetilde{G^\tau}$ acts on $G^\tau/G^\tau \cap B$ transitively, each cohomology sheaf of $\mathcal{F}|_{O(x)}$ is a local system on $G^\tau/G^\tau \cap B$. Therefore it is enough to show that $\mathrm{Loc}_{\widetilde{G^\tau}}(G^\tau/G^\tau \cap B)_{\mathrm{odd}} = 0$.

By Proposition 2.1(4), $G^\tau \cap B = T^\tau \cdot N_\tau$ for a unipotent group N_τ. Let $\widetilde{T^\tau} \subset \widetilde{G^\tau}$ be the preimage of T^τ in $\widetilde{G^\tau}$, then

$$\mathrm{Loc}_{\widetilde{G^\tau}}\big(G^\tau/T^\tau N_\tau\big)_{\mathrm{odd}} \xrightarrow{\sim} \mathrm{Loc}_{\widetilde{G^\tau}}\big(\widetilde{G^\tau}/\widetilde{T^\tau}\big)_{\mathrm{odd}} = \mathrm{Loc}_{\widetilde{T^\tau}}(\mathrm{Spec}\ k)_{\mathrm{odd}}.$$

The category $\mathrm{Loc}_{\widetilde{T^\tau}}(\mathrm{Spec}\ k)$ is equivalent to the category of $\mathbb{Q}'_\ell$-representations of $\pi_0(\widetilde{T^\tau})$. Therefore, for $\mathrm{Loc}_{\widetilde{T^\tau}}(\mathrm{Spec}\ k)_{\mathrm{odd}}$ to be nonzero, $\mu_2^{\ker}$ must not lie in the neutral component of $\widetilde{T^\tau}$. In the following lemma, we shall show that this is not case, therefore proving that $\mathcal{F}|_{O(x)} = 0$ whenever $O(x)$ is not the open orbit.

Lemma 3.5 *When $[\tau] \neq -1 \in W$, there exists a coroot $\alpha^\vee$ of G, invariant under τ, such that $\alpha^\vee : \mathbb{G}_m \to T^\tau$ does not lift to $\mathbb{G}_m \to \widetilde{T^\tau}$. In particular, $\mu_2^{\ker}$ lies in the neutral component of $\widetilde{T^\tau}$.*

Proof If $G=\mathrm{SL}_2$, then $[\tau]=1$ and the positive coroot $\alpha^\vee$ meets the requirement. In the following we assume that G is not of type A_1.

As in the situation of Lemma 2.4, we introduce the Levi subgroup $L\subset G$ with root system Φ^τ_G. Since $[\tau]\neq -1\in W$, $\mathfrak{t}^\tau\neq 0$ and $\Phi^\tau_G\neq\varnothing$ because they have to span $\mathfrak{t}^\tau$ by Lemma 2.4(1). Let L_1 be a simple factor of L which is not a torus. Let $\mathfrak{l}_1=\mathrm{Lie}L_1$. By Lemma 2.4(2), $\tau|_{L_1}$ is nontrivial, hence there exists a root α of L_1 which is not a root of L^τ_1; i.e., τ acts on the root space $\mathfrak{g}_\alpha$ by -1.

Assume either G is simply-laced or this α is a long root if G is of type G_2. Give a grading of $\mathfrak{g}$ according to the adjoint action of the coroot $\alpha^\vee$, so that $\mathrm{Ad}(\alpha^\vee(s))$ acts on $\mathfrak{g}(d)$ by s^d. Since α is a long root of G, we have $\mathfrak{g}(d)=0$ for $|d|>2$, and that $\mathfrak{g}(2)=\mathfrak{g}_\alpha$. Since τ acts on $\mathfrak{g}_\alpha$ by -1, $\mathfrak{g}^\tau\cap\mathfrak{g}(2)=0$, therefore the action of $\mathrm{Ad}(\alpha^\vee)$ on $\mathfrak{g}^\tau$ has weights $-1,0,1$. Suppose $\alpha^\vee$ lifted to a cocharacter of $\widetilde{T}^\tau$, it would give an element in the coroot lattice of G^τ which is minuscule. By the table in Sect. 2.5.1, all simple factors of G^τ are of type A or D when G is oddly-laced. An examination of root systems of type A and D shows that no element in the coroot lattice of G^τ is minuscule. Therefore $\alpha^\vee$ is not liftable to $\widetilde{T}^\tau$. For simply-laced G, this finishes the proof of the lemma.

It remains to consider the case $G=G_2$, and L is the Levi corresponding to a short root α of G. Using the grading given by $\alpha^\vee$, we have $\mathfrak{g}^\tau(3)\neq 0$ and $\mathfrak{g}^\tau(1)\neq 0$. However, $\widetilde{G}^\tau\cong\mathrm{SL}_2\times\mathrm{SL}_2$, and any homomorphism $\mathbb{G}_m\to\mathrm{SL}_2$ acts by even weights under the adjoint representation. Therefore $\alpha^\vee$ is not liftable to $\widetilde{T}^\tau$, and the G_2 case is also settled. □

3.4.2 *$\mathcal{F}$ is zero outside $[\widetilde{K}_0\backslash f\ell_G]$*

Recall the Birkhoff decomposition (3.1). Each double coset $\widetilde{w}\in W_K\backslash\widetilde{W}/W_K$ gives a point $\mathcal{P}_{\widetilde{w}}\in\mathrm{Bun}_G(\mathbf{P}_0,\mathbf{P}_\infty)$, whose stabilizer is $\mathrm{Stab}(\widetilde{w})=\Gamma_0\cap\mathrm{Ad}(\widetilde{w})\mathbf{P}_\infty$. We have a canonical homomorphism by evaluating at $t=0$

$$\mathrm{ev}_0:\mathrm{Stab}(\widetilde{w})\subset\Gamma_0\to K_0.$$

Changing the $\mathbf{P}_0$-level to $\widetilde{\mathbf{P}}_0$-level, we get a similar Birkhoff decomposition for $\mathrm{Bun}_G(\widetilde{\mathbf{P}}_0,\mathbf{P}_\infty)$. The double coset $\widetilde{w}$ also gives a point $\widetilde{\mathcal{P}}_{\widetilde{w}}\in\mathrm{Bun}_G(\widetilde{\mathbf{P}}_0,\mathbf{P}_\infty)$ with stabilizer

$$\widetilde{\mathrm{Stab}}(\widetilde{w}):=\mathrm{Stab}(\widetilde{w})\times_{K_0}\widetilde{K}_0.$$

Let $\mathrm{Bun}_{\widetilde{w}}$ be the preimage of the stratum $\{\widetilde{\mathcal{P}}_{\widetilde{w}}\}/\widetilde{\mathrm{Stab}}(\widetilde{w})$ in $\mathrm{Bun}=\mathrm{Bun}_G(\widetilde{\mathbf{P}}_0,\mathbf{I}_1,\mathbf{P}_\infty)$. The open stratum corresponding to the unit coset in $W_K\backslash\widetilde{W}/W_K$ is $[\widetilde{K}_0\backslash f\ell_G]$, which has been dealt with in Sect. 3.4.1. Our goal here is to show that $D^b(\mathrm{Bun}_{\widetilde{w}})_{\mathrm{odd}}=0$ whenever $\widetilde{w}$ is not the unit coset in $W_K\backslash\widetilde{W}/W_K$. This together with Sect. 3.4.1 finishes the proof of Theorem 3.2.

Similar to the description of the preimage of the unit double coset in Bun in Sect. 3.2.4, we have an isomorphism

$$\mathrm{Bun}_{\widetilde{w}} \cong \left[\widetilde{\mathrm{Stab}}(\widetilde{w})\backslash f\ell_G\right]$$

where $\widetilde{\mathrm{Stab}}(\widetilde{w})$ acts through

$$\widetilde{\mathrm{Stab}}(\widetilde{w}) \to \mathrm{Stab}(\widetilde{w}) = \Gamma_0 \cap \mathrm{Ad}(\widetilde{w})\mathbf{P}_\infty \subset \Gamma_0 \subset G\left[t, t^{-1}\right] \xrightarrow{t=1} G. \quad (3.13)$$

Therefore

$$D^b(\mathrm{Bun}_{\widetilde{w}})_{\mathrm{odd}} \cong D^b_{\widetilde{\mathrm{Stab}}(\widetilde{w})}(f\ell_G)_{\mathrm{odd}}$$

We need to show that the latter category is zero.

Let $G(\widetilde{w}) \subset G$ be the image of the map (3.13). This is a subgroup of G containing T, whose root system we now describe.

Recall from Sect. 2.3 that the parahoric subgroup $\mathbf{P}_{\frac{1}{2}\rho^\vee}$ is defined using the facet containing $\frac{1}{2}\rho^\vee$ in the apartment $\mathfrak{A}(T)$. Now $\mathbf{P}_\infty$ has the same type as $\mathbf{P}_{\frac{1}{2}\rho^\vee}$, so it is defined using the facet containing another point $x \in \mathfrak{A}(T)$ which is in the $\widetilde{W}$-orbit of $\frac{1}{2}\rho^\vee$ (recall $\widetilde{W}$ acts on $\mathfrak{A}(T)$ by affine transformations). Since G is simply-connected, the stabilizer of x is precisely the Weyl group of W_K. Affine roots of $L_\infty G$ are affine functions on $\mathfrak{A}(T)$. Affine roots α appearing in Γ_0 are those such that $\alpha(x) \geq 0$; affine roots α appearing in $\mathrm{Ad}(\widetilde{w})\mathbf{P}_\infty$ are those such that $\alpha(\widetilde{w}x) \leq 0$. Therefore affine roots appearing in $\mathrm{Stab}(\widetilde{w})$ are those such that $\alpha(x) \geq 0$ and $\alpha(\widetilde{w}x) \leq 0$. Write such an affine root as $\alpha = \gamma + n\delta$ where $\gamma \in \Phi_G \cup \{0\}$, δ is the imaginary root of $L_\infty G$ and $n \in \mathbb{Z}$. Evaluating at $t = 1$ gives γ as a root appearing in $G(\widetilde{w})$. In conclusion, we see that $\gamma \in \Phi_G$ appears in $G(\widetilde{w})$ if and only if there exists $n \in \mathbb{Z}$ such that $\gamma(x) + n \geq 0$ and $\gamma(\widetilde{w}x) + n \leq 0$. Since x is in the $\widetilde{W}$-orbit of $\frac{1}{2}\rho^\vee$, $\gamma(x), \gamma(\widetilde{w}x) \in \frac{1}{2}\mathbb{Z}$, therefore such an integer n exists if

- either $\langle\gamma, x - \widetilde{w}x\rangle > 0$;
- or $\langle\gamma, x - \widetilde{w}x\rangle = 0$ and $\gamma(x) \in \mathbb{Z}$.

Note that $\lambda = x - \widetilde{w}x$ is a well defined vector in $\mathbb{X}_*(T)_{\mathbb{Q}}$, and $\langle\gamma, x - \widetilde{w}x\rangle$ is the pairing between $\mathbb{X}^*(T)_{\mathbb{Q}}$ and $\mathbb{X}_*(T)_{\mathbb{Q}}$. Since $\gamma(x) \in \mathbb{Z}$ if and only if $\gamma \in \Phi_{K_0} \cup \{0\}$, the roots of $G(\widetilde{w})$ are

$$\left\{\gamma \in \Phi_G | \langle\gamma, \lambda\rangle \geq 0, \text{ and, when equality holds, } \gamma \in \Phi_{K_0}\right\}.$$

Let $P_\lambda \supset G(\widetilde{w})$ be the parabolic subgroup of G containing T whose roots consist of all γ such that $\langle\gamma, \lambda\rangle \geq 0$. Then we have a Levi decomposition $P_\lambda = N_\lambda L_\lambda$ where N_λ is the unipotent radical of P_λ and L_λ is the Levi subgroup of P_λ containing T whose roots are $\{\gamma \in \Phi_G | \langle\gamma, \lambda\rangle = 0\}$. The above discussion gives $G(\widetilde{w}) = N_\lambda(L_\lambda \cap K_0)$ (we identify K_0 as a subgroup of G using (3.6)).

Notice that $L_\lambda \cap K_0 \subset G(\widetilde{w})$ has a canonical lifting to a subgroup of $\mathrm{Stab}(\widetilde{w})$: this is the subgroup generated by T and containing real affine roots α of the form $\alpha(x) = \alpha(\widetilde{w}x) = 0$. Therefore, we can form $\widetilde{L_\lambda \cap K_0} = (L_\lambda \cap K_0) \times_{K_0} \widetilde{K}_0$, which is both a subgroup and a quotient group of $\widetilde{\mathrm{Stab}}(\widetilde{w})$.

Next we analyze the $G(\widetilde{w})$-orbits on $f\ell_G$. Let W_λ be the Weyl group of L_λ. Bruhat decomposition gives

$$\begin{aligned}\left[G(\widetilde{w})\backslash f\ell_G\right] &= \bigsqcup_{v \in W_\lambda \backslash W} \left[G(\widetilde{w})\backslash P_\lambda v B/B\right] \twoheadleftarrow \bigsqcup_{v \in W_\lambda \backslash W} \left[(L_\lambda \cap K_0)\backslash L_\lambda v B/B\right] \\ &= \bigsqcup_{v \in W_\lambda \backslash W} \left[(L_\lambda \cap K_0)\backslash f\ell_{L_\lambda}\right].\end{aligned}$$

The arrow above is a map between stacks which is bijective on the level of points. The last equality uses the fact that $L_\lambda v B/B = L_\lambda/(L_\lambda \cap \mathrm{Ad}(v)B)$ is the flag variety of L_λ. Therefore, in order to show that $D^b_{\widetilde{\mathrm{Stab}}(\widetilde{w})}(f\ell_G)_{\mathrm{odd}} = 0$, it suffices to show that $D^b_{\widetilde{L_\lambda \cap K_0}}(f\ell_{L_\lambda})_{\mathrm{odd}} = 0$. For $\widetilde{w}$ not equal to the unit W_K-double coset, $\lambda = x - \widetilde{w}x \neq 0$, therefore L_λ is a *proper* Levi subgroup of G. So our goal becomes to show that $D^b_{\widetilde{L \cap K_0}}(f\ell_L)_{\mathrm{odd}} = 0$ for all proper Levi subgroups $L \subset G$ containing T.

Since $K_0 = G^{\tau_0}$ where $\tau_0 = \exp(x) \in T^{\mathrm{ad}}$, and τ_0 also acts on L, we have $L \cap K_0 = L^{\tau_0}$. Fix a Borel $B_L \subset L$ containing T. Consider an L^{τ_0}-orbit $O(x) = L^{\tau_0} x B_L/B_L \subset f\ell_L$. By Proposition 2.1(1) again, up to right multiplying x by an element in B_L, we may assume that the involution $\tau = x\tau_0 x^{-1}$ lies in $N_{L^{\mathrm{ad}}}(T^{\mathrm{ad}})$. By the same argument as in Sect. 3.4.1, we reduce to show that $\mathrm{Loc}_{\widetilde{T^\tau}}(\mathrm{pt})_{\mathrm{odd}} = 0$, where $\widetilde{T^\tau} = T^\tau \times_{K_0} \widetilde{K}_0$. In Sect. 3.4.1, we have shown that $D^b_{\widetilde{T^\tau}}(\mathrm{pt})_{\mathrm{odd}} = 0$ provided $[\tau] \neq -1 \in W$. In our case, since $\tau \in N_{L^{\mathrm{ad}}}(T^{\mathrm{ad}})$ and L is proper, $[\tau]$ can never be -1. This finishes the proof.

4 Construction of the motives

In this section, we apply geometric Hecke operators to the automorphic sheaf $j_!\mathcal{F}_\chi$ found in the previous section to get a $\widehat{G}$-local system $\mathcal{E}_\chi$ on $\mathbb{P}^1 - \{0, 1, \infty\}$. We will also show that the local system $\mathcal{E}_\chi$ is motivic. We recall that G is of type A_1, D_{2n}, E_7, E_8 or G_2.

4.1 Geometric Hecke operators

4.1.1 The geometric Satake equivalence

We briefly review the main result of [31]. We have defined the loop group LG and its parahoric subgroup L^+G in Sect. 2.2. Let $\mathrm{Aut}_{\mathcal{O}}$ be the group scheme of formal change of variables: $R \mapsto \{$continuous (with respect to the t-adic topology) R-linear automorphisms of $R[[t]]\}$.

We usually denote the quotient LG/L^+G by Gr, the *affine Grassmannian*. This is an ind-scheme which is a union of projective varieties of increasing dimension. For each dominant coweight $\lambda \in \mathbb{X}_*(T)^{\mathrm{dom}}$, the L^+G-orbit containing $t^\lambda \in \mathrm{Gr}$ is denoted by Gr_λ. The closure of Gr_λ is called an *affine Schubert variety* and is denoted by $\mathrm{Gr}_{\leq\lambda}$. The dimension of Gr_λ is $\langle 2\rho, \lambda\rangle$, which is an even number since G is assumed to be simply-connected.

The geometric Satake category $\mathrm{Sat}^{\mathrm{geom}}$ is the category of $(L^+G)_{\overline{k}}$ equivariant $\mathbb{Q}_\ell$-perverse sheaves on $\mathrm{Gr}_{\overline{k}}$ whose supports are finite union of affine Schubert varieties. The superscript $^{\mathrm{geom}}$ is to indicate that sheaves in $\mathrm{Sat}^{\mathrm{geom}}$ are over the geometric fiber $\mathrm{Gr}_{\overline{k}}$ rather than Gr itself. As was shown in [31, Appendix], $\mathrm{Sat}^{\mathrm{geom}}$ is in fact equivalent to the category of perverse sheaves on $\mathrm{Gr}_{\overline{k}}$ which are constant along $L^+G_{\overline{k}}$-orbits, and every object in $\mathrm{Sat}^{\mathrm{geom}}$ carries a canonical equivariant structure under $\mathrm{Aut}_{\mathcal{O},\overline{k}}$. The category $\mathrm{Sat}^{\mathrm{geom}}$ is equipped with a convolution product $* : \mathrm{Sat}^{\mathrm{geom}} \times \mathrm{Sat}^{\mathrm{geom}} \to \mathrm{Sat}^{\mathrm{geom}}$ which makes it a Tannakian category over $\mathbb{Q}_\ell$. The global section functor

$$\mathrm{H}^* : \mathrm{Sat}^{\mathrm{geom}} \to \mathrm{Vec}_{\mathbb{Q}_\ell}$$
$$\mathcal{K} \mapsto \mathrm{H}^*(\mathrm{Gr}_{\overline{k}}, \mathcal{K})$$

is a fiber functor. The Tannakian group $\mathrm{Aut}^{\otimes}(\mathrm{H}^*)$ is isomorphic to the Langland dual group $\widehat{G}$ of G. Here $\widehat{G}$ is a split reductive group over $\mathbb{Q}_\ell$. All this is proved in [31, Sect. 5-7].

Let $\widetilde{\mathrm{Sat}}$ be the full subcategory of perverse sheaves on Gr which belong to $\mathrm{Sat}^{\mathrm{geom}}$ after pulling back to $\mathrm{Gr}_{\overline{k}}$. Then $\widetilde{\mathrm{Sat}}$ also has a convolution product. Let $\mathrm{IC}_\lambda \in \widetilde{\mathrm{Sat}}$ be the (normalized) intersection cohomology sheaf of $\mathrm{Gr}_{\leq\lambda}$:

$$\mathrm{IC}_\lambda := j_{\lambda,!*}\mathbb{Q}_\ell\big[\langle 2\rho, \lambda\rangle\big]\big(\langle \rho, \lambda\rangle\big),$$

where $j_\lambda : \mathrm{Gr}_\lambda \hookrightarrow \mathrm{Gr}$ is the inclusion. We remarked earlier that $\langle 2\rho, \lambda\rangle$ is even so $\langle \rho, \lambda\rangle$ is an integer; therefore no half Tate twist is needed.

Let $\mathrm{Sat} \subset \widetilde{\mathrm{Sat}}$ be the full subcategory consisting of finite direct sums of IC_λ's for the various $\lambda \in \mathbb{X}_*(T)^{\mathrm{dom}}$. We claim that Sat is closed under convolution. In [3, Sect. 3.5], Arkhipov and Bezrukavnikov show this is the case when k is a finite field, which implies the general case of $\mathrm{char}(k) > 0$. When $\mathrm{char}(k) = 0$, we only need to consider $k = \mathbb{Q}$. However, since $\mathrm{Sat}^{\mathrm{geom}}$ is a semisimple category with simple objects $\mathrm{IC}_{\lambda,\overline{\mathbb{Q}}}$, we know $\mathrm{IC}_\lambda * \mathrm{IC}_\mu \cong \bigoplus_\nu \mathrm{IC}_\nu \otimes V^\nu_{\lambda,\mu}$ for $\mathbb{Q}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-modules $V^\nu_{\lambda,\mu} = \mathrm{Hom}_{\mathrm{Gr}}(\mathrm{IC}_\nu, \mathrm{IC}_\lambda * \mathrm{IC}_\mu)$. By the standard argument of choosing an integral model, and the claim for finite fields, we see that the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on $V^\nu_{\lambda,\mu}$ is trivial when restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ for almost all primes p. By Chebotarev density, this implies that the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on $V^{\nu}_{\lambda,\mu}$ is trivial, and the claim is proved. It is then easy to see that the pullback functor gives a tensor equivalence $\mathrm{Sat} \xrightarrow{\sim} \mathrm{Sat}^{\mathrm{geom}}$, hence also a tensor equivalence

$$\mathrm{Sat} \xrightarrow{\sim} \mathrm{Rep}(\widehat{G}, \mathbb{Q}_\ell). \tag{4.1}$$

4.1.2 The Hecke correspondence

Consider the following correspondence

$$\begin{array}{ccc} & \mathrm{Hk} & \\ \overleftarrow{h} \swarrow & & \searrow \overrightarrow{h} \\ \mathrm{Bun} \times (\mathbb{P}^1 - \{0, 1, \infty\}) & & \mathrm{Bun} \times (\mathbb{P}^1 - \{0, 1, \infty\}) \end{array} \tag{4.2}$$

We need to explain some notations. The stack Hk is the functor which sends R to the category of tuples $(x, \mathcal{P}, \mathcal{P}', \iota)$ where $x \in (\mathbb{P}^1 - \{0, 1, \infty\})(R)$, $\mathcal{P}, \mathcal{P}' \in \mathrm{Bun}(R)$ and ι is an isomorphism between the $\mathcal{P}$ and $\mathcal{P}'$ away from the graph of x. In particular, ι preserves the level structures at 0, 1 and ∞. The maps $\overleftarrow{h}$ and $\overrightarrow{h}$ are defined by

$$\overleftarrow{h}\big(x, \mathcal{P}, \mathcal{P}', \iota\big) = (\mathcal{P}, x); \qquad \overrightarrow{h}\big(x, \mathcal{P}, \mathcal{P}', \iota\big) = \big(\mathcal{P}', x\big).$$

4.1.3 Geometric Hecke operators

The geometric fibers of $\overrightarrow{h}$ over $\mathrm{Bun} \times (\mathbb{P}^1 - \{0, 1, \infty\})$ are (non-canonically) isomorphic to the affine Grassmannian Gr_G. Locally in the smooth topology on $\mathrm{Bun} \times (\mathbb{P}^1 - \{0, 1, \infty\})$ this fibration is trivial (see [21, Remark 4.1]). For $\mathcal{K} \in \mathrm{Sat}$, the local triviality of the fibration $\overrightarrow{h}$ allows us to define a complex $\mathcal{K}_{\mathrm{Hk}} \in D^b(\mathrm{Hk}, \mathbb{Q}_\ell)$ on Hk whose restriction to each fiber of $\overrightarrow{h}$ is isomorphic to $\mathcal{K}$.

The *universal geometric Hecke operator* is defined as a bifunctor

$$\begin{aligned} \widetilde{\mathbb{T}} : \mathrm{Sat} \times D^b\big(\mathrm{Bun} \times \big(\mathbb{P}^1 - \{0, 1, \infty\}\big)\big) &\to D^b\big(\mathrm{Bun} \times \big(\mathbb{P}^1 - \{0, 1, \infty\}\big)\big) \\ (\mathcal{K}, \mathcal{F}) &\mapsto \overrightarrow{h}_!\big(\overleftarrow{h}^* \mathcal{F} \otimes_{\mathbb{Q}_\ell} \mathcal{K}_{\mathrm{Hk}}\big). \end{aligned} \tag{4.3}$$

Note that $\mathcal{K}_{\mathrm{Hk}}$ always has $\mathbb{Q}_\ell$-coefficients. The bifunctor $\widetilde{\mathbb{T}}$ is compatible with the tensor structure $*$ on Sat in the sense that the functor

$$\mathrm{Sat} \to \mathrm{End}\big(D^b\big(\mathrm{Bun} \times \big(\mathbb{P}^1 - \{0, 1, \infty\}\big)\big)\big)$$

$$\mathcal{K} \mapsto \left(\mathcal{F} \mapsto \widetilde{\mathbb{T}}(\mathcal{K}, \mathcal{F})\right)$$

has a natural structure of a monoidal functor (the monoidal structure on the target is given by composition of endofunctors). We more often use the following functor

$$\begin{aligned} \mathbb{T} : \mathrm{Sat} \times D^b(\mathrm{Bun}) &\to D^b\left(\mathrm{Bun} \times \left(\mathbb{P}^1 - \{0, 1, \infty\}\right)\right) \\ \mathcal{F} &\mapsto \widetilde{\mathbb{T}}\left(\mathcal{K}, \mathcal{F} \boxtimes \underline{\mathbb{Q}'_\ell}_{\mathbb{P}^1 - \{0,1,\infty\}}\right). \end{aligned}$$

4.2 Eigen local systems

Definition 4.1 A *Hecke eigensheaf* is a triple $(\mathcal{F}, \mathcal{E}, \epsilon)$ where

- $\mathcal{F} \in D^b(\mathrm{Bun})$;
- $\mathcal{E} : \mathrm{Sat} \to \mathrm{Loc}(\mathbb{P}^1 - \{0, 1, \infty\})$ is a tensor functor;
- ϵ is a system of isomorphisms $\epsilon(\mathcal{K})$ for each $\mathcal{K} \in \mathrm{Sat}$

$$\epsilon(\mathcal{K}) : \mathbb{T}(\mathcal{K}, \mathcal{F}) \xrightarrow{\sim} \mathcal{F} \boxtimes \mathcal{E}(\mathcal{K}) \tag{4.4}$$

  which is compatible with the tensor structures (see [17, discussion after Proposition 2.8]).

The functor $\mathcal{E}$ in the definition defines a $\widehat{G}$-local system on $\mathbb{P}^1 - \{0, 1, \infty\}$, which is called the *eigen local system* of $\mathcal{F}$. When we say $\mathcal{F} \in D^b(\mathrm{Bun})$ is a Hecke eigensheaf, it means there exists $(\mathcal{E}, \epsilon)$ as above making the triple $(\mathcal{F}, \mathcal{E}, \epsilon)$ a Hecke eigensheaf in the sense of Definition 4.1.

The construction of the local system in the Main Theorem is given in the following theorem.

Theorem 4.2 *Suppose G is of type A_1, D_{2n}, E_7, E_8 or G_2. Assume (3.7) holds.*

(1) *Assume (2.8) also holds. Then for every odd character χ of $\widetilde{A}_0$, the object $j_!\mathcal{F}_\chi = j_*\mathcal{F}_\chi \in D^b(\mathrm{Bun})_{\mathrm{odd}}$ is a Hecke eigensheaf with eigen local system*

$$\mathcal{E}'_\chi : \mathrm{Sat} \cong \mathrm{Rep}(\widehat{G}, \mathbb{Q}_\ell) \to \mathrm{Loc}\left(\mathbb{P}^1 - \{0, 1, \infty\}, \mathbb{Q}'_\ell\right).$$

(2) *Suppose G is of type A_1, D_{4n+2} or E_7 and we do* not *assume (2.8). Then $\mathcal{E}_{\chi, \mathbb{Q}'_\ell}$ is* a priori *a $\widehat{G}(\mathbb{Q}'_\ell)$-local system on $\mathbb{P}^1_{k'} - \{0, 1, \infty\}$ where $k' = k(\sqrt{-1})$. There is a canonical way to descend $\mathcal{E}'_\chi$ to a $\widehat{G}(\mathbb{Q}'_\ell)$-local system on $\mathbb{P}^1_k - \{0, 1, \infty\}$.*

(3) *The $\widehat{G}(\mathbb{Q}'_\ell)$-local system $\mathcal{E}'_\chi$ descends canonically to a $\widehat{G}(\mathbb{Q}_\ell)$-local system $\mathcal{E}_\chi$.*

(4) *If k is a finite field, $\mathcal{E}_\chi(\mathcal{K})$ is pure of weight zero for any $\mathcal{K} \in$* Sat.

The proof of this theorem occupies Sect. 4.4. In the rest of this subsection, assuming this theorem, we work step by step towards an explicit description of the eigen local system $\mathcal{E}_\chi$.

4.2.1 The Beilinson-Drinfeld Grassmannian

Beilinson and Drinfeld defined a global analog of the affine Grassmannian, which fits into a Cartesian diagram

$$\begin{array}{ccc} \mathrm{GR} & \longrightarrow & \mathrm{Hk} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overrightarrow{h}} \\ \mathbb{P}^1 - \{0, 1, \infty\} & \xrightarrow{u_0 \times \mathrm{id}} & \mathrm{Bun} \times (\mathbb{P}^1 - \{0, 1, \infty\}) \end{array}$$

Using the moduli interpretation of Hk, we see that GR classifies triples $(x, \mathcal{P}, \iota)$ where $x \in (\mathbb{P}^1 - \{0, 1, \infty\})(R)$, $\mathcal{P} \in \mathrm{Bun}(R)$ and ι is an isomorphism $\mathcal{P}|_{\mathbb{P}^1_R - \Gamma(x)} \stackrel{\sim}{\to} \mathcal{P}_{u_0}|_{\mathbb{P}^1_R - \Gamma(x)}$, where $\Gamma(x) \subset \mathbb{P}^1_R$ is the graph of x and $\mathcal{P}_{u_0} \in \mathrm{Bun}(k)$ is the object corresponding to u_0. Fixing x the fiber of GR over x is denoted by Gr_x, which is an ind-scheme over R. When x is a closed point, we have $\mathrm{Gr}_x \stackrel{\sim}{\to} \mathrm{Gr} \otimes_k k(x)$.

For every $\mathcal{K} \in \mathrm{Sat}$, there is a corresponding object $\mathcal{K}_{\mathrm{GR}} \in D^b(\mathrm{GR}, \mathbb{Q}_\ell)$ obtained by spreading it over $\mathbb{P}^1 - \{0, 1, \infty\}$: for the construction, we may either take $\mathcal{K}_{\mathrm{GR}}$ to be the restriction of $\mathcal{K}_{\mathrm{Hk}}$, or use [16, Sect. 2.1.3]. For each geometric point $x \in \mathbb{P}^1 - \{0, 1, \infty\}$, the restriction of $\mathcal{K}_{\mathrm{GR}}$ to the fiber Gr_x is isomorphic to $\mathcal{K}$ (base changed to $\mathrm{Gr} \otimes_k k(x)$).

4.2.2 Calculation of the eigen local system $\mathcal{E}_\chi$

From now on till Sect. 4.2.3 we work under both assumptions (3.7) and (2.8).

Define $\widetilde{\mathrm{GR}}^U$ by the Cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathrm{GR}}^U & \longrightarrow & \mathrm{Hk} \\ \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle (\overleftarrow{h}, \overrightarrow{h})} \\ \mathbb{P}^1 - \{0, 1, \infty\} & \xrightarrow{u_0 \times u_0 \times \mathrm{id}} & \mathrm{Bun} \times \mathrm{Bun} \times (\mathbb{P}^1 - \{0, 1, \infty\}) \end{array} \tag{4.5}$$

The morphism $u_0 \times u_0 \times \mathrm{id} : \mathbb{P}^1 - \{0, 1, \infty\} \to \mathrm{Bun} \times \mathrm{Bun} \times (\mathbb{P}^1 - \{0, 1, \infty\})$ factors through the $(\widetilde{A} \times \widetilde{A})$-torsor $\mathbb{P}^1 - \{0, 1, \infty\} \to [\widetilde{K}_0 \backslash U]^2 \times (\mathbb{P}^1 -$

$\{0, 1, \infty\}$), therefore $\widetilde{\mathrm{GR}^U}$ carries a natural $\widetilde{A} \times \widetilde{A}$-action. When we need to distinguish the two copies of $\widetilde{A}$, we denote the first one by $\widetilde{A}(1)$, which acts on $\widetilde{\mathrm{GR}^U}$ via base change through $\overleftarrow{h}$; similarly we denote the second copy of $\widetilde{A}$ by $\widetilde{A}(2)$, which acts via base change through $\overrightarrow{h}$. Note that the μ_2^{ker} in either copy of $\widetilde{A}$ acts in the same way on $\widetilde{\mathrm{GR}^U}$.

The ind-scheme $\widetilde{\mathrm{GR}^U}$ fits into a diagram

$$\begin{array}{ccccccccc}
\widetilde{\mathrm{GR}^U} & \xrightarrow{\nu} & \mathrm{GR}^U & \xhookrightarrow{j_{\mathrm{GR}}} & \mathrm{GR} & \longrightarrow & \mathrm{Hk} & & \\
\swarrow & & \swarrow{\scriptstyle \omega^U} & & \swarrow{\scriptstyle \omega} \quad \searrow{\scriptstyle \pi} & & & \searrow{\scriptstyle \overrightarrow{h}} & \\
\mathrm{Spec}\, k & \xrightarrow{u_0} & [\widetilde{K}_0 \backslash U] & \hookrightarrow & \mathrm{Bun} & & \mathbb{P}^1 - \{0,1,\infty\} & \xrightarrow{u_0 \times \mathrm{id}} & \mathrm{Bun} \times (\mathbb{P}^1 - \{0,1,\infty\})
\end{array} \tag{4.6}$$

where all squares are Cartesian by definition. The arrows ν and u_0 are both $\widetilde{A}(1)$-torsors. Using the moduli interpretation given in Sect. 4.2.1, $\omega : \mathrm{GR} \to \mathrm{Hk} \xrightarrow{\overleftarrow{h}} \mathrm{Bun}$ sends the triple $(x, \mathcal{P}, \iota)$ to $\mathcal{P} \in \mathrm{Bun}$. Denote the projections to $\mathbb{P}^1 - \{0, 1, \infty\}$ by

$$\pi^U : \mathrm{GR}^U \to \mathbb{P}^1 - \{0, 1, \infty\}; \qquad \widetilde{\pi} : \widetilde{\mathrm{GR}^U} \to \mathbb{P}^1 - \{0, 1, \infty\}.$$

Since $\widetilde{A} \times \widetilde{A}$ acts on the fibers of $\nu_{\mathrm{Hk}} : \widetilde{\mathrm{GR}^U} \to \mathrm{Hk}$, the complex $\widetilde{\pi}_! \nu^* \mathcal{K}_{\mathrm{GR}, \mathbb{Q}'_\ell} = \widetilde{\pi}_! \nu^*_{\mathrm{Hk}} \mathcal{K}_{\mathrm{Hk}}$ is $\widetilde{A} \times \widetilde{A}$-equivariant. Let $(\widetilde{\pi}_! \nu^* \mathcal{K}_{\mathrm{GR}, \mathbb{Q}'_\ell})_{\mathrm{odd}}$ be the direct summand on which μ_2^{ker} acts by the sign representation (via either copy of $\widetilde{A}$).

On the other hand, $\widetilde{A}(\bar{k}) \times \widetilde{A}(\bar{k})$ acts on the $\widetilde{A}(\bar{k})$ via $(a_1, a_2) \cdot a = a_1 a a_2^{-1}$, hence it acts on the group algebra $\mathbb{Q}'_\ell[\widetilde{A}(\bar{k})]$. Then $\mathbb{Q}'_\ell[\widetilde{A}(\bar{k})]$ is an $\Gamma^{(2)} := (\widetilde{A}(\bar{k}) \times \widetilde{A}(\bar{k})) \rtimes \mathrm{Gal}(\bar{k}/k)$-module. We may decompose it under the action of $\widetilde{A}_0$ via the embedding into $\widetilde{A}(2)$:

$$\mathbb{Q}'_\ell\big[\widetilde{A}(\bar{k})\big] = \bigoplus_{\chi \in \widetilde{A}^*_{0,\mathrm{odd}}} \mathbb{Q}'_\ell\big[\widetilde{A}(\bar{k})\big]_\chi .$$

Each direct summand $\mathbb{Q}'_\ell[\widetilde{A}(\bar{k})]_\chi$ is still a $\Gamma^{(2)}$-module. The analog of Lemma 3.3 allows us to view $\mathbb{Q}'_\ell[\widetilde{A}(\bar{k})]_\chi$ as an $(\widetilde{A} \times \widetilde{A})$-equivariant local system on $\mathrm{Spec}\, k$, which we denote simply by $\mathbb{Q}'_\ell[\widetilde{A}]_\chi$.

Lemma 4.3 *There is a canonical isomorphism in* $\mathrm{Loc}_{\widetilde{A} \times \widetilde{A}}(\mathbb{P}^1 - \{0, 1, \infty\})$:

$$\big(\widetilde{\pi}_! \nu^* \mathcal{K}_{\mathrm{GR}}\big)_{\mathrm{odd}} \otimes_{\mathbb{Q}_\ell} \mathbb{Q}'_\ell \cong \bigoplus_{\chi \in \widetilde{A}^*_{0,\mathrm{odd}}} \mathbb{Q}'_\ell[\widetilde{A}]_\chi \otimes \mathcal{E}'_\chi(\mathcal{K}) \tag{4.7}$$

Consequently, we have an isomorphism in $\mathrm{Loc}(\mathbb{P}^1 - \{0, 1, \infty\})$

$$\mathcal{E}'_\chi(\mathcal{K}) \cong \left(\mathbb{Q}'_\ell[\widetilde{A}]^*_\chi \otimes_{\mathbb{Q}_\ell} \left(\widetilde{\pi}_! \nu^* \mathcal{K}_{\mathrm{GR}}\right)_{\mathrm{odd}}\right)^{\widetilde{A}\times\widetilde{A}}. \tag{4.8}$$

Here the $(\widetilde{A} \times \widetilde{A})$*-invariants of a sheaf is taken in the sense of Sect.* 3.3.4

Proof By the definition of the geometric Hecke operator (4.3) and proper base change, we have

$$(u_0 \times \mathrm{id})^* \mathbb{T}(\mathcal{K}, j_! \mathcal{F}_\chi) = \pi_!\left(\omega^* j_! \mathcal{F}_\chi \otimes_{\mathbb{Q}_\ell} \mathcal{K}_{\mathrm{GR}}\right) = \pi_!^U\left(\omega^{U,*} \mathcal{F}_\chi \otimes_{\mathbb{Q}_\ell} \mathcal{K}_{\mathrm{GR}}\right). \tag{4.9}$$

By the definition of the Hecke eigen property (4.4), we have

$$(u_0 \times \mathrm{id})^* \mathbb{T}(\mathcal{K}, j_! \mathcal{F}_\chi) \cong (u_0 \times \mathrm{id})^* \left(\mathcal{F}_\chi \boxtimes_{\mathbb{Q}_\ell} \mathcal{E}_\chi(\mathcal{K})\right) \cong V_\chi \otimes \mathcal{E}'_\chi(\mathcal{K}). \tag{4.10}$$

Combining (4.9) and (4.10), we get an isomorphism

$$\pi_!^U\left(\omega^{U,*} \mathcal{F}_\chi \otimes_{\mathbb{Q}_\ell} \mathcal{K}_{\mathrm{GR}}\right) \cong V_\chi \otimes \mathcal{E}'_\chi(\mathcal{K}). \tag{4.11}$$

This isomorphism is $\widetilde{A}(2)$-equivariant with $\widetilde{A}(2)$ acting on V_χ on the right side.

Since both ν and $u_0 : \mathrm{Spec}\, k \to [\widetilde{K}_0 \backslash f\ell_G]$ are $\widetilde{A}(1)$-torsors, $u_{0,*}\mathbb{Q}'_\ell$ and $\nu_*\mathbb{Q}'_\ell$ carry $\widetilde{A}(1)$-actions, so it makes sense to extract the direct summands $(u_{0,*}\mathbb{Q}'_\ell)_{\mathrm{odd}}$ and $(\nu_*\mathbb{Q}'_\ell)_{\mathrm{odd}}$ on which $\mu_2^{\ker}$ acts by the sign representation. We have an $\widetilde{A}(1)$-equivariant decomposition $(u_{0,*}\mathbb{Q}'_\ell)_{\mathrm{odd}} = \bigoplus_{\chi \text{ odd}} V_\chi^* \otimes \mathcal{F}_\chi$ where $\widetilde{A}(1)$ acts on V_χ^*. Pulling back along ω^U, we get $(\nu_*\mathbb{Q}'_\ell)_{\mathrm{odd}} \cong \bigoplus_{\chi \text{ odd}} V_\chi^* \otimes \omega^{U,*}\mathcal{F}_\chi$. Therefore we get an $\widetilde{A} \times \widetilde{A}$-equivariant isomorphism

$$\begin{aligned}\left(\widetilde{\pi}_! \nu^* \mathcal{K}_{\mathrm{GR}}\right)_{\mathrm{odd}} \otimes_{\mathbb{Q}_\ell} \mathbb{Q}'_\ell &\cong \pi_!^U\left(\left(\nu_*\mathbb{Q}'_\ell\right)_{\mathrm{odd}} \otimes_{\mathbb{Q}_\ell} \mathcal{K}_{\mathrm{GR}}\right) \\ &\cong \bigoplus_{\chi \in \widetilde{A}^*_{0,\mathrm{odd}}} \pi_!^U\left(V_\chi^* \otimes \omega^{U,*}\mathcal{F}_\chi \otimes_{\mathbb{Q}_\ell} \mathcal{K}_{GR}\right)\end{aligned}$$

Applying (4.11) to the right side above, we get an $\widetilde{A} \times \widetilde{A}$-equivariant isomorphism

$$\left(\widetilde{\pi}_! \nu^* \mathcal{K}_{\mathrm{GR}}\right)_{\mathrm{odd}} \cong \bigoplus_{\chi \in \widetilde{A}^*_{0,\mathrm{odd}}} V_\chi^* \otimes V_\chi \otimes \mathcal{E}'_\chi(\mathcal{K}). \tag{4.12}$$

On the right side, $\widetilde{A}(1)$ acts on V_χ^* and $\widetilde{A}(2)$ acts on V_χ. Note that $\mathbb{Q}'_\ell[\widetilde{A}(\bar{k})]_\chi = V_\chi^* \otimes V_\chi$ as $\Gamma^{(2)}$-modules. Therefore the right side of (4.12) is the same as the right side of (4.7). Since $\mathbb{Q}'_\ell[\widetilde{A}]_\chi$ is irreducible as a representation of $\widetilde{A}(\bar{k}) \times \widetilde{A}(\bar{k})$, (4.8) follows from (4.7). This finishes the proof of the lemma. □

4.2.3 Quasi-minuscule Schubert variety

We specialize to the case $\lambda = \theta^\vee$, the coroot corresponding to the highest root θ of G. This is called a *quasi-minuscule weight* of $\widehat{G}$ because in the weight decomposition of the irreducible representation $V_{\theta^\vee}$ of $\widehat{G}$, all nonzero weights are in the Weyl group orbit of $\theta^\vee$. When $\widehat{G}$ is simply-laced, $V_{\theta^\vee}$ is the adjoint representation of $\widehat{G}$.

For basic properties of the variety $\mathrm{Gr}_{x,\leq\theta^\vee}$ we refer to [21, Sect. 5.3, especially Lemma 5.22]. In particular, it has dimension $\langle 2\rho, \theta^\vee\rangle = 2h^\vee - 2$, where $h^\vee$ is the dual Coxeter number of G. It consists of two L_x^+G-orbits: the base point $\mathrm{Gr}_{x,0}$ (the only singularity) and its complement $\mathrm{Gr}_{x,\theta^\vee}$, which is in fact isomorphic to the variety Y introduced in Sect. 1.4. The open subset $\mathrm{Gr}^U_{x,\theta^\vee}$ is a union $\mathrm{Gr}_{x,0} \cup (Y - D_x)$, where D_x is an ample divisor of $\mathrm{Gr}_{x,\leq\theta^\vee}$ depending algebraically on x (for the ampleness of D_x see the last paragraph of the proof of Lemma 4.8).

Let $\widetilde{Y} \subset \widetilde{\mathrm{GR}^U}$ be the preimage of $\mathrm{GR}_{\theta^\vee}$ in $\widetilde{\mathrm{GR}^U}$. Note that $\widetilde{Y}$ still carries the action of $\widetilde{A} \times \widetilde{A}$. The projection $\eta : \widetilde{Y} \to \mathbb{P}^1 - \{0, 1, \infty\}$ is smooth with fibers $\widetilde{Y}_x$ which are $\widetilde{A}(1)$-torsors over $Y - D_x$.

We abbreviate $\mathcal{E}'_\chi(\mathrm{IC}_{\theta^\vee})$ by $\mathcal{E}'_{\chi,\mathrm{qm}}$ (a $\mathbb{Q}'_\ell$-local system of rank $\dim V_{\theta^\vee}$).

Proposition 4.4 *There is a canonical $(\widetilde{A} \times \widetilde{A})$-equivariant isomorphism of local systems over $\mathbb{P}^1 - \{0, 1, \infty\}$*

$$\big(\mathbf{R}^{2h^\vee-2}\eta_!\mathbb{Q}'_\ell\big)_{\mathrm{odd}}\big(h^\vee - 1\big) \cong \bigoplus_{\chi\in\widetilde{A}^*_{0,\mathrm{odd}}} \mathbb{Q}'_\ell[\widetilde{A}]_\chi \otimes \mathcal{E}'_{\chi,\mathrm{qm}}. \tag{4.13}$$

Consequently, we have an isomorphism of local systems over $\mathbb{P}^1 - \{0, 1, \infty\}$

$$\mathcal{E}'_{\chi,\mathrm{qm}} \cong \big(\mathbb{Q}'_\ell[\widetilde{A}]^*_\chi \otimes \big(\mathbf{R}^{2h^\vee-2}\eta_!\mathbb{Q}'_\ell\big)_{\mathrm{odd}}\big)^{\widetilde{A}\times\widetilde{A}}\big(h^\vee - 1\big). \tag{4.14}$$

Proof Denote the preimage of $\mathrm{GR}_{\leq\theta^\vee}$ in $\widetilde{\mathrm{GR}^U}$ by $\widetilde{Q}$. Let $\mathcal{K} = \mathrm{IC}_{\theta^\vee}$. Then the spread-out $\mathcal{K}_{\mathrm{GR}}$ is the intersection complex of $\mathrm{GR}_{\leq\theta^\vee}$ up to a shift. Since $\nu : \widetilde{Q} \to \mathrm{GR}_{\theta^\vee}$ is étale, we have

$$\nu^*\mathcal{K}_{\mathrm{GR}} \cong \mathrm{IC}_{\widetilde{Q}}\big[2h^\vee - 2\big]\big(h^\vee - 1\big).$$

Here we have normalized the intersection complex $\mathrm{IC}_{\widetilde{Q}}$ to lie in cohomological degrees $0, \ldots, 2\dim\widetilde{Q}$ and to be pure of weight 0 as a complex. We still use $\widetilde{\pi}$ to denote the projection $\widetilde{Q} \to \mathbb{P}^1 - \{0, 1, \infty\}$. Then Lemma 4.3 applied to $\mathcal{K}_{\mathrm{GR}}$ gives an $(\widetilde{A} \times \widetilde{A})$-equivariant isomorphism

$$(\widetilde{\pi}_!\mathrm{IC}_{\widetilde{Q}})_{\mathrm{odd}} \otimes_{\mathbb{Q}_\ell} \mathbb{Q}'_\ell\big[2h^\vee - 2\big]\big(h^\vee - 1\big) \cong \bigoplus_{\chi\in\widetilde{A}^*_{0,\mathrm{odd}}} \mathbb{Q}'_\ell[\widetilde{A}]_\chi \otimes \mathcal{E}'_{\chi,\mathrm{qm}}. \tag{4.15}$$

In particular, $(\widetilde{\pi}_! \mathrm{IC}_{\widetilde{Q}})_{\mathrm{odd}}$ is concentrated in the middle degree $2h^\vee - 2$, and is a local system.

Now we relate the intersection cohomology of $\widetilde{Q}$ to the usual cohomology of $\widetilde{Y}$. From the discussion in Sect. 4.2.3, we can write $\mathrm{GR}_{\leq\theta^\vee} = \mathrm{GR}_0 \cup \mathrm{GR}_{\theta^\vee}$. Let $\widetilde{Z}$ be the preimage of GR_0 in $\widetilde{Q}$, then $\widetilde{Q} = \widetilde{Z} \cup \widetilde{Y}$. Let $j : \widetilde{Y} \hookrightarrow \widetilde{Q}$ and $i : \widetilde{Z} \hookrightarrow \widetilde{Q}$ be natural embeddings. We have a distinguished triangle

$$j_! \mathbb{Q}_\ell\left[2h^\vee - 2\right]\left(h^\vee - 1\right) \to \mathrm{IC}_{\widetilde{Q}}\left[2h^\vee - 2\right]\left(h^\vee - 1\right) \to C \otimes i_* \underline{\mathbb{Q}_\ell}_{\widetilde{Z}} \to$$

where C is the stalk of $\mathrm{IC}_{\widetilde{Q}}[2h^\vee - 2](h^\vee - 1)$ along $\widetilde{Z}$, which is the same as the stalk of $\mathcal{K} = \mathrm{IC}_{\theta^\vee}$ at Gr_0. By the parity vanishing of the intersection cohomology complex on the affine Grassmannian (see [16, A.7]), C lies in degrees ≤ -2, therefore, taking compactly supported cohomology of the above distinguished triangle we get an isomorphism

$$\mathbf{R}^{2h^\vee - 2} \eta_! \mathbb{Q}_\ell \xrightarrow{\sim} \mathbf{R}^{2h^\vee - 2} \widetilde{\pi}_! \mathrm{IC}_{\widetilde{Q}}.$$

Combining with (4.15) we get (4.13). The other isomorphism (4.14) follows from (4.13) for the same reason that (4.8) follows from (4.7). □

In the next proposition, we show that when $k = \mathbb{Q}$, $\mathcal{E}_\chi$ has an integral model which interpolates the situations for $k = \mathbb{F}_p$. When we need to emphasize the base field we are working with, we write $\mathcal{E}_{\chi,k}$ for the eigen local system over $\mathbb{P}^1_k - \{0, 1, \infty\}$.

Proposition 4.5 *There is a positive integer N such that the $\widehat{G}(\mathbb{Q}_\ell)$-local system $\mathcal{E}_{\chi,\mathbb{Q}}$ over $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ extends to a $\widehat{G}(\mathbb{Q}_\ell)$-local system $\underline{\mathcal{E}}_\chi$ over $\mathbb{P}^1_{\mathbb{Z}[1/2\ell N]} - \{0, 1, \infty\}$. Moreover, for any geometric point* Spec $k \to$ Spec$\mathbb{Z}[1/2\ell N]$, *the restriction of $\underline{\mathcal{E}}_\chi$ to $\mathbb{P}^1_k - \{0, 1, \infty\}$ is isomorphic to $\mathcal{E}_{\chi,k}$.*

Proof We define

$$\mathbb{Q}' := \begin{cases} \mathbb{Q} & \text{if } G \text{ is of type } D_{4n}, E_8 \text{ or } G_2 \\ \mathbb{Q}(i) & \text{if } G \text{ is of type } A_1, D_{4n+2} \text{ or } E_7 \end{cases} \tag{4.16}$$

Let $\mathbb{Z}'$ be the ring of integers in $\mathbb{Q}'$. We will first extend $\mathcal{E}'_{\chi,\mathrm{qm},\mathbb{Q}'}$ (a local system over $\mathbb{P}^1_{\mathbb{Q}'} - \{0, 1, \infty\}$ with $\mathbb{Q}'_\ell$-coefficients of rank $d = \dim V_{\theta^\vee}$) to $\mathbb{P}^1_{\mathbb{Z}'[1/2\ell N]} - \{0, 1, \infty\}$ for some N.

In what follows, we use underlined symbols to denote the integral versions of the spaces, sheaves, etc. Over $\mathbb{Z}'[1/2]$, the groups $G, K, \ldots$ and the spaces $\mathrm{Bun}, U, \ldots$ have integral models $\underline{G}, \underline{K}, \ldots, \underline{\mathrm{Bun}}, \underline{U}, \ldots$. All these spaces are defined in a natural way such that their k-fibers (k is any field with $\mathrm{char}(k) \neq$

2) are the same as the corresponding spaces over the base field k we defined before.

As discussed in Sect. 3.2.5, the rational point u_0 extends to a point $\underline{u}_0 : \operatorname{Spec}\mathbb{Z}'[1/2N_0] \to \underline{U}$. The stabilizer of $\underline{u}_0$ under $\underline{\widetilde{K}_0}$ is a finite flat group scheme $\underline{\widetilde{A}}$ over $\mathbb{Z}'[1/2N_0]$. We may define an integral version of $\widetilde{Y}$ (in Sect. 4.2.3) by forming the Cartesian diagram

$$\begin{array}{ccc} \underline{\widetilde{Y}} & \longrightarrow & \underline{\mathrm{GR}}_{\theta^\vee} \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{Z}'[1/2N_0] & \xrightarrow{\underline{u}_0} & \underline{\mathrm{Bun}} \end{array}$$

Let $\underline{\eta} : \underline{\widetilde{Y}} \to \mathbb{P}^1_{\mathbb{Z}'[1/2\ell N_0]} - \{0, 1, \infty\}$ be the projection. By enlarging N_0 to some positive integer N, we may assume $(\mathbf{R}_c^{2h^\vee-2}\underline{\eta}_!\mathbb{Q}'_\ell)_{\mathrm{odd}}$ is a $\mathbb{Q}'_\ell$-local system over $\mathbb{P}^1_{\mathbb{Z}'[1/2\ell N]} - \{0, 1, \infty\}$ since it is a local system over $\mathbb{P}^1_{\mathbb{Q}'} - \{0, 1, \infty\}$ by (4.13). Let

$$\underline{\mathcal{E}}'_{\chi,\mathrm{qm}} := \left(\mathbb{Q}'_\ell\left[\widetilde{A}(\mathbb{Q})\right]^*_\chi \otimes \left(\mathbf{R}_c^{2h^\vee-2}\underline{\eta}_!\mathbb{Q}'_\ell\right)_{\mathrm{odd}}\right)^{\underline{\widetilde{A}}\times\underline{\widetilde{A}}}$$

We need to make sense of the operation of taking invariants under the action of the group scheme $\underline{\widetilde{A}}$. Note that $\underline{\widetilde{A}}$ is finite étale over $\mathbb{Z}'[1/2N_0]$, hence $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}')$ acts on $\widetilde{A}(\overline{\mathbb{Q}})$ via its quotient $\pi_1(\mathbb{Z}'[1/2N_0])$, therefore the construction of invariants by descent in Sect. 3.3.4 still works over the base $\mathbb{Z}'[1/2N_0]$. By proper base change and (4.14), the restriction of $\underline{\mathcal{E}}'_{\chi,\mathrm{qm}}$ to $\mathbb{P}^1_k - \{0, 1, \infty\}$ is the eigen local system $\mathcal{E}'_{\chi,\mathrm{qm},k}$ for any field-valued point $\operatorname{Spec} k \to \operatorname{Spec}\mathbb{Z}'[1/2\ell N_0]$.

The monodromy representation of $\underline{\mathcal{E}}'_{\chi,\mathrm{qm}}$ takes the form

$$\rho' : \pi_1\left(\mathbb{P}^1_{\mathbb{Z}'[1/2\ell N]} - \{0, 1, \infty\}, *\right) \to \mathrm{GL}_d\left(\mathbb{Q}'_\ell\right),$$

where $*$ is a fixed geometric base point which we omit in the sequel. Since the $\mathbb{Q}'$-fiber of $\underline{\mathcal{E}}'_{\chi,\mathrm{qm}}$ is $\mathcal{E}'_{\chi,\mathrm{qm},\mathbb{Q}'}$, we have a commutative diagram

$$\begin{array}{ccccc} \pi_1(\mathbb{P}^1_{\mathbb{Q}'} - \{0, 1, \infty\}) & \hookrightarrow & \pi_1(\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}) & \xrightarrow{\rho_{\mathbb{Q}}} & \widehat{G}(\mathbb{Q}_\ell) \\ \downarrow s' & & \downarrow s & \overset{\rho}{\nearrow} & \int a \\ \pi_1(\mathbb{P}^1_{\mathbb{Z}'[1/2\ell N]} - \{0, 1, \infty\}) & \hookrightarrow & \pi_1(\mathbb{P}^1_{\mathbb{Z}[1/2\ell N]} - \{0, 1, \infty\}) & \dashrightarrow & \mathrm{GL}_d(\mathbb{Q}'_\ell) \end{array} \qquad (4.17)$$

Here $\rho_{\mathbb{Q}}$ is the monodromy representation of $\mathcal{E}_{\chi,\mathbb{Q}}$ and the vertical map a is the homomorphism giving the quasi-minuscule representation of $\widehat{G}$. Since the vertical arrows s' and s are surjections with the same kernel, there is a unique way to fill in the dotted arrows. This dotted arrow ρ gives the desired $\widehat{G}(\mathbb{Q}_\ell)$-local system $\underline{\mathcal{E}}_\chi$ over $\mathbb{P}^1_{\mathbb{Z}[1/2\ell N]} - \{0, 1, \infty\}$. □

4.3 Description of the motives

4.3.1 Motives of an open variety

In this section, we assume the base field k is a number field. For every smooth projective variety X over k, and every $i \in \mathbb{Z}$, there is a well-defined motive $h^i(X) \in \mathrm{Mot}_k$ such that under the ℓ-adic cohomology functor, $h^i(X)$ is sent to $\mathrm{H}^i(X_{\overline{k}}, \mathbb{Q}_\ell)$.

More generally, suppose X is a smooth quasi-projective variety over k, one can still define an object $h^i_{c,\mathrm{pur}}(X) \in \mathrm{Mot}_k(\mathbb{Q})$ as follows. Let $\overline{X}$ be a compactification of X over $\mathbb{Q}$, which is a smooth projective variety such that $D = \overline{X} - X$ is a union of smooth divisors $\bigcup_{s\in S} D_s$ with normal crossings. Such an $\overline{X}$ exists by Hironaka's resolution of singularities. Since $\mathrm{Mot}_k(L)$ is an abelian category by Jannsen [22, Theorem 1], we can take kernels of maps. Let

$$h^i_{c,\mathrm{pur}}(X) = \ker\left(h^i(\overline{X}) \to \bigoplus_{s\in S} h^i(D_s)\right).$$

This may depend on the choice of the compactification. However we will see below that the ℓ-adic realization of $h^i_{c,\mathrm{pur}}(X)$ only depends on X. Consider the following maps induced by inclusion and restriction

$$\mathrm{H}^i_c(X_{\overline{k}}, \mathbb{Q}_\ell) \to \mathrm{H}^i(\overline{X}_{\overline{k}}, \mathbb{Q}_\ell) \to \bigoplus_{s\in S} \mathrm{H}^i(D_{s,\overline{k}}, \mathbb{Q}_\ell).$$

Each cohomology group carries a weight filtration (we can choose an integral model and look at the weights given by Frob_v for almost all v) and the maps are strictly compatible with the weight filtrations. Taking the weight i pieces, we get

$$\mathrm{Gr}^W_i \mathrm{H}^i_c(X_{\overline{k}}, \mathbb{Q}_\ell) = \ker\left(\mathrm{H}^i(\overline{X}_{\overline{k}}, \mathbb{Q}_\ell) \to \bigoplus_{s\in S} \mathrm{H}^i(D_{\overline{k}}, \mathbb{Q}_\ell)\right) = \mathrm{H}\big(h^i_{c,\mathrm{pur}}(X), \mathbb{Q}_\ell\big).$$

Since the ℓ-adic realization functor is exact (because $\mathrm{Mot}_k(\mathbb{Q})$ is semisimple by Jannsen [22, Theorem 1]), it sends $h^i_{c,\mathrm{pur}}(X)$ to the $\mathrm{Gal}(\overline{k}/k)$-module $\mathrm{Gr}^W_i \mathrm{H}^i_c(X_{\overline{k}}, \mathbb{Q}_\ell)$.

Suppose the smooth quasi-projective variety X is equipped with an action of a finite group scheme A over k, then we may first find an A-equivariant projective embedding. In fact, let act : $A \times X \to X$ and pr : $A \times X \to X$ be the action and projection map. If $\mathcal{L}$ is an ample line bundle on X, then its average under the A-action $\mathcal{L}_A = \det(\mathrm{act}_* \mathrm{pr}^* \mathcal{L})$ is again ample and A-equivariant. Using a high power of $\mathcal{L}_A$ we have an A-equivariant projective embedding $X \hookrightarrow \mathbb{P}^N$, and the closure of its image gives an A-equivariant compactification $\overline{X}$ of X. Using the equivariant version of Hironaka's resolution of singularities, we may assume that $\overline{X} - X = \bigcup_{s \in S} D_s$ is again a union of smooth divisors with normal crossings, and the whole situation is A-equivariant.

Every *closed point* $a \in |A|$ gives a self-correspondence $\Gamma(a)$ of $(X, \overline{X}, D_s)$ which is the graph of the a-action. All such correspondences span an algebra isomorphic to $\mathbb{Q}[A(\bar{k})]^{\mathrm{Gal}(\bar{k}/k)}$, and its action on $h^i_{c,\mathrm{pur}}(X)$ gives a homomorphism

$$\mathbb{Q}\big[A(\bar{k})\big]^{\mathrm{Gal}(\bar{k}/k)} \to \mathrm{End}_{\mathrm{Mot}_k(\mathbb{Q})}\big(h^i_{c,\mathrm{pur}}(X)\big).$$

If L is a number field and $e \in L[A(\bar{k})]^{\mathrm{Gal}(\bar{k}/k)}$ is an idempotent, then $eh^i_{c,\mathrm{pur}}(X)$ is an object in $\mathrm{Mot}_k(L)$.

4.3.2 The idempotents

We assume the number field k satisfies the assumption (2.8). Fix $\chi \in \widetilde{A}^*_{0,\mathrm{odd}}$. Let φ_χ be the characters of irreducible representations V_χ. For $G = D_{4n}, E_8$ or G_2, φ_χ takes $\mathbb{Q}$-values because $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts trivially on the set of irreducible characters; for $G = A_1, D_{4n+2}, E_7$ the character φ_χ takes $\mathbb{Q}(i)$-values. We define $\mathbb{Q}'$ by (4.16), so φ_χ takes values in $\mathbb{Q}'$. We will consider the category $\mathrm{Mot}_k(\mathbb{Q}')$ of motives over k with coefficients in $\mathbb{Q}'$.

Under the $\mathrm{Gal}(\bar{k}/k)$-action on $\widetilde{A}(\bar{k})$, φ_χ is also invariant because this action fixes the central character χ. We can make the following idempotent element in $\mathbb{Q}'[\widetilde{A}(\bar{k}) \times \widetilde{A}(\bar{k})]$

$$e_\chi(a_1, a_2) := \frac{1}{2^{2(\mathrm{rank}\ G+1)}} \sum_{(a_1,a_2) \in \widetilde{A}(\bar{k})} \varphi_\chi\big(a_1 a_2^{-1}\big).$$

Since φ_χ is constant on $\mathrm{Gal}(\bar{k}/k)$-orbits of $\widetilde{A}(\bar{k})$, we actually have $e_\chi \in \mathbb{Q}'[\widetilde{A}(\bar{k}) \times \widetilde{A}(\bar{k})]^{\mathrm{Gal}(\bar{k}/k)}$. The action of e_χ on the $\widetilde{A}(\bar{k}) \times \widetilde{A}(\bar{k})$-module $\mathbb{Q}'_\ell[\widetilde{A}(\bar{k})]_\chi$ is the projector onto $\mathrm{id} \in \mathrm{End}(V_\chi) = \overline{\mathbb{Q}}_\ell[\widetilde{A}(\bar{k})]_\chi$. The action of e_χ on $\mathbb{Q}'_\ell[\widetilde{A}(\bar{k})]_{\chi'}$ is zero if $\chi' \neq \chi$. Let

$$M_{\chi,x} := e_\chi h^{2h^\vee-2}_{c,\mathrm{pur}}(\widetilde{Y}_x)\big(h^\vee - 1\big) \in \mathrm{Mot}_k\big(\mathbb{Q}'\big).$$

Proposition 4.6 *Assume* (2.8) *holds for the number field* k. *There is an isomorphism of* $\mathrm{Gal}(\overline{k}/k)$*-modules*

$$\mathrm{H}\big(M_{\chi,x},\mathbb{Q}'_\ell\big)\cong\big(\mathcal{E}'_{\chi,\mathrm{qm}}\big)_x.$$

In particular, the $\mathrm{Gal}(\overline{k}/k)$*-representation* $(\mathcal{E}_{\chi,\mathrm{qm}})_x$ *is motivic.*

Proof Taking stalk of (4.13) at $x\in\mathbb{P}^1(k)-\{0,1,\infty\}$, we get an isomorphism of $\mathrm{Gal}(\overline{k}/k)$-modules:

$$\mathrm{H}_c^{2h^\vee-2}\big(\widetilde{Y}_x,\mathbb{Q}'_\ell\big)_{\mathrm{odd}}\big(h^\vee-1\big)\cong\bigoplus_{\chi\in\widetilde{A}^*_{0,\mathrm{odd}}}\mathbb{Q}'_\ell[\widetilde{A}]_\chi\otimes\big(\mathcal{E}'_{\chi,\mathrm{qm}}\big)_x. \tag{4.18}$$

By Proposition 4.5, $(\mathcal{E}'_{\chi,\mathrm{qm}})_x$ is unramified at large enough primes p, and is pure of weight zero under Frob_p according the purity result proved in Theorem 4.2(4). Using (4.18), we see that the same purity property holds for $\mathrm{H}_c^{2h^\vee-2}(\widetilde{Y}_x,\mathbb{Q}'_\ell)_{\mathrm{odd}}(h^\vee-1)$. Moreover, e_χ projects $\mathrm{H}_c^{2h^\vee-2}(\widetilde{Y}_x,\mathbb{Q}'_\ell)$ to a subspace of the odd part, therefore

$$\begin{aligned}\mathrm{H}\big(M_{\chi,x},\mathbb{Q}'_\ell\big)&\cong e_\chi\mathrm{Gr}^W_{2h^\vee-2}\mathrm{H}_c^{2h^\vee-2}\big(\widetilde{Y}_x,\mathbb{Q}'_\ell\big)\big(h^\vee-1\big)\\&=e_\chi\mathrm{H}_c^{2h^\vee-2}\big(\widetilde{Y}_x,\mathbb{Q}'_\ell\big)_{\mathrm{odd}}\big(h^\vee-1\big).\end{aligned}$$

Making e_χ act on the right side of (4.18), we get

$$\mathrm{H}\big(M_{\chi,x},\mathbb{Q}'_\ell\big)\cong e_\chi\mathbb{Q}'_\ell[\widetilde{A}]_\chi\otimes\big(\mathcal{E}'_{\chi,\mathrm{qm}}\big)_x=\big(\mathcal{E}'_{\chi,\mathrm{qm}}\big)_x.$$

□

4.4 Proof of Theorem 4.2

Proposition 4.7 *Let* $\mathcal{K}\in\mathrm{Sat}$ *and* $\mathcal{F}\in\mathrm{Loc}_{\widetilde{K}_0}(U)_{\mathrm{odd}}$.

(1) $(u_0\times\mathrm{id})^*\mathbb{T}(\mathcal{K},j_!\mathcal{F})[1]\in D^b(\mathbb{P}^1-\{0,1,\infty\})$ *is a perverse sheaf.*
(2) *If* k *is a finite field*, $(u_0\times\mathrm{id})^*\mathbb{T}(\mathcal{K},j_!\mathcal{F})$ *is pure of weight zero.*

Proof The proof is a variant of [21, Sect. 4.1]. Let λ be a large enough coweight of G such that $\mathcal{K}$ is supported in $\mathrm{Gr}_{\le\lambda}$. The spread-out complex $\mathcal{K}_{\mathrm{GR}}$ is in perverse degree one on GR (since its restriction to each geometric fiber Gr_x is perverse) and it is pure of weight zero. We restrict the diagram (4.6) to $\mathrm{GR}_{\le\lambda}$ without changing notations of the morphisms. By (4.9) we have

$$(u_0\times\mathrm{id})^*\mathbb{T}(\mathcal{K},j_!\mathcal{F})\cong\pi^U_!\big(\omega^{U,*}\mathcal{F}\otimes\mathcal{K}_{\mathrm{GR},\mathbb{Q}'_\ell}\big).$$

From this we see that $\omega^{U,*}\mathcal{F} \otimes \mathcal{K}_{\mathrm{GR},\mathbb{Q}'_\ell}$ is also in perverse degree one and pure of weight zero. In Lemma 4.8 below we will show that $\pi^U : \mathrm{GR}^U_{\le\lambda} \to \mathbb{P}^1 - \{0, 1, \infty\}$ is affine, therefore $(u_0 \times \mathrm{id})^*\mathbb{T}(\mathcal{K}, j_!\mathcal{F}) \in {}^pD^{\ge 1}(\mathrm{Bun} \times (\mathbb{P}^1 - \{0, 1, \infty\}))$ by [4, Thérème 4.1.1]. By [9, Variant 6.2.3 of Théorème 3.3.1], it has weight ≤ 0.

On the other hand, consider the Cartesian diagram

$$\begin{array}{ccccc}
\mathrm{GR}^U & \xrightarrow{v^U} & \mathrm{Hk}^U & \xrightarrow{\overleftarrow{h}^U} & [\widetilde{K}_0\backslash U] \\
\big\downarrow{\scriptstyle j_{\mathrm{GR}}} & & \big\downarrow{\scriptstyle j_{\mathrm{Hk}}} & & \big\downarrow{\scriptstyle j} \\
\mathrm{GR} & \xrightarrow{v} & \mathrm{Hk} & \xrightarrow{\overleftarrow{h}} & \mathrm{Bun}
\end{array}$$

Since $\overleftarrow{h}$ is a locally trivial fibration in smooth topology by [21, Remark 4.1], the natural transformation $\overleftarrow{h}^* j_* \to j_{\mathrm{Hk},*} \overleftarrow{h}^{U,*}$ is an isomorphism. Therefore

$$\omega^* j_* = v^* \overleftarrow{h}^* j_* \xrightarrow{\sim} v^* j_{\mathrm{Hk},*} \overleftarrow{h}^{U,*} \xrightarrow{\sim} j_{\mathrm{GR},*} v^{U,*} \overleftarrow{h}^{U,*} = j_{\mathrm{GR},*}\omega^{U,*},$$

where the second isomorphism uses the fact that v is étale. Hence

$$\begin{aligned}
&(u_0 \times \mathrm{id})^*\mathbb{T}(\mathcal{K}, j_*\mathcal{F}) \\
&\quad\cong \pi_!\big(\omega^* j_*\mathcal{F} \otimes \mathcal{K}_{\mathrm{GR},\mathbb{Q}'_\ell}\big) \cong \pi_*\big(\big(j_{\mathrm{GR},*}\omega^{U,*}\mathcal{F}\big) \otimes \mathcal{K}_{\mathrm{GR},\mathbb{Q}'_\ell}\big) \\
&\quad\cong \pi_* j_{\mathrm{GR},*}\big(\omega^{U,*}\mathcal{F} \otimes \mathcal{K}_{\mathrm{GR},\mathbb{Q}'_\ell}\big) = \pi^U_*\big(\omega^{U,*}\mathcal{F} \otimes \mathcal{K}_{\mathrm{GR},\mathbb{Q}'_\ell}\big).
\end{aligned} \tag{4.19}$$

Here $\pi_! = \pi_*$ because $\pi : \mathrm{GR}_{\le\lambda} \to \mathbb{P}^1 - \{0, 1, \infty\}$ is proper. Since π^U is affine and $\omega^{U,*}\mathcal{F} \otimes \mathcal{K}_{\mathrm{GR},\mathbb{Q}'_\ell}$ is in perverse degree 1, we have $(u_0 \times \mathrm{id})^*\mathbb{T}(\mathcal{K}, j_*\mathcal{F}) \in {}^pD^{\le 1}(\mathbb{P}^1 - \{0, 1, \infty\})$ by [4, Corollaire 4.1.2]. Since $\omega^{U,*}\mathcal{F} \otimes \mathcal{K}_{\mathrm{GR},\mathbb{Q}'_\ell}$ is pure of weight 0, by the dual statement of [9, 6.2.3], $(u_0 \times \mathrm{id})^*\mathbb{T}(\mathcal{K}, j_*\mathcal{F})$ has weight ≥ 0.

Since $j_!\mathcal{F} = j_*\mathcal{F}$ by Theorem 3.2, the above argument shows that $(u_0 \times \mathrm{id})^*\mathbb{T}(\mathcal{K}, j_*\mathcal{F})$ is concentrated in perverse degree 1 and is pure of weight 0. □

Lemma 4.8 *The morphism* $\pi^U : \mathrm{GR}^U_{\le\lambda} \to \mathbb{P}^1 - \{0, 1, \infty\}$ *is affine.*

Proof Fix a point $x \in (\mathbb{P}^1 - \{0, 1, \infty\})(R)$ for some finitely generated k-algebra R. We will argue that the fiber $\mathrm{Gr}^U_{x,\le\lambda}$ of $\mathrm{GR}^U_{\le\lambda}$ over x is an affine scheme.

The embedding j factors as $[\widetilde{K}_0\backslash U] \hookrightarrow [\widetilde{K}_0\backslash f\ell_G] \xrightarrow{j_1} \mathrm{Bun}$, therefore we have $\mathrm{Gr}^U_{x,\le\lambda} \hookrightarrow \mathrm{GR}^\star_{x,\le\lambda} \hookrightarrow \mathrm{Gr}_{x,\le\lambda}$, where $\mathrm{Gr}^\star_{x,\le\lambda} = \omega^{-1}([\widetilde{K}_0\backslash f\ell_G])$. Since

U is itself affine, so are the open embedding $[\widetilde{K}_0\backslash U] \hookrightarrow [\widetilde{K}_0\backslash f\ell_G]$ and its base change $\mathrm{Gr}^U_{x,\leq\lambda} \hookrightarrow \mathrm{Gr}^\star_{x,\leq\lambda}$. Therefore it suffices to show that $\mathrm{Gr}^\star_{x,\leq\lambda}$ is affine.

Consider the morphism

$$\beta : \mathrm{Gr}_{x,\leq\lambda} \xrightarrow{\omega} \mathrm{Bun} \to \mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{P}_\infty) \to \mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty).$$

By definition, $\mathrm{Gr}^\star_{x,\leq\lambda}$ is the preimage of $[\widetilde{K}_0\backslash f\ell_G] \subset \mathrm{Bun}$ under $\overleftarrow{h}$, hence the preimage of $[\{\star\}/K_0] \subset \mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$. By Lemma 3.1, the open substack $[\{\star\}/K_0]$ of $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$ is the non-vanishing locus of a section of a line bundle $\mathcal{L}$. Therefore, $\mathrm{Gr}^\star_{x,\leq\lambda}$ is the non-vanishing locus of a section of $\beta^*\mathcal{L}$. In order to show that $\mathrm{Gr}^\star_{x,\leq\lambda}$ is affine, it suffices to show that $\beta^*\mathcal{L}$ is ample on $\mathrm{Gr}_{x,\leq\lambda}$ relative to the base $\mathrm{Spec} R$. By [13, Corollary 12], the relative Picard group of Gr_x over $\mathrm{Spec}\ R$ is isomorphic to $\mathbb{Z}$, hence any line bundle on Gr_x with a nonzero section must be relatively ample. In particular, $\beta^*\mathcal{L}$ is ample on Gr_x, hence on its closed subscheme $\mathrm{Gr}_{x,\leq\lambda}$. This finishes the proof of the lemma. □

4.4.1 Preservation of the character χ

We first show a general result which works for all G satisfying (2.1). Let $\widetilde{ZG}$ be the preimage of the center of G (which is contained in K_0) in $\widetilde{K}_0$. The automorphism group of every object $\mathcal{P} \in \mathrm{Bun}_G(\mathbf{P}_0, \mathbf{I}_1, \mathbf{P}_\infty)$ contains ZG; similarly, the automorphism group of every object $\mathcal{P} \in \mathrm{Bun}$ contains $\widetilde{ZG}$. In other words, the classifying stack $\mathbb{B}(\widetilde{ZG})$ acts on the stack Bun. Moreover, the whole Hecke correspondence diagram (4.2) is $\mathbb{B}(\widetilde{ZG})$-equivariant. For every character $\psi : \widetilde{ZG} \to \overline{\mathbb{Q}}_\ell^\times$, we have the subcategory $D^b(\mathrm{Bun})_\psi \subset D^b(\mathrm{Bun})$ consisting of complexes on which $\widetilde{ZG}$ acts through ψ. By the $\mathbb{B}(\widetilde{ZG})$-equivariance of the diagram (the action is trivial on $\mathcal{K} \in \mathrm{Sat}$), the geometric Hecke operators $\mathbb{T}(\mathcal{K}, -)$ necessarily send $D^b(\mathrm{Bun})_\psi$ to $D^b(\mathrm{Bun} \times (\mathbb{P}^1 - \{0, 1, \infty\}))_\psi$.

Now back to the situation of Theorem 4.2, where we have $\widetilde{ZG} = \widetilde{A}_0$. For any $\chi \in \widetilde{A}_0(\bar{k})^*_{\mathrm{odd}}$ and any $\mathcal{K} \in \mathrm{Sat}$, the above discussion shows that $\mathbb{T}(\mathcal{K}, j_!\mathcal{F}_\chi) \in D^b(\mathrm{Bun} \times (\mathbb{P}^1 - \{0, 1, \infty\}))_\chi$.

4.4.2 Hecke eigen property with assumption (2.8)

By Proposition 4.7(1), $\mathbb{T}(\mathcal{K}, j_!\mathcal{F}_\chi)$ is concentrated in perverse degree 1 so Lemma 3.4 is applicable. Therefore we can write canonically

$$\mathbb{T}(\mathcal{K}, j_!\mathcal{F}_\chi) = \mathbb{T}(\mathcal{K}, j_!\mathcal{F}_\chi)_\chi \cong j_!\mathcal{F}_\chi \boxtimes \left(V_\chi^* \otimes (u_0 \times \mathrm{id})^*\mathbb{T}(\mathcal{K}, j_!\mathcal{F}_\chi)\right)^{\widetilde{A}}$$

Here the first equality follows from the discussion in Sect. 4.4.1. We define

$$\mathcal{E}'_\chi(\mathcal{K}) := \big(V_\chi^* \otimes (u_0 \times \mathrm{id})^* \mathbb{T}(\mathcal{K}, j_! \mathcal{F}_\chi)\big)^{\widetilde{A}} \tag{4.20}$$

which is a $\mathbb{Q}'_\ell$-complex on $\mathbb{P}^1 - \{0, 1, \infty\}$ concentrated in perverse degree 1.

Recall from Remark 2.8 that the construction of $\mathcal{F}_\chi$ (or the Γ-module V_χ) is not canonical: we are free to twist it by a continuous character ψ of $\mathrm{Gal}(\overline{k}/k)$. However, we have a canonical isomorphism

$$\big(V_\chi(\psi)^* \otimes (u_0 \times \mathrm{id})^* \mathbb{T}\big(\mathcal{K}, j_! \mathcal{F}_\chi(\psi)\big)\big)^{\widetilde{A}} \cong \big(V_\chi^* \otimes (u_0 \times \mathrm{id})^* \mathbb{T}(\mathcal{K}, j_! \mathcal{F}_\chi)\big)^{\widetilde{A}}.$$

Therefore $\mathcal{E}'_\chi(\mathcal{K})$ is canonical (i.e., independent of the choice involved in defining $\mathcal{F}_\chi$).

Using the monoidal structure of $\mathbb{T}(-, j_!\mathcal{F}_\chi)$ as spelled out in Sect. 4.1, we get canonical isomorphisms

$$\varphi_{\mathcal{K}_1, \mathcal{K}_2} : \mathcal{E}'_\chi(\mathcal{K}_1) \otimes \mathcal{E}'_\chi(\mathcal{K}_2) \xrightarrow{\sim} \mathcal{E}'_\chi(\mathcal{K}_1 * \mathcal{K}_2) \tag{4.21}$$

for any two objects $\mathcal{K}_1, \mathcal{K}_2 \in \mathrm{Sat}$. These isomorphisms are compatible with the associativity and commutativity constraints (for the commutativity constraint, we use Mirković and Vilonen's construction of the fusion product for the Satake category). Once we have the tensor property (4.21), we can use the argument of [21, Sect. 4.2] to show that each $\mathcal{E}'_\chi(\mathcal{K})$ must be a local system on $\mathbb{P}^1 - \{0, 1, \infty\}$. Therefore the assignment $\mathcal{K} \mapsto \mathcal{E}'_\chi(\mathcal{K})$ gives a tensor functor

$$\mathcal{E}'_\chi : \mathrm{Sat} \to \mathrm{Loc}\big(\mathbb{P}^1 - \{0, 1, \infty\}, \mathbb{Q}'_\ell\big)$$

which serves as the eigen local system of the Hecke eigensheaf $j_!\mathcal{F}_\chi$. This proves statement (1) of Theorem 4.2. The purity statement (4) of Theorem 4.2 follows from Proposition 4.7(2).

Lemma 4.9 *Assume (2.8) holds for k. Then there is a canonical isomorphism of $\widehat{G}(\mathbb{Q}'_\ell)$-local systems $\mathcal{E}'_\chi \cong \mathcal{E}'_{\overline{\chi}}$.*

Proof For a stack S over $\mathbb{P}^1 - \{0, 1, \infty\}$ with structure morphism $s : S \to \mathbb{P}^1 - \{0, 1, \infty\}$, we denote by $\mathbb{D}^{\mathrm{rel}} : D^b(S)^{\mathrm{opp}} \to D^b(S)$ the relative Verdier duality functor with respect to the morphism s: $\mathbb{D}^{\mathrm{rel}}(\mathcal{F}) = \mathbf{R}\underline{\mathrm{Hom}}_S(\mathcal{F}, \mathbb{D}_s)$, where $\mathbb{D}_s = s^! \overline{\mathbb{Q}}_\ell$ is the relative dualizing complex of s.

We use $(-)^\vee$ to denote the duality functor on a rigid tensor category, such as Sat or the tensor category of local systems over a certain base.

Applying duality to (4.20), we get an isomorphism

$$\mathcal{E}'_\chi(\mathcal{K})^\vee = \big(V_\chi^* \otimes (u_0 \times \mathrm{id})^* \mathbb{T}(\mathcal{K}, j_! \mathcal{F}_\chi)\big)^{\widetilde{A}, \vee}$$

$$\cong \big(V_\chi \otimes \mathbb{D}^{\mathrm{rel}}(u_0 \times \mathrm{id})^*\mathbb{T}(\mathcal{K}, j_!\mathcal{F}_\chi)\big)^{\widetilde{A}}. \tag{4.22}$$

By (4.9), we have

$$\begin{aligned}\mathbb{D}^{\mathrm{rel}}(u_0 \times \mathrm{id})^*\mathbb{T}(\mathcal{K}, j_!\mathcal{F}_\chi) &\cong \mathbb{D}^{\mathrm{rel}}\pi^U_!\big(\omega^{U,*}\mathcal{F} \otimes \mathcal{K}_{\mathrm{GR},\mathbb{Q}'_\ell}\big)\\ &\cong \pi^U_*\big(\omega^{U,*}\mathcal{F}^\vee_\chi \otimes \mathbb{D}^{\mathrm{rel}}\mathcal{K}_{\mathrm{GR},\mathbb{Q}'_\ell}\big)\\ &\cong \pi^U_*\big(\omega^{U,*}\mathcal{F}^\vee_\chi \otimes (\mathbb{D}\mathcal{K})_{\mathbb{Q}'_\ell}\big)\\ &\cong (u_0 \times \mathrm{id})^*\mathbb{T}\big(\mathbb{D}\mathcal{K}, j_!\mathcal{F}^\vee_\chi\big).\end{aligned}$$

The last equality follows from (4.19). Plugging this into (4.22), we get a canonical isomorphism

$$\mathcal{E}'_\chi(\mathcal{K})^\vee \cong \big(V_\chi \otimes (u_0 \times \mathrm{id})^*\mathbb{T}\big(\mathbb{D}\mathcal{K}, j_!\mathcal{F}^\vee_\chi\big)\big)^{\widetilde{A}}. \tag{4.23}$$

Fix a Γ'-isomorphism $\beta : V^*_\chi \cong V_{\overline{\chi}}(\psi)$ for some character ψ of $\mathrm{Gal}(\overline{k}/k')$. Note that β is unique up to a scalar, and it also induces an isomorphism $\mathcal{F}^\vee_\chi \cong \mathcal{F}_{\overline{\chi}}(\psi)$. Using β we may rewrite (4.23) as

$$\mathcal{E}'_\chi(\mathcal{K})^\vee \stackrel{\sim}{\to} \big(V_{\overline{\chi}}(\psi)^* \otimes (u_0 \times \mathrm{id})^*\mathbb{T}\big(\mathbb{D}\mathcal{K}, j_!\mathcal{F}_{\overline{\chi}}(\psi)\big)\big)^{\widetilde{A}} \cong \mathcal{E}'_{\overline{\chi}}(\mathbb{D}\mathcal{K}), \tag{4.24}$$

which is independent of the choice of β because we used β twice. One checks that (4.24) is compatible with the convolution structure of Sat, hence giving an isomorphism of tensor functors

$$\mathcal{E}'_\chi \stackrel{\sim}{\to} \big(\mathcal{E}'_{\overline{\chi}} \circ \mathbb{D}\big)^\vee : \mathrm{Sat} \to \mathrm{Loc}\big(\mathbb{P}^1 - \{0, 1, \infty\}, \mathbb{Q}'_\ell\big). \tag{4.25}$$

The Verdier duality functor $\mathbb{D}$ is a contravariant tensor functor. Composed with the duality $(-)^\vee$ on Sat (as a rigid tensor category), $\mathbb{D}(-)^\vee : \mathrm{Sat} \to \mathrm{Sat}$ is a tensor functor which naturally commutes with the fiber functor of taking cohomology $\mathrm{H}^* : \mathrm{Sat} \to \mathrm{Vec}$. Therefore, identifying Sat with $\mathrm{Rep}(\widehat{G})$, the functor $\mathbb{D}(-)^\vee$ is induced from an automorphism τ of $\widehat{G}$. Since $\mathbb{D}$ acts as identity on objects; the duality $(-)^\vee$ on $\mathrm{Sat} \cong \mathrm{Rep}(\widehat{G})$ also acts as identity on objects since $-1 \in W$, we conclude that the tensor functor $\mathbb{D}(-)^\vee$ also acts as identity on objects. This means the automorphism τ of $\widehat{G}$ is inner, and hence there is an isomorphism of tensor functors

$$\mathbb{D}(-)^\vee \stackrel{\sim}{\to} \mathrm{id}_{\mathrm{Sat}} : \mathrm{Sat} \to \mathrm{Sat}. \tag{4.26}$$

Moreover, such a tensor isomorphism δ is *unique* because the automorphism group of $\mathrm{id}_{\mathrm{Sat}}$ is $Z\widehat{G}$ which is trivial.

Using (4.25), (4.26), and the fact that $\mathcal{E}'_\chi$ intertwines the duality on Sat and $\mathrm{Loc}(\mathbb{P}^1 - \{0, 1, \infty\})$, we get a canonical isomorphism of tensor functors

$$\mathcal{E}'_\chi \xrightarrow{\sim} \left(\mathcal{E}'_{\overline{\chi}} \circ \mathbb{D}\right)^\vee \xrightarrow{\sim} \mathcal{E}'_{\overline{\chi}} \circ \left(\mathbb{D}(-)^\vee\right) \cong \mathcal{E}'_{\overline{\chi}}. \qquad \square$$

4.4.3 Descent of the base field

We now prove statement (2) in Theorem 4.2. We only need to consider the case $\sqrt{-1} \notin k$ and G is of type A_1, D_{4n+2} or E_7. Let $k' = k(\sqrt{-1})$. In the previous section we constructed a $\widehat{G}(\mathbb{Q}'_\ell)$-local system $\mathcal{E}_{\chi,k',\mathbb{Q}'_\ell}$ over $\mathbb{P}^1_{k'} - \{0, 1, \infty\}$. For a stack X over k, we denote its base change to k' by $X_{k'}$ and let $\sigma : X_{k'} \to X_{k'}$ be the k-involution induced from the nontrivial element in $\mathrm{Gal}(k'/k)$.

Let $\Gamma' = \widetilde{A}(\overline{k}) \rtimes \mathrm{Gal}(\overline{k}/k')$, which is a subgroup of Γ of index 2. The construction in Sect. 2.6.4 only gives a Γ'-module V_χ since we assumed (2.8) there. Since $\widetilde{A}_0$ contains a copy of μ_4, the action of the involution $\sigma \in \mathrm{Gal}(k'/k)$ changes the central character χ to $\overline{\chi}$. Therefore we have an isomorphism of Γ'-modules $\alpha : \sigma^* V_\chi \xrightarrow{\sim} V_{\overline{\chi}}(\psi)$ for some character ψ of $\mathrm{Gal}(\overline{k}/k')$. Note that α is unique up to a scalar, and it induces an isomorphism $\sigma^*\mathcal{F}_\chi \cong \mathcal{F}_{\overline{\chi}}(\psi)$. Using α we get an isomorphism

$$\begin{aligned}
&\sigma^*\left(V_\chi^* \otimes (u_0 \times \mathrm{id})^*\mathbb{T}(\mathcal{K}, j_!\mathcal{F}_\chi)\right)^{\widetilde{A}} \\
&\quad \xrightarrow{\sim} \left(V_{\overline{\chi}}(\psi)^* \otimes (u_0 \times \mathrm{id})^*\mathbb{T}\left(\mathcal{K}, j_!\mathcal{F}_{\overline{\chi}}(\psi)\right)\right)^{\widetilde{A}} \\
&\quad \xrightarrow{\sim} \left(V_{\overline{\chi}}^* \otimes (u_0 \times \mathrm{id})^*\mathbb{T}(\mathcal{K}, j_!\mathcal{F}_{\overline{\chi}})\right)^{\widetilde{A}}
\end{aligned}$$

which is independent of the choice of α because we used α twice. Using (4.20), we get a canonical isomorphism

$$\sigma^*\mathcal{E}'_{\chi,k'}(\mathcal{K}) \xrightarrow{\sim} \mathcal{E}'_{\overline{\chi},k'}(\mathcal{K})$$

which is compatible with the convolution on $\mathcal{K} \in \mathrm{Sat}$. Therefore we get an isomorphism of $\widehat{G}(\mathbb{Q}'_\ell)$-local systems $\sigma^*\mathcal{E}'_{\chi,k'} \xrightarrow{\sim} \mathcal{E}'_{\overline{\chi},k'}$. Combining this with Lemma 4.9, we get a canonical isomorphism of $\widehat{G}(\mathbb{Q}'_\ell)$-local systems $\sigma^*\mathcal{E}'_{\chi,k'} \xrightarrow{\sim} \mathcal{E}'_{\chi,k'}$. Canonicity guarantees that this isomorphism is involutive and hence gives a descent datum for each $\mathcal{E}'_{\chi,k'}$ from $\mathbb{P}^1_{k'} - \{0, 1, \infty\}$ to $\mathbb{P}^1_k - \{0, 1, \infty\}$, giving the desired $\widehat{G}(\mathbb{Q}'_\ell)$-local system $\mathcal{E}'_\chi$ over $\mathbb{P}^1_k - \{0, 1, \infty\}$. This finishes the proof of Theorem 4.2(2).

4.4.4 Rationality of the coefficients

Finally we prove the statement (3) in Theorem 4.2. We only need to work in the case $\mathbb{Q}'_\ell \neq \mathbb{Q}_\ell$. Let $\tau \in \mathrm{Gal}(\mathbb{Q}'_\ell/\mathbb{Q}_\ell)$ be the nontrivial element. For any $\mathbb{Q}'_\ell$-linear object $\mathcal{H}$ such as a vector space or complex of sheaves, we use ${}^\tau\mathcal{H}$ to denote $\mathbb{Q}'_\ell \otimes_{\mathbb{Q}'_\ell,\tau} \mathcal{H}$.

First suppose G is of type D_{4n}, E_8 or G_2, then all $\chi \in \widetilde{A}^*_{0,\mathrm{odd}}$ take values in ± 1. The Γ-module ${}^\tau V_\chi$ is again an irreducible $\widetilde{A}(\overline{k})$-module with central character χ, therefore there exists some character ψ of $\mathrm{Gal}(\overline{k}/k)$ and an isomorphism $\alpha : {}^\tau V_\chi \cong V_\chi(\psi)$, well defined up to a scalar in $E^\times$. This α also induces an isomorphism ${}^\tau\mathcal{F}_\chi \xrightarrow{\sim} \mathcal{F}_\chi(\psi)$. Since $\mathcal{K}$ is defined with $\mathbb{Q}_\ell$-coefficients, we have an isomorphism

$$ {}^\tau\big(V_\chi^* \otimes (u_0 \times \mathrm{id})^*\mathbb{T}(\mathcal{K}, j_!\mathcal{F}_\chi)\big)^{\widetilde{A}} \xrightarrow{\sim} \big(V_\chi(\psi)^* \otimes (u_0 \times \mathrm{id})^*\mathbb{T}\big(\mathcal{K}, j_!\mathcal{F}_\chi(\psi)\big)\big)^{\widetilde{A}} \tag{4.27} $$

which is independent of the choice of α because we used α twice. This gives a canonical isomorphism

$$ \varphi_\tau(\mathcal{K}) : {}^\tau\mathcal{E}'_\chi(\mathcal{K}) \cong \mathcal{E}'_\chi(\mathcal{K}). $$

Canonicity guarantees that $\varphi_\tau(\mathcal{K})$ is compatible with the tensor structure as $\mathcal{K}$ varies and that $\varphi_\tau^2 = \mathrm{id}$. Therefore $\mathcal{E}'_\chi$ descends to a $\widehat{G}(\mathbb{Q}_\ell)$-local system $\mathcal{E}_\chi$ on $\mathbb{P}^1 - \{0, 1, \infty\}$.

Now suppose G is of type A_1, D_{4n+2} or E_7. The involution τ changes the central character χ to $\overline{\chi}$. The same argument as in the previous case shows that there is a canonical isomorphism between $\widehat{G}(\mathbb{Q}'_\ell)$-local systems on $\mathbb{P}^1_{k'} - \{0, 1, \infty\}$ $(k' = k(\sqrt{-1}))$ ${}^\tau\mathcal{E}'_{\chi,k'} \xrightarrow{\sim} \mathcal{E}'_{\overline{\chi},k'}$. Combined with Lemma 4.9, we get a canonical isomorphism ${}^\tau\mathcal{E}'_{\chi,k'} \xrightarrow{\sim} \mathcal{E}'_{\chi,k'}$. Canonicity guarantees this is involutive, so it gives a descent datum of $\mathcal{E}'_{\chi,k'}$ to a $\widehat{G}(\mathbb{Q}_\ell)$-local system $\mathcal{E}_{\chi,k'}$ on $\mathbb{P}^1_{k'} - \{0, 1, \infty\}$. We then use the same argument of Sect. 4.4.3 to show that this $\widehat{G}(\mathbb{Q}_\ell)$-local system descends to $\mathbb{P}^1_k - \{0, 1, \infty\}$. This finishes the proof of Theorem 4.2.

5 Local and global monodromy

We recall that G is of type A_1, D_{2n}, E_7, E_8 or G_2. The goal of this section is to show that the geometric monodromy of the eigen local system $\mathcal{E}_\chi$ is Zariski dense in $\widehat{G}$ when $\widehat{G}$ is of type A_1, E_7, E_8 or G_2. From this, we deduce the existence of motives with ℓ-adic motivic Galois group of type $\widehat{G}$ and solve the inverse Galois problem for $\widehat{G}(\mathbb{F}_\ell)$ when $\widehat{G} = G_2$ or E_8.

For most part of this section, we fix an odd character χ of $\widetilde{A}_0(\bar{k})$, and denote $\mathcal{E}_\chi$ simply by $\mathcal{E}$. The results in this section will be insensitive to χ.

To emphasize the dependence on the base field, we use $\mathcal{E}_k$ to denote the local system $\mathcal{E}$ over $\mathbb{P}^1_k - \{0, 1, \infty\}$. The monodromy representation of $\mathcal{E}_k$ is (again we often omit the base point from π_1)

$$\rho_k : \pi_1\big(\mathbb{P}^1_k - \{0, 1, \infty\}\big) \to \widehat{G}(\mathbb{Q}_\ell).$$

To show that the image of ρ_k is "big", we first exhibit nontrivial elements in the image by studying the local monodromy of ρ_k around the punctures.

5.1 Remarks on Gaitsgory's nearby cycles

To study the local monodromy of $\mathcal{E}$, we need a parahoric variant of Gaitsgory's nearby cycle construction [16]. In this subsection, k is algebraically closed.

5.1.1 Hecke operators at ramified points

Let X be a complete smooth connected curve over k and S be a finite set of closed points on X. Let $\{\mathbf{P}_x\}$ be a set of level structures, one for each $x \in S$. For each $x \in S$, we have a Hecke correspondence

$$\begin{array}{ccc} & \mathrm{Hk}_x & \\ \swarrow^{\overleftarrow{h}_x} & & \searrow^{\overrightarrow{h}_x} \\ \mathrm{Bun}_G(\mathbf{P}_x; x \in S) & & \mathrm{Bun}_G(\mathbf{P}_x; x \in S) \end{array} \tag{5.1}$$

which classifies triples $(\mathcal{P}, \mathcal{P}', \iota)$ where $\mathcal{P}, \mathcal{P}' \in \mathrm{Bun}_G(\mathbf{P}_x; x \in S)$ and $\iota : \mathcal{P}|_{X-S} \xrightarrow{\sim} \mathcal{P}'|_{X-S}$ preserving the level structures at $S - \{x\}$.

Let $\mathrm{Fl}_{\mathbf{P}_x} = L_x G/\mathbf{P}_x$ be the affine partial flag variety associated to $\mathbf{P}_x$. Analogous to the convolution product in the Satake category, we may define a monoidal structure on the category $D^b_{\mathbf{P}_x}(\mathrm{Fl}_{\mathbf{P}_x})$ (see [16, Appendix A.4] for how to make sense of this equivariant derived category). Similar to the situation in Sect. 4.1.3, $\overrightarrow{h}_x$ is a locally trivial fibration with fibers isomorphic to $\mathrm{Fl}_{\mathbf{P}_x}$, and for $\mathcal{K} \in D^b_{\mathbf{P}_x}(\mathrm{Fl}_{\mathbf{P}_x})$ one can define a spread-out of $\mathcal{K}$ to $\mathcal{K}_{\mathrm{Hk}_x} \in D^b(\mathrm{Hk}_x)$. For $\mathcal{K} \in D^b_{\mathbf{P}_x}(\mathrm{Fl}_{\mathbf{P}_x})$ and $\mathcal{F} \in D^b(\mathrm{Bun}_G(\mathbf{P}_x; x \in S))$, we define

$$\mathbb{T}_x(\mathcal{K}, \mathcal{F}) := \overrightarrow{h}_{x,!}\big(\overleftarrow{h}^*_x \mathcal{F} \otimes \mathcal{K}_{\mathrm{Hk}_x}\big).$$

This defines a monoidal action of $D^b_{\mathbf{P}_x}(\mathrm{Fl}_{\mathbf{P}_x})$ on $D^b(\mathrm{Bun}_G(\mathbf{P}_x; x \in S))$.

We also need the Hecke modifications at two points, one moving point not in S and the other is $x \in S$. Let $S^x = S - \{x\}$, we have a diagram

$$\begin{array}{ccc} & \mathrm{Hk}_{X-S^x} & \\ \overleftarrow{h} \swarrow & & \searrow \overrightarrow{h} \\ \mathrm{Bun}_G(\mathbf{P}_x; x \in S) & & \mathrm{Bun}_G(\mathbf{P}_x; x \in S) \times (X - S^x) \end{array}$$

The stack Hk_{X-S^x} classifies $(y, \mathcal{P}, \mathcal{P}', \iota)$ where $y \in X - S^x, \mathcal{P}, \mathcal{P}' \in \mathrm{Bun}_G(\mathbf{P}_x; x \in S)$ and $\iota : \mathcal{P}|_{X-S-\{y\}} \xrightarrow{\sim} \mathcal{P}'|_{X-S-\{y\}}$ preserving the level structures at $S - \{x\}$. In particular, Hk_x defined in (5.1) is the fiber of Hk_{X-S^x} over x.

5.1.2 *Parahoric version of Gaitsgory's construction*

We recall the setting of [16]. Fixing a base point $u_0 \in \mathrm{Bun}_G(\mathbf{P}_x; x \in S)(k)$. Let $\mathrm{GR}_{X-S^x} = \overrightarrow{h}^{-1}(\{u_0\} \times (X - S^x))$. The family $\mathrm{GR}_{X-S^x} \to X - S^x$ interpolates $\mathrm{Gr}_y \times \mathrm{Fl}_{\mathbf{P}_x}$ (for $y \notin S$) and $\mathrm{Fl}_{\mathbf{P}_x}$ at x:

$$\begin{array}{ccccc} \mathrm{Fl}_{\mathbf{P}_x} & \longrightarrow & \mathrm{GR}_{X-S^x} & \longleftarrow & \mathrm{GR}_{X-S} \times \mathrm{Fl}_{\mathbf{P}_x} \\ \downarrow & & \downarrow & & \downarrow \\ \{x\} & \longrightarrow & X - S^x & \longleftarrow & X - S \end{array}$$

The nearby cycles functor defines a functor

$$\Psi_{\mathbf{P}_x} : \mathrm{Sat} \times D^b(\mathrm{Fl}_{\mathbf{P}_x}) \to D^b(\mathrm{Fl}_{\mathbf{P}_x})$$

sending $(\mathcal{K}, \mathcal{F})$ to the nearby cycles of $\mathcal{K}_{\mathrm{GR}} \boxtimes \mathcal{F}$, where $\mathcal{K}_{\mathrm{GR}} \in D^b(\mathrm{GR}_{X-S})$ is the spread-out of $\mathcal{K} \in \mathrm{Sat}$ as in Sect. 4.2.1. Setting $\mathcal{F}$ to be the skyscraper sheaf $\delta_{\mathbf{P}_x}$ at the base point of $\mathrm{Fl}_{\mathbf{P}_x}$, we get a t-exact functor (by the exactness of nearby cycles functor)

$$\begin{aligned} Z_{\mathbf{P}_x} : \mathrm{Sat} &\to \mathrm{Perv}(\mathrm{Fl}_{\mathbf{P}_x}) \\ \mathcal{K} &\mapsto \Psi_{\mathbf{P}_x}(\mathcal{K}, \delta_{\mathbf{P}_x}). \end{aligned}$$

When $\mathbf{P}_x = \mathbf{I}_x$, the $\mathrm{Fl}_{\mathbf{P}_x}$ becomes the affine flag variety Fl_x, and we recover Gaitsgory's original nearby cycles functor

$$Z_x := Z_{\mathbf{I}_x} : \mathrm{Sat} \to \mathrm{Perv}(\mathrm{Fl}_x)$$

In [16, Theorem 1], Gaitsgory proves that $Z_x(\mathcal{K})$ is left-$\mathbf{I}_x$-equivariant, convolution exact and central: for any $\mathcal{F} \in \mathrm{Perv}_{\mathbf{I}_x}(\mathrm{Fl}_x)$, $Z_x(\mathcal{K}) * \mathcal{F}$ is perverse and there is a canonical isomorphism $Z_x(\mathcal{K}) * \mathcal{F} \cong \mathcal{F} * Z_x(\mathcal{K})$. Here $*$ denotes the convolution product on $D^b_{\mathbf{I}_x}(\mathrm{Fl}_x)$.

Now we relate $Z_{\mathbf{P}_x}$ to Z_x. Define GR'_{X-S^x} in a similar way as GR_{X-S^x}, except that we replace $\mathbf{P}_x$ by $\mathbf{I}_x$. We have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Fl}_x & \hookrightarrow & \mathrm{GR}'_{X-S^x} & \hookleftarrow & \mathrm{GR}_{X-S} \times \mathrm{Fl}_x \\
\downarrow{\scriptstyle p_x} & & \downarrow{\scriptstyle p_{X-S^x}} & & \downarrow{\scriptstyle \mathrm{id}\times p_x} \\
\mathrm{Fl}_{\mathbf{P}_x} & \hookrightarrow & \mathrm{GR}_{X-S^x} & \hookleftarrow & \mathrm{GR}_{X-S} \times \mathrm{Fl}_{\mathbf{P}_x} \\
\downarrow & & \downarrow & & \downarrow \\
\{x\} & \hookrightarrow & X - S^x & \hookleftarrow & X - S
\end{array}$$

where all squares are Cartesian and p_x and p_{X-S^x} are proper. Since nearby cycles commute with proper base change, we have an $\mathrm{Gal}(F_x^s/F_x)$-equivariant isomorphism

$$Z_{\mathbf{P}_x}(\mathcal{K}) = \Psi(\mathcal{K} \boxtimes \delta_{\mathbf{P}_x}) = \Psi(\mathcal{K} \boxtimes p_{x,*}\delta_{\mathbf{I}_x}) \cong p_{x,*}\Psi(\mathcal{K} \boxtimes \delta_{\mathbf{I}_x}) = p_{x,*}Z_x(\mathcal{K}). \tag{5.2}$$

Let $C_{\mathbf{P}_x} \in D^b_{\mathbf{I}_x}(\mathrm{Fl}_x)$ be the constant sheaf supported on $\mathbf{P}_x/\mathbf{I}_x$, which is perverse up to a shift. Therefore we have a canonical isomorphism

$$p_x^* Z_{\mathbf{P}_x}(\mathcal{K}) = p_x^* p_{x,*} Z_x(\mathcal{K}) = Z_x(\mathcal{K}) * C_{\mathbf{P}_x} \cong C_{\mathbf{P}_x} * Z_x(\mathcal{K}) \tag{5.3}$$

The last equality above uses the centrality of $Z_x(\mathcal{K})$. Any object of the form $C_{\mathbf{P}_x} * (-)$ is equivariant under the left action of $\mathbf{P}_x$ on Fl_x because $C_{\mathbf{P}_x}$ is, therefore $p_x^* Z_{\mathbf{P}_x}(\mathcal{K})$ is also left $\mathbf{P}_x$-equivariant, hence descends to an object in $\mathrm{Perv}_{\mathbf{P}_x}(\mathrm{Fl}_{\mathbf{P}_x})$.

Let $\mathcal{F} \in D^b(\mathrm{Bun}_G(\mathbf{P}_x; x \in S))$ be a Hecke eigensheaf with eigen local system $\mathcal{E} : \mathrm{Sat} \to \mathrm{Loc}(X - S)$.

Lemma 5.1 *Let $x \in S$ and let $I_x = \mathrm{Gal}(F_x^s/F_x)$ be the inertia group at x (recall the residue field k is algebraically closed). For each $\mathcal{K} \in \mathrm{Sat}$, there is a canonical isomorphism of I_x-modules*

$$\mathcal{F}_{u_0} \otimes \mathcal{E}(\mathcal{K})|_{\mathrm{Spec} F_x^s} \cong \left(\mathbb{T}_x\left(Z_{\mathbf{P}_x}(\mathcal{K}), \mathcal{F}\right)\right)_{u_0}$$

where on the right side I_x acts on the nearby cycles $Z_{\mathbf{P}_x}(\mathcal{K})$.

Proof The argument is the same as in [21, Sect. 4.3]. □

Corollary 5.2 *For any $\mathcal{K} \in \mathrm{Sat}$, $\mathcal{E}(\mathcal{K}) \in \mathrm{Loc}(X - S)$ is tamely ramified and the monodromy at every $x \in S$ is unipotent.*

Proof Pick a point $u_0 \in \mathrm{Bun}_G(\mathbf{P}_x; x \in S)$ such that $\mathcal{F}_{u_0} \neq 0$. By Gaitsgory's result [16, Proposition 7], I_x acts on $Z_x(\mathcal{K})$ tamely and unipotently, hence the same is true on $Z_{\mathbf{P}_x}(\mathcal{K})$ by (5.2), and on $\mathcal{E}(\mathcal{K})|_{\mathrm{Spec} F_x^s}$ by Lemma 5.1. □

In our case, $S = \{0, 1, \infty\}$ and we will consider the moduli stack Bun instead of $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{I}_1, \mathbf{P}_\infty)$. The category $D^b(\mathrm{Bun})_{\mathrm{odd}}$ is preserved by the Hecke operators at 1 or ∞ because the oddness condition only involves the level structure at 0. Corollary 5.2 shows that $\mathcal{E}$ is tame at 1 and ∞.

5.2 Local monodromy

In this subsection we give descriptions of the local geometric monodromy of ρ_k around the punctures. We assume k is algebraically closed. Recall that I_x denotes the inertia group at a closed point x of $\mathbb{P}^1$. Let I_x^{tame} be the tame quotient of I_x.

Proposition 5.3 *Under the homomorphism ρ_k, a topological generator of I_1^{tame} gets mapped to a regular unipotent element in $\widehat{G}(\mathbb{Q}_\ell)$.*

Proof We would like to use the argument of [21, Sect. 4.3]. The only thing we need to show is that, for each irreducible object $\mathrm{IC}_{\widetilde{w}} \in D^b_{\mathbf{I}_1}(\mathrm{Fl}_1)$ (indexed by an element $\widetilde{w}$ in the affine Weyl group $\widetilde{W}$), where $\widetilde{w} \neq 1$, we have $\mathbb{T}_1(\mathrm{IC}_{\widetilde{w}}, \mathcal{F}) = 0$ for any object $\mathcal{F} \in D^b(\mathrm{Bun})_{\mathrm{odd}}$. Since $\widetilde{w} \neq 1$, there exists a simple reflection s_i such that $\widetilde{w} = \widetilde{w}' s_i$ and $\ell(\widetilde{w}) = \ell(\widetilde{w}') + 1$. Let $\mathbf{P}_{1,i}$ be the parahoric subgroup of $L_1 G$ generated by $\mathbf{I}_1$ and the root subgroup of $-\alpha_i$ (α_i is the simple root corresponding to s_i). Then $\mathrm{IC}_{\widetilde{w}}$ is the pullback of a shifted perverse sheaf on $\mathrm{Fl}_{\mathbf{P}_{1,i}}$. This implies that $\mathbb{T}_1(\mathrm{IC}_{\widetilde{w}}, \mathcal{F}) \in D^b(\mathrm{Bun})_{\mathrm{odd}}$ is the pullback of an odd complex on $\mathrm{Bun}(\widetilde{\mathbf{P}}_0, \mathbf{P}_{1,i}, \mathbf{P}_\infty)$. We now claim that the category $D^b(\mathrm{Bun}(\widetilde{\mathbf{P}}_0, \mathbf{P}_{1,i}, \mathbf{P}_\infty))_{\mathrm{odd}}$ is zero, which then implies $\mathbb{T}_1(\mathrm{IC}_{\widetilde{w}}, \mathcal{F}) = 0$ and completes the proof.

Let $\mathcal{H} \in D^b(\mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{P}_{1,i}, \mathbf{P}_\infty))_{\mathrm{odd}}$ be a nonzero object. We view $\mathcal{H}$ as a $\widetilde{K}_0$-equivariant complex on $\mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_{1,i}, \mathbf{P}_\infty)$. Let $v : \mathrm{Spec}\, k \to \mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_{1,i}, \mathbf{P}_\infty)$ be a point where the stalk of $\mathcal{H}$ is nonzero. Let $q_i : \mathrm{Bun}^+ \to \mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_{1,i}, \mathbf{P}_\infty)$ be the projection, whose fibers are isomorphic to $\mathbf{P}_{1,i}/\mathbf{I}_1 \cong \mathbb{P}^1$. Then $q_i^{-1}(v) \subset U$ because $\mathcal{H}$ has to vanish outside

$U \subset \mathrm{Bun}^+$ be Theorem 3.2. We have the following Cartesian diagram

$$\begin{array}{ccccccc} \mathbb{P}^1 & \longrightarrow & [\mathbb{P}^1/\mathrm{Aut}(v)] & \overset{i'_v}{\hookrightarrow} & U & \hookrightarrow & \mathrm{Bun}^+ \\ \downarrow & & \downarrow & & & & \downarrow q_i \\ \{v\} & \longrightarrow & [\{v\}/\mathrm{Aut}(v)] & & \overset{i_v}{\hookrightarrow} & & \mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_{1,i}, \mathbf{P}_\infty) \end{array}$$

Since i_v is representable, so is its base change i'_v, which implies that the action of $\mathrm{Aut}(v)$ on $\mathbb{P}^1$ is free. On the other hand, the morphism $\mathbb{P}^1 \to [\mathbb{P}^1/\mathrm{Aut}(v)] \hookrightarrow U$ has to be constant because $\mathbb{P}^1$ is proper while U is affine. Therefore $\mathrm{Aut}(v)$ acts on $\mathbb{P}^1$ both freely and transitively. This implies $\mathbb{P}^1$ is a torsor under the algebraic group $\mathrm{Aut}(v)$, which is not possible. Hence $\mathcal{H}$ has to be zero everywhere. □

5.2.1 An involution in $\widehat{G}$

By the construction of the canonical double cover in Sect. 2.5.2, we have $\mathbb{X}_*(T)/\mathbb{X}_*(\widetilde{T}) \cong \mathbb{Z}/2\mathbb{Z}$, where $\widetilde{T}$ is the preimage of T in $\widetilde{K}_0$. This defines an order two character

$$\mathbb{X}_*(T) \twoheadrightarrow \mathbb{X}_*(T)/\mathbb{X}_*(\widetilde{T}) \cong \mathbb{Z}/2\mathbb{Z} \xrightarrow{\sim} \{\pm 1\}$$

and hence an element $\kappa \in \widehat{T}[2]$. One can check case by case that when G is of type A_1, D_{2n}, E_7, E_8 or G_2, κ is always a split Cartan involution in $\widehat{G}$ (cf. Proposition 2.2).

Proposition 5.4 *The local system $\mathcal{E}_k$ is tame at* 0. *Under the homomorphism ρ_k, a topological generator of I_0^{tame} gets mapped to an element with Jordan decomposition $g_s g_u \in \widehat{G}(\mathbb{Q}_\ell)$, where the semisimple part g_s is conjugate to the split Cartan involution $\kappa \in \widehat{T}[2]$.*

Proof Pulling back the double covering $\widetilde{\mathbf{P}}_0 \to \mathbf{P}_0$ to $\mathbf{I}_0 \subset \mathbf{P}_0$, we get a double covering $\widetilde{\mathbf{I}}_0 \to \mathbf{I}_0$. The reductive quotient of $\widetilde{\mathbf{I}}_0$ is $\widetilde{T}$. We will consider the moduli stack $\mathrm{Bun}_G(\widetilde{\mathbf{I}}_0, \mathbf{I}_1, \mathbf{P}_\infty)$, defined similarly as Bun. As we discussed in Sect. 3.3.1, we can define the category such as $D^b(\mathrm{Bun}_G(\widetilde{\mathbf{I}}_0, \mathbf{I}_1, \mathbf{P}_\infty))_{\mathrm{odd}}$ and $D^b(L_0G/\widetilde{\mathbf{I}}_0)_{\mathrm{odd}}$ etc. The inclusion $\widetilde{\mathbf{I}}_0 \hookrightarrow \widetilde{\mathbf{P}}_0$ gives a projection

$$p : \mathrm{Bun}_G(\widetilde{\mathbf{I}}_0, \mathbf{I}_1, \mathbf{P}_\infty) \to \mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{I}_1, \mathbf{P}_\infty) = \mathrm{Bun}.$$

Let $j_!\mathcal{F}$ be the Hecke eigensheaf with eigen local system $\mathcal{E}$. The complex $p^* j_!\mathcal{F}$ is clearly also a Hecke eigensheaf on $\mathrm{Bun}_G(\widetilde{\mathbf{I}}_0, \mathbf{I}_1, \mathbf{P}_\infty)$ for the Hecke operators on $\mathbb{P}^1 - \{0, 1, \infty\}$ with the same eigen local system $\mathcal{E}$. Therefore it suffices to prove a stronger statement: for any nonzero Hecke eigensheaf $\mathcal{F}$ on $\mathrm{Bun}_G(\widetilde{\mathbf{I}}_0, \mathbf{I}_1, \mathbf{P}_\infty)$ with eigen local system $\mathcal{E}$, the $\widehat{G}$-local system $\mathcal{E}$ is tamely ramified and the semisimple part of the local monodromy is conjugate to κ.

We need another variant of Gaitsgory's nearby cycles construction allowing sheaves which are monodromic with respect to the torus action. This variant is sketched in [6, Sects. 2.1 and 2.2]. Let us be more specific about the version we need. There is a family $\mathrm{GR}'_{\mathbb{P}^1-\{1,\infty\}}$ interpolating $\mathrm{Gr}_x \times L_0G/\widetilde{\mathbf{I}}_0$ and $L_0G/\widetilde{\mathbf{I}}_0$

$$\begin{array}{ccccc} L_0G/\widetilde{\mathbf{I}}_0 & \hookrightarrow & \mathrm{GR}'_{\mathbb{P}^1-\{1,\infty\}} & \hookleftarrow & \mathrm{GR}_{\mathbb{P}^1-\{0,1,\infty\}} \times L_0G/\widetilde{\mathbf{I}}_0 \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & \mathbb{P}^1-\{1,\infty\} & \hookleftarrow & \mathbb{P}^1-\{0,1,\infty\} \end{array}$$

Let $\delta_{\mathrm{odd}} \in \mathrm{Perv}(L_0G/\widetilde{\mathbf{I}}_0)$ be the rank one local system supported on $\mathbf{I}_0/\widetilde{\mathbf{I}}_0 \cong \mathbb{B}\mu_2^{\mathrm{ker}}$ corresponding to the sign representation of μ_2^{ker}. Using the nearby cycles of the above family, we define

$$\begin{aligned} Z'_0 : \mathrm{Sat} &\to \mathrm{Perv}_{\widetilde{\mathbf{I}}_0}(L_0G/\widetilde{\mathbf{I}}_0)_{\mathrm{odd}} \\ \mathcal{K} &\mapsto \Psi(\mathcal{K}_{\mathrm{GR}} \boxtimes \delta_{\mathrm{odd}}). \end{aligned}$$

Here the subscript "odd" in $\mathrm{Perv}_{\widetilde{\mathbf{I}}_0}(L_0G/\widetilde{\mathbf{I}}_0)_{\mathrm{odd}}$ means taking those objects on which both actions of μ_2^{ker} (from left and right) are through the sign representation. Using the same construction as in Sect. 5.1, the derived category $D^b_{\widetilde{\mathbf{I}}_0}(L_0G/\widetilde{\mathbf{I}}_0)_{\mathrm{odd}}$ still acts on $D^b(\mathrm{Bun}_G(\widetilde{\mathbf{I}}_0, \mathbf{I}_1, \mathbf{P}_\infty))_{\mathrm{odd}}$. We denote this action by

$$\begin{aligned} \mathbb{T}'_0 : D^b_{\widetilde{\mathbf{I}}_0}(L_0G/\widetilde{\mathbf{I}}_0)_{\mathrm{odd}} \times D^b\big(\mathrm{Bun}_G(\widetilde{\mathbf{I}}_0, \mathbf{I}_1, \mathbf{P}_\infty)\big)_{\mathrm{odd}} \\ \to D^b\big(\mathrm{Bun}_G(\widetilde{\mathbf{I}}_0, \mathbf{I}_1, \mathbf{P}_\infty)\big)_{\mathrm{odd}}. \end{aligned}$$

We also have a variant of Lemma 5.1: there is an I_0-equivariant isomorphism

$$\mathcal{F}_{u_0} \otimes_{\mathbb{Q}_\ell} \mathcal{E}(\mathcal{K})|_{\mathrm{Spec} F_0^s} \cong \big(\mathbb{T}'_0\big(Z'_0(\mathcal{K}), \mathcal{F}\big)\big)_{u_0} \tag{5.4}$$

for any $\mathcal{K} \in \mathrm{Sat}$. By [3, Sect. 5.2, Claim 2], the monodromy action on $Z'_0(\mathcal{K})$ factors through the tame quotient (the Claim in *loc.cit.* requires the family to live over $\mathbb{A}^1$ and carry a $\mathbb{G}_m$-action compatible with the rotation action on $\mathbb{A}^1$. In our situation, we can extend $\mathrm{GR}'_{\mathbb{P}^1-\{1,\infty\}}$ to $\mathrm{GR}'_{\mathbb{A}^1}$ by ignoring the level structure at 1, and the rotation action on $\mathbb{P}^1$ induces the desired action on $\mathrm{GR}'_{\mathbb{A}^1}$). Therefore the local system $\mathcal{E}(\mathcal{K})$ is tame at 0 for any $\mathcal{K}$.

Next we compute the semisimple part of the tame monodromy. The following argument is borrowed from [6, Sects. 2.4 and 2.5], to which we refer more details. There is a filtration F_λ on $Z'_0(\mathcal{K})$indexed by $\lambda \in \mathbb{X}_*(T)$ (partially ordered using the positive coroot lattice) with associated graded $\mathrm{Gr}^F_\lambda Z'_0(\mathcal{K})$ isomorphic to a direct sum of the Wakimoto sheaf J_λ. Moreover, the monodromy operator $m(\mathcal{K})$ on $Z'_0(\mathcal{K})$ preserves this filtration, and acts on $\mathrm{Gr}^F_\lambda Z'_0(\mathcal{K})$ by $\kappa(\lambda)$ (recall any element in $\widehat{T}(\mathbb{Q}_\ell)$ is a homomorphism $\mathbb{X}_*(T) \to \mathbb{Q}_\ell^\times$). Therefore, we can write $m(\mathcal{K})$ into Jordan normal form $m(\mathcal{K})_s m(\mathcal{K})_u = m(\mathcal{K})_u m(\mathcal{K})_s$. Here both $m(\mathcal{K})_u$ and $m(\mathcal{K})_s$ preserve the filtration F_λ, $m(\mathcal{K})_s$ acts on $\mathrm{Gr}^F_\lambda Z'_0(\mathcal{K})$ by the scalar $\kappa(\lambda)$ and $m(\mathcal{K})_u$ acts on $\mathrm{Gr}^F_\lambda Z'_0(\mathcal{K})$ as the identity. On the other hand, let $\zeta_0 \in I_0^{\mathrm{tame}}$ be a topological generator, and let $\rho_k(\zeta_0) = g_s g_u = g_u g_s$ be the Jordan decomposition of $\rho_k(\zeta_0)$ in $\widehat{G}(\mathbb{Q}_\ell)$. Since the isomorphism (5.4) intertwines the action of $\rho_k(\zeta_0)$ on $\mathcal{E}(\mathcal{K})|_{\mathrm{Spec} F_0^s}$ and the action of $m(\mathcal{K})$ on $Z'_0(\mathcal{K})$, it must also intertwine the g_s-action and the $m(\mathcal{K})_s$-action by the uniqueness of Jordan decomposition.

Let $\mathcal{A}$ be the full subcategory of $\mathrm{Perv}(\widetilde{\mathbf{I}}_0\backslash L_0G/\widetilde{\mathbf{I}}_0)_{\mathrm{odd}}$ consisting of those objects admitting filtrations with graded pieces isomorphic to Wakimoto sheaves. Let $\mathrm{Gr}\mathcal{A}$ be the full subcategory of $\mathcal{A}$ consisting of direct sums of Wakimoto sheaves. Then $\mathrm{Gr}\mathcal{A} \cong \mathrm{Rep}(\widehat{T})$ as tensor categories, and the functor

$$\mathrm{Rep}(\widehat{G}, \mathbb{Q}_\ell) \cong \mathrm{Sat} \xrightarrow{Z'_0} \mathcal{A} \xrightarrow{\oplus \mathrm{Gr}^F_\lambda} \mathrm{Gr}\mathcal{A} \cong \mathrm{Rep}(\widehat{T}, \mathbb{Q}_\ell)$$

is isomorphic to the restriction functor $\mathrm{Res}^{\widehat{G}}_{\widehat{T}} : \mathrm{Rep}(\widehat{G}) \to \mathrm{Rep}(\widehat{T})$ induced by the inclusion $\widehat{T} \hookrightarrow \widehat{G}$. The semisimple part of the monodromy operator $\{m(\mathcal{K})_s\}_{\mathcal{K}\in\mathrm{Sat}}$ acts as an automorphism of $\mathrm{Res}^{\widehat{G}}_{\widehat{T}}$, hence determines an element $\tau \in \widehat{T}(\mathbb{Q}_\ell) = \mathrm{Aut}^{\otimes}(\mathrm{Res}^{\widehat{G}}_{\widehat{T}})$. The above discussion has identified the action of τ on Wakimoto sheaves (which correspond to irreducible algebraic representations of $\widehat{T}$ under the equivalence $\mathrm{Gr}\mathcal{A} \cong \mathrm{Rep}(\widehat{T})$), hence $\tau = \kappa$. To summarize, $\{m(\mathcal{K})_s\}_{\mathcal{K}\in\mathrm{Sat}}$ gives a tensor automorphism of the fiber functor $\omega : \mathrm{Sat} \cong \mathrm{Rep}(\widehat{G}) \xrightarrow{\mathrm{Res}} \mathrm{Rep}(\widehat{T}) \to \mathrm{Vec}_{\mathbb{Q}_\ell}$ which is the same as κ.

On the other hand, g_s gives an automorphism of another fiber functor $\omega' : \mathrm{Sat} \to \mathrm{Vec}_{\mathbb{Q}_\ell}$ given by taking the fiber of $\mathcal{E}$ at $\mathrm{Spec} F_0^s$. The isomorphism (5.4) gives an isomorphism of the two fiber functors ω and ω' under which g_s corresponds to $m(\mathcal{K})_s$, therefore g_s is conjugate to $\tau = \kappa$. □

Proposition 5.5 *Suppose G is not of type A_1. Under the homomorphism ρ_k, a topological generator of I_∞^{tame} gets mapped to a unipotent element in $\widehat{G}(\mathbb{Q}_\ell)$ which is neither regular nor trivial.*

Proof Let $\zeta_\infty \in I_\infty^{\mathrm{tame}}$ be a topological generator. By Corollary 5.2, $\rho_k(\zeta_\infty)$ is unipotent. Let $N = \log(\rho_k(\zeta_\infty))$, which is a nilpotent element in $\widehat{\mathfrak{g}}$.

We first argue that $N \neq 0$. Suppose $N = 0$, then the local system $\mathcal{E}$ is unramified on $\mathbb{P}^1 - \{0, 1\}$ and tame at 0 (by Proposition 5.4) and at 1 (by Corollary 5.2). The tame fundamental group $\pi_1^{\mathrm{tame}}(\mathbb{P}^1 - \{0, 1\})$ is topologically generated by one element, which is both the generator of I_0^{tame} and I_1^{tame}. Since the local monodromy at 1 is unipotent element while the local monodromy at 0 is not according to Proposition 5.4, this is impossible. Therefore $N \neq 0$.

The rest of the proof is devoted to showing that N is not regular. Suppose it is, then it acts on $\widehat{\mathfrak{g}}$ with a Jordan block of size $2h - 1$: the lowest root space is sent isomorphically to the highest root space by $\mathrm{Ad}(N)^{2h-2}$ (h is the Coxeter number of $\widehat{G}$). Let γ be the coroot of G corresponding to the highest root of $\widehat{G}$. Then $\mathrm{IC}_\gamma \in \mathrm{Sat}$ corresponds to the adjoint representation of $\widehat{G}$ under the Satake equivalence (4.1). In Sect. 5.1 we recalled the parahoric variant of Gaitsgory's nearby cycles functor $Z_{\mathbf{P}_\infty}$. By Lemma 5.1, we have an I_∞-equivariant isomorphism

$$V_\chi \otimes \mathcal{E}(\mathrm{IC}_\gamma)|_{\mathrm{Spec} F_\infty^s} \cong \left(\mathbb{T}_\infty\left(Z_{\mathbf{P}_\infty}(\mathrm{IC}_\gamma), j_!\mathcal{F}_\chi\right)\right)_{u_0}, \tag{5.5}$$

where I_∞ acts trivially on V_χ. In particular, the action of N on the left side is reflected from the logarithm of the monodromy action on $Z_{\mathbf{P}_\infty}(\mathrm{IC}_\gamma)$. By (5.3), we have an isomorphism which respects the monodromy operators

$$p_\infty^* Z_{\mathbf{P}_\infty}(\mathrm{IC}_\gamma) \cong Z_\infty(\mathrm{IC}_\gamma) * C_{\mathbf{P}_\infty}.$$

Let M be the logarithm of the monodromy operator on $Z_\infty(\mathrm{IC}_\gamma)$, and M' be the induced endomorphism of $Z_\infty(\mathrm{IC}_\gamma) * C_{\mathbf{P}_\infty}$, which intertwines with the logarithm monodromy on $Z_{\mathbf{P}_x}(\mathrm{IC}_\gamma)$, hence with N via (5.5). Since $\mathrm{Ad}(N)^{2h-2} \neq 0$, we must have $M'^{2h-2} \neq 0$ on $Z_\infty(\mathrm{IC}_\gamma) * C_{\mathbf{P}_\infty}$, and $M^{2h-2} \neq 0$ on $Z_\infty(\mathrm{IC}_\gamma)$.

The following argument uses the theory of weights. For $\mathrm{char}(k) > 0$, every object in concern comes via base change from a finite base field $\mathbb{F}_q$, hence we may assume $k = \overline{\mathbb{F}}_q$ and use the weight theory of Weil sheaves as developed by Deligne in [9]. For $\mathrm{char}(k) = 0$, every object in concern comes via base change from a number field L, and we may still talk about weights by choosing a place v of L at which all objects have good reduction.

The object $Z_\infty(\mathrm{IC}_\gamma)$ has a Jordan-Holder series whose associated graded are Tate twists of irreducible objects $\mathrm{IC}_{\widetilde{w}}$ for $\widetilde{w} \in \widetilde{W}$. In [18, Theorem 1.1], Görtz and Haines give an estimate of the weights of the twists of $\mathrm{IC}_{\widetilde{w}}$ appearing in $Z_\infty(\mathrm{IC}_\gamma)$: $\mathrm{IC}_{\widetilde{w}}$ appears with weight in the range $[\ell(\widetilde{w}) - \ell(\gamma), \ell(\gamma) - \ell(\widetilde{w})]$ (note the different normalization we take here and in [18]: we normalize $\mathrm{IC}_{\widetilde{w}}$ and IC_γ to have weight zero while in [18] they have weight $\ell(\widetilde{w})$ and

$\ell(\gamma)$ respectively). Here $\ell(\gamma)$ is the length of the translation element γ in $\widetilde{W}$, and in fact $\ell(\gamma) = \langle 2\rho, \gamma\rangle = 2h - 2$. Since the logarithmic monodromy operator decreases weight by 2, the subquotients isomorphic to (a twist of) $\mathrm{IC}_{\widetilde{w}}$ are killed after applying M for $\ell(\gamma) - \ell(\widetilde{w}) = 2h - 2 - \ell(\widetilde{w})$ times (if this is negative, this means $\mathrm{IC}_{\widetilde{w}}$ does not appear in $Z_\infty(\mathrm{IC}_\gamma)$ at all). Therefore, only the skyscraper sheaf $\delta = \mathrm{IC}_e$ survives after applying M for $2h - 2$ times, i.e., M^{2h-2} factors as

$$M^{2h-2} : Z_\infty(\mathrm{IC}_\gamma) \twoheadrightarrow \delta(1-h) \hookrightarrow Z_\infty(\mathrm{IC}_\gamma)(2-2h)$$

Here the first arrow is the passage to $\mathrm{Gr}^W_{2h-2} Z_\infty(\mathrm{IC}_\gamma)$, the maximal weight quotient, and the second arrow is induced from $\delta(h-1) \cong \mathrm{Gr}^W_{2-2h} Z_\infty(\mathrm{IC}_\gamma) \hookrightarrow Z_\infty(\mathrm{IC}_\gamma)$, the inclusion of the lowest weight piece. Consequently, M'^{2h-2} factors as

$$M'^{2h-2} : Z_\infty(\mathrm{IC}_\gamma) * C_{\mathbf{P}_\infty} \to C_{\mathbf{P}_\infty}(1-h) \to Z_\infty(\mathrm{IC}_\gamma) * C_{\mathbf{P}_\infty}(2-2h). \quad (5.6)$$

Let w_K be the longest element of $W_K < \widetilde{W}$. Then $C_{\mathbf{P}_\infty}$ is equal to IC_{w_K} up to a shift and a Tate twist. Therefore we may replace $C_{\mathbf{P}_\infty}$ by IC_{w_K} in (5.6), and get a sequence of maps in the category $\mathcal{P} := \mathrm{Perv}_{\mathbf{I}_\infty}(\mathrm{Fl}_\infty)$.

To proceed, we need Lusztig's theory of two-sided cells and Bezrukavnikov's geometric result on cells. Lusztig defined a pre-order $\leq_{LR}$ on the affine Weyl group $\widetilde{W}$ by declaring that $w_1 \leq_{LR} w_2$ if IC_{w_1} appears as a simple constituent in the convolution $\mathrm{IC}_{w_2} * \mathcal{K}$ for some $\mathcal{K} \in \mathcal{P}$. This pre-order can be completed into a partial order, which then partitions $\widetilde{W}$ into finitely many equivalence classes called *two-sided cells* (*cells* for short). For details see Lusztig's paper [25].

Let $\underline{c}$ be the cell containing w_K. Let $\mathcal{P}_{\leq \underline{c}}$ (resp. $\mathcal{P}_{<\underline{c}}$) be the full subcategory of $\mathcal{P}$ generated (under extensions) by $\{\mathrm{IC}_{\widetilde{w}}\}$ where $\widetilde{w}$ belongs to some two-sided cell $\leq \underline{c}$ (resp. $< \underline{c}$). Let $\mathcal{P}_{\underline{c}} = \mathcal{P}_{\leq\underline{c}}/\mathcal{P}_{<\underline{c}}$ be the Serre quotient. Since $\mathrm{IC}_{w_K} \in \mathcal{P}_{\leq\underline{c}}$, we have $Z_\infty(\mathcal{K}) * \mathrm{IC}_{w_K} \subset \mathcal{P}_{\leq\underline{c}}$. The element $w_K \in \underline{c}$ is a *distinguished* (*or Duflo*) *involution* (see [26, Sect. 1.3] for definition, which uses the a-function on two sided-cells [25, Sect. 2.1]). In our case, we may apply [25, Proposition 2.4] to show $a(w_K) = \ell(w_K)$, which forces w_K to be a Duflo involution by definition. In [5, Sect. 4.3], Bezrukavnikov introduces a full subcategory $\mathcal{A}_{w_K} \subset \mathcal{P}_{\underline{c}}$ generated by the image of all subquotients of $Z_\infty(\mathcal{K}) * \mathrm{IC}_{w_K}$ in the category $\mathcal{P}_{\underline{c}}$, as $\mathcal{K}$ runs over Sat. It is proved there that $\mathcal{A}_{w_K}$ has a natural structure of a monoidal abelian category with unit object the image of IC_{w_K}, such that the functor

$$\begin{aligned}\mathrm{Res}_{w_K} : \mathrm{Sat} &\to \mathcal{A}_{w_K}\\ \mathcal{K} &\mapsto \left[Z_\infty(\mathcal{K}) * \mathrm{IC}_{w_K}\right]\end{aligned}$$

is monoidal. Here we use $[-]$ to denote the passage from $\mathcal{P}_{\leq \underline{c}}$ to $\mathcal{P}_{\underline{c}}$. By [5, Theorem 1], there is a subgroup $\widehat{H} \subset \widehat{G}$ and a unipotent element $u_{w_K} \in \widehat{G}(\overline{\mathbb{Q}}_\ell)$ commuting with $\widehat{H}$, and an equivalence of tensor categories $\Phi_{w_K} : \mathcal{A}_{w_K} \cong \mathrm{Rep}(\widehat{H})$ such that the following diagram is commutative (by a natural isomorphism)

$$\begin{array}{ccc} \mathrm{Sat} & \xrightarrow{\mathrm{Res}_{w_K}} & \mathcal{A}_{w_K} \\ \wr\downarrow \mathrm{Satake} & & \wr\downarrow \Phi_{w_K} \\ \mathrm{Rep}(\widehat{G}) & \xrightarrow{\mathrm{Res}^{\widehat{G}}_{\widehat{H}}} & \mathrm{Rep}(\widehat{H}) \end{array}$$

where $\mathrm{Res}^{\widehat{G}}_{\widehat{H}}$ is the restriction functor. Moreover, the monodromy operator on $Z_\infty(-)$ induces a natural automorphism of Res_{w_K}, which corresponds to the natural automorphism of u_{w_K} on $\mathrm{Res}^{\widehat{G}}_{\widehat{H}}$. There is a bijection defined by Lusztig [27, Theorem 4.8(b)]

$$\{\text{two sided cells of } \widetilde{W}\} \xrightarrow{\sim} \{\text{unipotent classes in } \widehat{G}\}$$

By [5, Theorem 2], the unipotent element u_{w_K} is in the unipotent class corresponding to $\underline{c}$ under Lusztig's bijection. Since G is not of type A_1, $\ell(w_K) > 0$, hence u_{w_K} is not the regular unipotent class (this is because $a(w_K) = \ell(w_K) > 0$, while the two-sided cell corresponding to the regular class has a-value equal to 0). Therefore $\log(u_{w_K})^{2h-2}$ is zero on $\widehat{\mathfrak{g}}$, because only regular nilpotent elements have a Jordan block of size $2h-1$ under the adjoint representation. Because $\log(u_{w_K})$ intertwines with the logarithmic monodromy on $\mathrm{Res}_{w_K}(-)$, the logarithmic monodromy M' on $[Z_\infty(\mathrm{IC}_\gamma) * \mathrm{IC}_{w_K}] = \mathrm{Res}_{w_K}(\mathrm{IC}_\gamma)$ must satisfy $M'^{2h-2} = 0$, as a morphism in the Serre quotient $\mathcal{P}_{\underline{c}}$. Hence M'^{2h-2} as a morphism in the category $\mathcal{P}_{\leq \underline{c}}$ factors as

$$M'^{2h-2} : Z_\infty(\mathrm{IC}_\gamma) * \mathrm{IC}_{w_K} \twoheadrightarrow Q \hookrightarrow Z_\infty(\mathrm{IC}_\gamma) * \mathrm{IC}_{w_K}(2-2h)$$

for some $Q \in \mathcal{P}_{<\underline{c}}$. However, we have another factorization (5.6), hence there must be an arrow $q : Q \to C_{\mathbf{P}_\infty}(1-h)$ through which the inclusion $Q \hookrightarrow Z_\infty(\mathrm{IC}_\gamma) * C_{\mathbf{P}_\infty}(2-2h)$ factors. However, $C_{\mathbf{P}_\infty}$ is an irreducible object of $\mathcal{P}_{\leq \underline{c}}$ which does not lie in $\mathcal{P}_{<\underline{c}}$ while $Q \in \mathcal{P}_{<\underline{c}}$, such an arrow q must be zero. This means $M'^{2h-2} = 0$ as a morphism in $\mathcal{P}_{\leq \underline{c}}$ and in $\mathcal{P}$, which contradicts our assumption. This proves that N is not regular and finishes the proof of the proposition. □

Remark 5.6 When G is of type A_1, $\mathbf{P}_\infty$ is also an Iwahori subgroup. The same argument as in Proposition 5.3 shows that the local monodromy at ∞ is also regular unipotent.

5.3 Global geometric monodromy

In this section, we continue to assume that k is algebraically closed. We study the Zariski closure of the image of ρ_k, also known as the *geometric monodromy group* of the local system $\mathcal{E}$.

Theorem 5.7 *Let $\widehat{G}_\rho$ be the Zariski closure of the image of ρ_k. Then the neutral component of $\widehat{G}_\rho$ is a semisimple group, and*

$$\widehat{G}_\rho \begin{cases} = \widehat{G} & \text{if } \widehat{G} \text{ is of type } A_1, E_7, E_8 \text{ or } G_2 \\ \supset \mathrm{SO}_{4n-1} & \text{if } \widehat{G} = \mathrm{PSO}_{4n}, n \geq 3 \\ \supset G_2 & \text{if } \widehat{G} = \mathrm{PSO}_8. \end{cases}$$

Proof Let $\widehat{H}$ be the neutral component of $\widehat{G}_\rho$. We first show that $\widehat{H}$ is a semisimple group.

For $\mathrm{char}(k) > 0$, the local system comes via base change from the base field $\mathbb{F}_q$. Then by Theorem 4.2(4), $\mathcal{E}(\mathcal{K})$ is pure of weight zero, hence geometrically semisimple by [4, Corollaire 5.4.6]. Our claim then follows from [9, Corollaire 1.3.9] (in [9], Deligne remarked that the proof only uses the fact that the local system is *geometrically* semisimple).

For $\mathrm{char}(k) = 0$, we may reduce to the case $k = \mathbb{C}$. Consider the description of $\mathcal{E}'(\mathcal{K}) = (u_0 \times \mathrm{id})^* \mathbb{T}(\mathcal{K}, j_* \mathcal{F}_\chi)$ given in the first equality of (4.19). The complex $\omega^* j_* \mathcal{F} \otimes \mathcal{K}_{\mathrm{GR}, \mathbb{Q}'_\ell}$ is a semisimple perverse sheaf because π is a locally trivial fibration with fibers Gr and that both $j_* \mathcal{F}_\chi$ and $\mathcal{K}$ are semisimple perverse sheaves. Therefore, $\mathcal{E}'(\mathcal{K})$ is the direct image of a semisimple perverse sheaf of geometric origin along the proper map $\mathrm{GR}_{\leq \lambda} \to \mathbb{P}^1 - \{0, 1, \infty\}$, it is semisimple by the Decomposition Theorem [4, Théorème 6.2.5]. This implies that $\widehat{H}$ is a reductive group. Now suppose $S^0 = \widehat{H}^{ab}$ is nontrivial. Then $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$ maps densely into an algebraic group $S(\mathbb{Q}_\ell)$ with neutral component S^0 being a torus. The group $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$ is topologically generated by the loops ζ_0, ζ_1 and ζ_∞ around the punctures. We have already seen from Proposition 5.3 and 5.5 that $\rho_k(\zeta_1)$ and $\rho_k(\zeta_\infty)$ are unipotent, hence have trivial image in $S(\mathbb{Q}_\ell)$. By Proposition 5.4, $\rho_k(\zeta_0)$ has semisimple part of order 2, therefore the image of $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$ in $S(\mathbb{Q}_\ell)$ is generated by an element of order at most 2, and cannot be Zariski dense. This contradiction implies that S^0 is trivial, i.e., $\widehat{H}$ is semisimple.

By Proposition 5.3, $\widehat{H}$ contains a regular unipotent element. Hence $\widehat{H}$ is a semisimple subgroup of $\widehat{G}$ containing a principal PGL_2. According to

Dynkin's classification (see [15, p. 1500]), either $\widehat{H}$ is the principal PGL_2, or $\widehat{H}=\widehat{G}$, or $\widehat{H}=\mathrm{SO}_{4n-1}$ if $\widehat{G}=\mathrm{PSO}_{4n}$, or $\widehat{H}=G_2$ if $\widehat{G}=\mathrm{PSO}_8$.

If G is not of type A_1, the unipotent monodromy at ∞ cannot lie in the principal PGL_2 because it is neither trivial nor regular by Proposition 5.5. Hence $\widehat{H}$ cannot be equal to the principal PGL_2 in this case. This finishes the proof. □

5.4 Image of Galois representations

In this subsection, we work with the base field $k=\mathbb{Q}$. For any rational number $x\in\mathbb{Q}-\{0,1\}$, we get a closed point $i_x:\operatorname{Spec}\mathbb{Q}\hookrightarrow\mathbb{P}^1_{\mathbb{Q}}-\{0,1,\infty\}$ and hence an embedding $i_{x,\#}:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\pi_1(\mathbb{P}^1_{\mathbb{Q}}-\{0,1,\infty\})$, well-defined up to conjugacy. Restricting the representation $\rho_{\mathbb{Q}}$ (attached to the $\widehat{G}$-local system $\mathcal{E}_{\mathbb{Q}}$) using $i_{x,\#}$, we get a continuous Galois representation

$$\rho_x:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\widehat{G}(\mathbb{Q}_\ell).$$

A variant of Hilbert irreducibility [39, Theorem 2] shows that for x away from a thin set of $\mathbb{Q}$, the image of ρ_x is the same as the image of $\rho_{\mathbb{Q}}$, and therefore Zariski dense if G is of type A_1, E_7, E_8 or G_2 by Theorem 5.7. We would like to give an effective criterion for ρ_x to have large image.

Proposition 5.8 *Let $\rho:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\widehat{G}(\mathbb{Q}_\ell)$ be a continuous ℓ-adic representation. For each prime p, let $\rho_p:=\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ be the restriction of ρ to the decomposition group at p. Suppose*

(1) *For almost all primes p, ρ_p is unramified and pure of weight* 0;
(2) *For some prime $p'\neq\ell$, $\rho_{p'}$ is tamely ramified, and a topological generator of the tame inertia group at p' maps to a regular unipotent element in $\widehat{G}(\mathbb{Q}_\ell)$;*
(3) *The representation ρ_ℓ is Hodge-Tate (i.e., for any algebraic representation V of $\widehat{G}$, the induced action of* $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ *on V is Hodge-Tate).*

Let $\widehat{G}_\rho\subset\widehat{G}$ be the Zariski closure of the image of ρ. Then the neutral component $\widehat{G}^\circ_\rho$ is a reductive subgroup of $\widehat{G}$ containing a principal PGL_2.

Proof The proof is partially inspired by Scholl's argument in [34, Proposition 3].

Let $R\subset\widehat{G}^\circ_\rho$ be the unipotent radical and let $\widehat{H}$ be the connected reductive quotient $\widehat{G}^\circ_\rho/R$. All these groups are over $\mathbb{Q}_\ell$.

Let V be an algebraic representation of $\widehat{G}$, viewed as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module via ρ. We first claim that for any subquotient V' of V as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module, $\det(V')$ is a finite order character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In fact, since ρ is Hodge-Tate, so is V'. Hence we can write $\det(V') = \chi_\ell^m \epsilon$ for some finite order character ϵ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and some integer m, where χ_ℓ is the ℓ-adic cyclotomic character. By (1), $\det(V')$ is pure of weight zero at almost all p, hence Frob_p acts on $\det(V')$ by a number with archimedean norm 1. However, $\chi_\ell(\mathrm{Frob}_p) = p$, hence $m = 0$. This implies $\det(V')$ is a finite order character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

By assumption $\widehat{G}^\circ_\rho(\mathbb{Q}_\ell)$ contains a regular unipotent element u, which is the image of a topological generator of the tame inertia group at p'. Let $N = \log(u) \in \widehat{\mathfrak{g}}$. Let s be the image of a (lifting of the) Frobenius at p'. Then $\mathrm{Ad}(s)N = p'N$. Since N is regular nilpotent, the element s is semisimple and well-defined up to conjugacy. The action of s on $\widehat{\mathfrak{g}}$ determines a grading

$$\widehat{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} \widehat{\mathfrak{g}}(n) \tag{5.7}$$

such that s acts on $\widehat{\mathfrak{g}}(n)$ by $(p')^n$, and $N : \widehat{\mathfrak{g}}(n) \to \widehat{\mathfrak{g}}(n+1)$. In fact, taking N to be the sum of simple root generators, this grading is the same as the grading by the height of the roots. In particular, $\widehat{\mathfrak{g}}(n) \neq 0$ only for $1 - h \le n \le h - 1$, where h is the Coxeter number of $\widehat{G}$, and $\dim \widehat{\mathfrak{g}}(1-h) = \dim \widehat{\mathfrak{g}}(h-1) = 1$. Moreover, the map

$$N^{2n} : \widehat{\mathfrak{g}}(-n) \to \widehat{\mathfrak{g}}(n) \tag{5.8}$$

is an isomorphism for any $n \ge 0$.

For any subquotient $\widehat{G}_\rho$-module V of $\widehat{\mathfrak{g}}$, we can similarly define a grading $V = \bigoplus_n V(n)$ under the action of s. We say V is *symmetric* under the s-action if $\dim V(-n) = \dim V(n)$ for any n.

The action of the unipotent group R on $\widehat{\mathfrak{g}}$ gives a canonical increasing filtration of $\widehat{\mathfrak{g}}$: $F_0\widehat{\mathfrak{g}} = 0$, $F_i\widehat{\mathfrak{g}}/F_{i-1}\widehat{\mathfrak{g}} = (\widehat{\mathfrak{g}}/F_{i-1}\widehat{\mathfrak{g}})^R$. Therefore $\widehat{G}_\rho$ acts on each $\mathrm{Gr}^F_i\widehat{\mathfrak{g}}$ via the reductive quotient $\widehat{G}_\rho/R$.

We claim that for each i, the associated graded $\widehat{G}_\rho$-modules $\mathrm{Gr}^F_i\widehat{\mathfrak{g}}$ are symmetric under the s-action. In fact, it suffices to show that each $F_i\widehat{\mathfrak{g}}$ is symmetric under the s-action. Since (5.8) is an isomorphism, $N^{2n} : F_i\widehat{\mathfrak{g}}(-n) \to F_i\widehat{\mathfrak{g}}(n)$ is injective. Hence $\dim F_i\widehat{\mathfrak{g}}(-n) \le \dim F_i\widehat{\mathfrak{g}}(n)$ for any $n \ge 0$. On the other hand, we argued that $\det(F_i\widehat{\mathfrak{g}})$ is a finite order character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, so s can only act on $\det(F_i\widehat{\mathfrak{g}})$ as identity. This implies that $\dim F_i\widehat{\mathfrak{g}}(-n) = \dim F_i\widehat{\mathfrak{g}}(n)$ for all n.

Let $\overline{N} \in \widehat{\mathfrak{h}} = \mathrm{Lie}\widehat{H}$ be the image of N. There is a unique i such that $\mathrm{Gr}^F_i\widehat{\mathfrak{g}}(-h+1) \neq 0$, hence $\mathrm{Gr}^F_i\widehat{\mathfrak{g}}(m) \neq 0$ by the Claim. The iteration

$$\overline{N}^{2h-2} : \mathrm{Gr}^F_i\widehat{\mathfrak{g}}(-h+1) \to \mathrm{Gr}^F_i\widehat{\mathfrak{g}}(h-1)$$

is necessarily an isomorphism: for otherwise $N^{2h-2}\widehat{\mathfrak{g}}(-h+1) = \widehat{\mathfrak{g}}(h-1)$ would fall inside $F_{i-1}\widehat{\mathfrak{g}}$, contradiction. Therefore $\overline{N}$ acts on $\mathrm{Gr}^F_i\widehat{\mathfrak{g}}$ with a Jordan block of size $2h-1$.

Let $\iota : \widehat{H} \to \widehat{G}_\rho^\circ$ be any section (a group homomorphism). Now $\iota(\overline{N})$ acts on $\mathrm{Gr}_i^F \widehat{\mathfrak{g}}$ with a Jordan block of length $2h-1$, therefore the action on $\widehat{\mathfrak{g}}$ has a Jordan block of size $\geq 2h-1$. However, the only nilpotent class in $\widehat{\mathfrak{g}}$ with a Jordan block of size $\geq 2h-1$ under the adjoint representation is the regular nilpotent class. Hence $\iota(\overline{N})$ is a regular nilpotent class in $\widehat{\mathfrak{g}}$. Since $\iota(\widehat{H})$ is a reductive group, it contains a principal $\mathrm{PGL}_2 \subset \widehat{G}$.

Now fix such a principal $\mathrm{PGL}_2 \subset \iota(\widehat{H})$. Consider the adjoint action of PGL_2 on $\widehat{\mathfrak{g}}$. The action of the maximal torus of PGL_2 induces a similar grading as in (5.7), with $\widehat{\mathfrak{g}}(0)$ a Cartan subalgebra of $\widehat{\mathfrak{g}}$. The Lie algebra $\mathfrak{r}$ of R is a PGL_2-submodule of $\widehat{\mathfrak{g}}$ consisting entirely of nilpotent elements. However, any nonzero PGL_2-submodule of $\widehat{\mathfrak{g}}$ has a nonzero intersection with $\widehat{\mathfrak{g}}(0)$, hence containing nonzero semisimple elements. This forces R to be trivial. Therefore $\widehat{G}_\rho^\circ = \widehat{H}$ is a reductive subgroup of $\widehat{G}$ containing a principal PGL_2. □

In Proposition 4.5 we have extended the local system $\mathcal{E}_{\mathbb{Q}}$ to $\mathbb{P}^1_{\mathbb{Z}[1/2\ell N]} - \{0, 1, \infty\}$ for some integer N.

Proposition 5.9 *Let G be of type A_1, E_7, E_8 or G_2.*

1. *Let (a, b) be nonzero coprime integers such that both $a-b$ and b have prime divisors not dividing $2\ell N$. Let $x = \frac{a}{b}$. Then the image of ρ_x is Zariski dense in $\widehat{G}(\mathbb{Q}_\ell)$.*
2. *There are infinitely many rational numbers $x \in \mathbb{Q} - \{0, 1\}$ such that ρ_x are mutually non-isomorphic and all have Zariski dense image in $\widehat{G}(\mathbb{Q}_\ell)$.*

Proof (1) For each prime p, let $I_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be the inertia group. Let $Z \subset \mathbb{P}^1_{\mathbb{Z}[1/2\ell N]}$ be the closure of the $\mathbb{Q}$-point x, then the projection $\pi : Z \to \mathrm{Spec}\,\mathbb{Z}[1/2\ell N]$ is an isomorphism. For any prime $p \nmid 2\ell N$, the reduction of Z at p is a point $x_p \in \mathbb{P}^1_{\mathbb{F}_p} - \{0, 1, \infty\}$. Let $\rho_{\mathbb{F}_p} : \pi_1(\mathbb{P}^1_{\mathbb{F}_p} - \{0, 1, \infty\}) \to \widehat{G}(\mathbb{Q}_\ell)$ be the monodromy representation of the restriction of integral model $\underline{\mathcal{E}}$ to $\mathbb{P}^1_{\mathbb{F}_p} - \{0, 1, \infty\}$. Then the proof of [11, Proposition A.4] shows the following fact: $\rho_x(I_p)$ is contained in the image of $\rho_{\mathbb{F}_p}|_{I_{x_p}}$ ($I_{x_p} \subset \mathrm{Gal}(\mathbb{F}_p(t)^s/\mathbb{F}_p(t))$ being the inertia of the point $x_p \in \mathbb{P}^1(\mathbb{F}_p)$); moreover, if $\rho_{\mathbb{F}_p}|_{I_{x_p}}$ is tame and unipotent, $\rho_x|_{I_p}$ is also tame and maps to the same unipotent class. Using the calculation of the local monodromy in Sect. 5.2, we have for any $p \nmid 2\ell N$,

- If $p \mid a-b$ (hence $x_p = 1$), then $\rho_x(I_p)$ contains a regular unipotent element;
- If $p \mid b$ (hence $x_p = \infty$) and G is not of type A_1, then $\rho_x(I_p)$ contains a unipotent element which is neither trivial nor regular.
- If p does not divide $ab(a-b)$, then ρ_x is unramified at p.

We check that ρ_x satisfies the assumptions of Proposition 5.8: the unramifiedness of condition (1) is checked above and the purity is proved in Theorem 4.2(4); condition (2) is also checked above since we do have a prime $p \mid a-b$ and $p \nmid 2\ell N$; the Hodge-Tate condition (3) follows from the motivic interpretation (Proposition 4.6) and the theorem of Faltings [12]. Applying Proposition 5.8 to ρ_x, we conclude that the Zariski closure $\widehat{G}_{\rho_x}$ of the image of ρ_x is a reductive subgroup of $\widehat{G}$ containing a principal PGL_2. Moreover, if G is not of type A_1, it contains another unipotent element which is neither regular nor trivial. The argument of Theorem 5.7 then shows that the derived group of $\widehat{G}_{\rho_x}$ (which is semisimple and contains a principal PGL_2) is already the whole $\widehat{G}$.

(2) Choose an increasing sequence of prime numbers $2\ell N < p_1 < p_2 < \cdots$. Define $x_i = \frac{p_i + p_{i+1}}{p_{i+1}}$, $i = 1, 2, \ldots$. Then x_i satisfies the conditions in (1). Moreover, by the previous discussion, the places where ρ_{x_i} has regular unipotent monodromy are p_i together with possibly certain places dividing $2\ell N$. Therefore these ρ_{x_i} are mutually non-isomorphic and all have Zariski dense image in $\widehat{G}(\mathbb{Q}_\ell)$. □

5.5 Conjectural properties of the local system

The base field k is algebraically closed in this subsection. When G is of type A_1, our description of the local monodromy of $\mathcal{E}$ at 0, 1 and ∞ give complete information (note that at 0, the monodromy has to be κ because it is regular semisimple). When G is of type D_{2n}, E_7, E_8 or G_2, our description of the local monodromy at 0 and ∞ in Propositions 5.4 and 5.5 is not complete. In this subsection, we give a conjectural complete description of the local monodromy at 0 and ∞. Assuming this conjectural description, we deduce that the local system $\mathcal{E}$ is cohomologically rigid.

5.5.1 Local monodromy

Lusztig defined a map in [28]

$$\{\text{conjugacy classes in } W\} \to \{\text{unipotent classes in } \widehat{G}\}$$

In particular, if $-1 \in W$, it gets mapped to a unipotent class v in $\widehat{G}$, which has the property that

$$\dim Z_{\widehat{G}}(v) = \#\Phi_G/2.$$

We tabulate these unipotent classes using the Bala-Carter classification (see [8])

Type of G	the unipotent class v
D_{2n}	Jordan blocks $(1, 2, \dots, 2, 3)$
E_7	$4A_1$
E_8	$4A_1$
G_2	$\widetilde{A}_1$

Conjecture 5.10 *Under ρ_k, a topological generator of the tame inertia group I_∞^{tame} gets mapped to a unipotent element in the unipotent class v.*

We also make the following conjecture on the local monodromy at 0.

Conjecture 5.11 *Under ρ_k, a topological generator of the tame inertia group I_0^{tame} gets mapped to an element conjugate to $\kappa \in \widehat{T}[2]$ in Sect. 5.2.1. In other words, the unipotent part of the local monodromy at 0 is trivial.*

Remark 5.12 Conjecture 5.10 and Conjecture 5.11 would imply that in the case of G_2, our local system is isomorphic to the one constructed by Dettweiler and Reiter in [11], because they checked that there is up to isomorphism only one such local system with the same local monodromy as theirs.

5.5.2 Rigidity

Assume G is of type E_7, E_8 or G_2. Consider the adjoint local system $\mathrm{Ad}(\mathcal{E})$ associated to the adjoint representation of $\widehat{G}$. Let $j_{!*}\mathrm{Ad}(\mathcal{E})$ be the middle extension of $\mathrm{Ad}(\mathcal{E})$ to $\mathbb{P}^1$. Then we have an exact sequence (cf. [21, proof of Proposition 5.3])

$$\begin{aligned} 0 &\to \widehat{\mathfrak{g}}^{I_0} \oplus \widehat{\mathfrak{g}}^{I_1} \oplus \widehat{\mathfrak{g}}^{I_\infty} \to \mathrm{H}^1_c\big(\mathbb{P}^1 - \{0, 1, \infty\}, \mathrm{Ad}(\mathcal{E})\big) \\ &\to \mathrm{H}^1\big(\mathbb{P}^1, j_{!*}\mathrm{Ad}(\mathcal{E})\big) \to 0. \end{aligned} \tag{5.9}$$

Conjecture 5.13 *The local system $\mathcal{E}$ is* cohomologically rigid, i.e., $\mathrm{H}^1(\mathbb{P}^1, j_{!*}\mathrm{Ad}(\mathcal{E})) = 0$.

We make the following simple observation.

Lemma 5.14 *Any two of the three Conjectures 5.10, 5.11 and 5.13 imply the other.*

Proof Since $\mathrm{Ad}(\mathcal{E})$ is tame and has no global sections (by the Zariski density proved in Theorem 5.7), $\dim \mathrm{H}^1_c(\mathbb{P}^1 - \{0, 1, \infty\}, \mathrm{Ad}(\mathcal{E})) = -\chi(\mathbb{P}^1 - \{0, 1, \infty\}) \dim \widehat{G} = \dim \widehat{G}$ by the Grothendieck-Ogg-Shafarevich formula.

Also note that $\widehat{\mathfrak{g}}^{I_1} = \operatorname{rank} \widehat{G}$ because a generator of I_1^{tame} maps to a regular unipotent element by Proposition 5.3. Therefore the exact sequence (5.9) implies

$$\dim \widehat{\mathfrak{g}}^{I_0} + \dim \widehat{\mathfrak{g}}^{I_\infty} + \dim \mathrm{H}^1\big(\mathbb{P}^1, j_{!*}\mathrm{Ad}(\mathcal{E})\big) = \dim \widehat{G} - \operatorname{rank} \widehat{G} = \#\Phi_G.$$

Conjectures 5.10 and 5.11 imply Conjecture 5.13 because then $\widehat{\mathfrak{g}}^{I_\infty} = \widehat{\mathfrak{g}}^{v}$ has dimension $\#\Phi_G/2$ and $\widehat{\mathfrak{g}}^{I_0} = \widehat{\mathfrak{g}}^{\kappa}$ also has dimension $\#\Phi_G/2$ because κ is a split Cartan involution (see Sect. 5.2.1).

Conjectures 5.11 and 5.13 imply Conjecture 5.10 because then $\dim \widehat{\mathfrak{g}}^{I_0} = \#\Phi_G - \dim \widehat{\mathfrak{g}}^{v} = \#\Phi_G/2$ while $\widehat{\mathfrak{g}}^{I_0} \subset \widehat{\mathfrak{g}}^{\kappa}$ (which has dimension $\#\Phi_G/2$) by Proposition 5.4. These then force $\widehat{\mathfrak{g}}^{I_0} = \widehat{\mathfrak{g}}^{\kappa}$ and the unipotent part of the local monodromy at 0 must be trivial.

Conjectures 5.10 and 5.13 imply Conjecture 5.11 because then $\dim \widehat{\mathfrak{g}}^{I_\infty} = \#\Phi_G - \dim \widehat{\mathfrak{g}}^{\kappa} = \#\Phi_G/2$ and v is the only unipotent class with this property by checking tables in [8]. □

5.5.3 Global monodromy

Theorem 5.7 does not completely determine the geometric monodromy group of $\mathcal{E}$ in type D_{2n}. When $n > 2$,the nontrivial pinned automorphism σ of G permutes the odd central characters $\widetilde{A}^*_{0,\mathrm{odd}}$ nontrivially. For those χ which are fixed by σ, we expect the geometric monodromy of $\mathcal{E}_\chi$ to be $\widehat{G}^\sigma = \mathrm{SO}_{4n-1}$ (when $n > 2$), for the same reason as [21, Sect. 6.1]. How about those χ which are not fixed by σ? In the case G is of type D_4, is the geometric monodromy G_2 or SO_7 or PSO_8?

5.6 Application to the inverse Galois problem

In this subsection, we use the local system $\mathcal{E}$ to give affirmative answers to new cases of the inverse Galois problem as stated in Theorem 1.4. We also hint on the possibility of using rigidity method to give an alternative proof our results on the inverse Galois problem.

5.6.1 The Betti realization of the local system

The proof of Theorem 1.4 requires a counterpart of the local system $\mathcal{E}$ in analytic topology. To this end, we work with $k = \mathbb{C}$ and use the analytic topology instead of the étale topology. We change ℓ-adic cohomology to singular cohomology. Theorem 4.2 still holds in this situation. In particular, for each odd character χ of $\widetilde{A}_0$, we get a representation of the topological fundamental group

$$\rho_\chi^{\text{top}} : \pi_1^{\text{top}}\big(\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}\big) \to \widehat{G}(\mathbb{Q}).$$

5.6.2 Proof of Theorem 1.4

Let G be of type E_8 or G_2. We write ρ_χ^{top} simply as ρ^{top}.

Let $\mathbb{Z}' = \mathbb{Z}[i]$ and $\mathbb{Z}'_\ell = \mathbb{Z}_\ell[i]$. In the motivic interpretation of $\mathcal{E}^{\mathrm{qm}}$ given in Proposition 4.4, we may work with $\mathbb{Z}'$-coefficients in the analytic topology (resp. $\mathbb{Z}'_\ell$-coefficients in the étale topology). For sufficiently large prime ℓ, $(\mathbf{R}^{2h^\vee-2}\eta_!\mathbb{Z}'_\ell)_{\mathrm{odd}}$ is a free $\mathbb{Z}'_\ell$-module of rank $d = \dim V^{\mathrm{qm}}$. Also, there exists an integer N_1 such that the direct image in the analytic topology $(\mathbf{R}^{2h^\vee-2}\eta_!^{\mathrm{an}}\mathbb{Z}[\frac{1}{N_1}])_{\mathrm{odd}}$ a free $\mathbb{Z}'[1/N_1]$-module of rank d. Upon enlarging N_1, $\widehat{G}(\mathbb{Q}) \cap \mathrm{GL}_d(\mathbb{Z}'[\frac{1}{N_1}])$ is equal to $\widehat{G}(\mathbb{Z}[\frac{1}{N_1}])$ coming from the Chevalley group scheme structure of $\widehat{G}$. Similarly, for large primes ℓ, $\mathrm{GL}_n(\mathbb{Z}'_\ell) \cap \widehat{G}(\mathbb{Q}_\ell) = \widehat{G}(\mathbb{Z}_\ell)$. For large primes $\ell \nmid N_1$, the comparison isomorphism between singular cohomology and ℓ-adic cohomology gives a commutative diagram

$$
\begin{array}{ccccc}
\pi^{\mathrm{top}}(\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}) & \xrightarrow{\rho^{\mathrm{top}}} & \mathrm{GL}_n(\mathbb{Z}'[\frac{1}{N_1}]) \cap \widehat{G}(\mathbb{Q}) = \widehat{G}(\mathbb{Z}[\frac{1}{N_1}]) & \xrightarrow{\bmod \ell} & \widehat{G}(\mathbb{F}_\ell) \\
\downarrow & & \downarrow & \nearrow_{r_\ell} & \\
\pi_1(\mathbb{P}^1_{\mathbb{Q}} - \{0,1,\infty\}) & \xrightarrow{\rho_{\mathbb{Q}}} & \mathrm{GL}_n(\mathbb{Z}'_\ell) \cap \widehat{G}(\mathbb{Q}_\ell) = \widehat{G}(\mathbb{Z}_\ell) & &
\end{array}
\tag{5.10}
$$

We claim that the first row in the above diagram is surjective for large ℓ. Let $\Pi \subset \widehat{G}(\mathbb{Q})$ be the image of ρ^{top}, which is finitely generated because $\pi_1^{\mathrm{top}}(\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\})$ is a free group of rank 2. Since $\rho_{\overline{\mathbb{Q}}}$ has Zariski dense image in $\widehat{G}(\mathbb{Q}_\ell)$ (by Theorem 5.7) and $\pi_1^{\mathrm{top}}(\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\})$ is dense in $\pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\})$, ρ^{top} also has Zariski dense image in $\widehat{G}(\mathbb{Q})$. Therefore $\Pi \subset \widehat{G}(\mathbb{Q})$ is both finitely generated and Zariski dense. By [32, Theorem in the Introduction], for sufficiently large prime ℓ, the reduction modulo ℓ of Π surjects onto $\widehat{G}(\mathbb{F}_\ell)$.

Consequently, in the diagram (5.10), the composition $r_\ell \circ \rho_{\mathbb{Q}}$ is also surjective for large ℓ. Finally we apply Hilbert irreducibility theorem (see [37, Theorem 3.4.1]) to conclude that for any number field L, there exists $x \in L - \{0,1\}$ such that $r_\ell \circ \rho_x : \mathrm{Gal}(\overline{\mathbb{Q}}/L) \to \widehat{G}(\mathbb{Z}_\ell) \to \widehat{G}(\mathbb{F}_\ell)$ is surjective. This solves the inverse Galois problem over number fields for the finite simple group $\widehat{G}(\mathbb{F}_\ell)$.

Remark 5.15 When $\widehat{G} = E_7^{\mathrm{ad}}$, the same argument as above shows that for sufficiently large ℓ, there exists $x \in L - \{0,1\}$ (L is any given number field) such that $r_\ell \circ \rho_x : \mathrm{Gal}(\overline{\mathbb{Q}}/L) \to E_7^{\mathrm{ad}}(\mathbb{F}_\ell)$ is either surjective or has image equal to the finite simple group $E_7^{\mathrm{sc}}(\mathbb{F}_\ell)/\mu_2$.

5.6.3 *Rigid triples*

In inverse Galois theory, people use "rigid triples" in a finite group Γ to construct étale Γ-coverings of $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$, and hence get Γ as a Galois group over $\mathbb{Q}$ using Hilbert irreducibility. For details see Serre's book [37].

Let C_1 be the regular unipotent class in $\widehat{G}(\mathbb{F}_\ell)$. Let C_∞ be the unipotent class of v in $\widehat{G}(\mathbb{F}_\ell)$ defined in Sect. 5.5.1. Let C_0 be the conjugacy class of the reduction of κ in $\widehat{G}(\mathbb{F}_\ell)$.

Conjecture 5.16 *Let $\widehat{G}$ be of type E_8. Then for sufficiently large prime ℓ, (C_0, C_1, C_∞) form a strictly rigid triple in $\widehat{G}(\mathbb{F}_\ell)$. In other words, the equation*

$$g_0 g_1 g_\infty = 1, \quad g_i \in C_i \text{ for } i = 0, 1, \infty$$

has a unique solution up to conjugacy in $\widehat{G}(\mathbb{F}_\ell)$, and any such solution $\{g_0, g_1, g_\infty\}$ generate $\widehat{G}(\mathbb{F}_\ell)$.

Note added in proof: Guralnick and Malle [20] recently proved this conjecture for primes $\ell \geq 7$. The G_2-analog of the conjecture was proved much earlier by Thompson [40] for $\ell = 5$ and by Feit and Fong [14] for $\ell > 5$. Therefore, according to these works, we know that $G_2(\mathbb{F}_\ell)$ $(\ell \geq 5)$ and $E_8(\mathbb{F}_\ell)$ $(\ell \geq 7)$ are Galois groups over $\mathbb{Q}$.

Acknowledgement The author would like to thank B. Gross for many discussions and encouragement. In particular, the application to the inverse Galois problem was suggested by him. The author also thanks D. Gaitsgory, R. Guralnick, S. Junecue, N. Katz, G. Lusztig, B.-C. Ngô, D. Vogan, L. Xiao and an anonymous referee for helpful discussions or suggestions. The work is partially supported by the NSF grants DMS-0969470 and DMS-1261660.

References

1. Adams, J.: Nonlinear covers of real groups. Int. Math. Res. Not. **75**, 4031–4047 (2004)
2. Adams, J., Barbasch, D., Paul, A., Trapa, P., Vogan, D.A. Jr.: Unitary Shimura correspondences for split real groups. J. Am. Math. Soc. **20**(3), 701–751 (2007)
3. Arkhipov, S., Bezrukavnikov, R.: Perverse sheaves on affine flags and Langlands dual group (with an appendix by R. Bezrukavnikov and I. Mirković). Isr. J. Math. **170**, 135–183 (2009)
4. Beilinson, A., Bernstein, J., Deligne, P.: Faisceaux pervers. In: Analysis and Topology on Singular Spaces, i. Astérisque, vol. 100, pp. 5–171. Soc. Math. France, Paris (1982)
5. Bezrukavnikov, R.: On tensor categories attached to cells in affine Weyl groups. In: Representation Theory of Algebraic Groups and Quantum Groups. Adv. Stud. Pure Math., vol. 40, pp. 69–90. Math. Soc. Japan, Tokyo (2004)
6. Bezrukavnikov, R., Finkelberg, M., Ostrik, V.: On tensor categories attached to cells in affine Weyl groups. III. Isr. J. Math. **170**, 207–234 (2009)
7. Bourbaki, N.: Éléments de Mathématique. Fasc. XXXIV. Groupes et Algèbres de Lie. Chapitre IV–VI. Actualités Scientifiques et Industrielles, vol. 1337. Hermann, Paris (1968)

8. Carter, R.W.: Finite Groups of Lie Type. Conjugacy Classes and Complex Characters. Pure and Applied Mathematics. Wiley, New York (1985)
9. Deligne, P.: La conjecture de Weil. II. Publ. Math. IHÉS **52**, 137–252 (1980)
10. Deligne, P., Milne, J.S.: Tannakian categories. Lect. Notes Math. **900**, 101–228 (1982)
11. Dettweiler, M., Reiter, S.: Rigid local systems and motives of type G_2. With an appendix by M. Dettweiler and N.M. Katz. Compos. Math. **146**(4), 929–963 (2010)
12. Faltings, G.: p-adic Hodge theory. J. Am. Math. Soc. **1**, 255–299 (1988)
13. Faltings, G.: Algebraic loop groups and moduli spaces of bundles. J. Eur. Math. Soc. **5**, 41–68 (2003)
14. Feit, W., Fong, P.: Rational rigidity of $G_2(p)$ for any prime $p > 5$. In: Proceedings of the Rutgers Group Theory Year, 1983–1984, New Brunswick, N.J., 1983–1984, pp. 323–326. Cambridge University Press, Cambridge (1985)
15. Frenkel, E., Gross, B.: A rigid irregular connection on the projective line. Ann. Math. (2) **170**(3), 1469–1512 (2009)
16. Gaitsgory, D.: Construction of central elements in the affine Hecke algebra via nearby cycles. Invent. Math. **144**(2), 253–280 (2001)
17. Gaitsgory, D.: On de Jong's conjecture. Isr. J. Math. **157**, 155–191 (2007)
18. Görtz, U., Haines, T.J.: Bounds on weights of nearby cycles and Wakimoto sheaves on affine flag manifolds. Manuscr. Math. **120**(4), 347–358 (2006)
19. Gross, B.H., Savin, G.: Motives with Galois group of type G_2: an exceptional theta-correspondence. Compos. Math. **114**(2), 153–217 (1998)
20. Guralnick, R., Malle, G.: Rational rigidity for $E_8(p)$. Preprint. arXiv:1207.1464
21. Heinloth, J., Ngô, B.-C., Yun, Z.: Kloosterman sheaves for reductive groups. Ann. Math. (2) **177**(1), 241–310 (2013)
22. Jannsen, U.: Motives, numerical equivalence, and semi-simplicity. Invent. Math. **107**(3), 447–452 (1992)
23. Laszlo, Y., Olsson, M.: The six operations for sheaves on Artin stacks. I. Finite coefficients. Publ. Math. Inst. Hautes Études Sci. **107**, 109–168 (2008)
24. Laszlo, Y., Olsson, M.: The six operations for sheaves on Artin stacks. II. Adic coefficients. Publ. Math. Inst. Hautes Études Sci. **107**, 169–210 (2008)
25. Lusztig, G.: Cells in affine Weyl groups. In: Algebraic Groups and Related Topics, Kyoto/Nagoya, 1983. Adv. Stud. Pure Math., vol. 6, pp. 255–287. North-Holland, Amsterdam (1985)
26. Lusztig, G.: Cells in affine Weyl groups. II. J. Algebra **109**(2), 536–548 (1987)
27. Lusztig, G.: Cells in affine Weyl groups. IV. J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math. **36**(2), 297–328 (1989)
28. Lusztig, G.: From conjugacy classes in the Weyl group to unipotent classes. Represent. Theory **15**, 494–530 (2011)
29. Malle, G., Matzat, B.H.: Inverse Galois Theory. Springer Monographs in Mathematics. Springer, Berlin (1999)
30. Milne, J.S.: Shimura varieties and motives. In: Motives, Seattle, WA, 1991. Proc. Sympos. Pure Math., vol. 55, Part 2, pp. 447–523. Am. Math. Soc., Providence (1994)
31. Mirković, I., Vilonen, K.: Geometric Langlands duality and representations of algebraic groups over commutative rings. Ann. Math. (2) **166**(1), 95–143 (2007)
32. Matthews, C.R., Vaserstein, L.N., Weisfeiler, B.: Congruence properties of Zariski-dense subgroups. I. Proc. Lond. Math. Soc. (3) **48**(3), 514–532 (1984)
33. Reeder, M., Yu, J.-K.: Epipelagic representations and invariant theory. J. Am. Math. Soc. (in press)
34. Scholl, A.J.: On some ℓ-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ attached to noncongruence subgroups. Bull. Lond. Math. Soc. **38**(4), 561–567 (2006)
35. Serre, J.-P.: Linear Representations of Finite Groups. Graduate Texts in Mathematics, vol. 42. Springer, New York (1977). Translated from the second French edition by Leonard L. Scott

36. Serre, J.-P.: Propriétés conjecturales des groupes de Galois motiviques et des représentations ℓ-adiques. In: Motives, Seattle, WA, 1991. Proc. Sympos. Pure Math., vol. 55, Part 1, pp. 377–400. Am. Math. Soc., Providence (1994)
37. Serre, J.-P.: Topics in Galois Theory. Lecture Notes Prepared by Henri Darmon. Research Notes in Mathematics, vol. 1. Jones & Bartlett Publishers, Boston (1992)
38. Springer, T.A.: Some results on algebraic groups with involutions. In: Algebraic Groups and Related Topics, Kyoto/Nagoya, 1983. Adv. Stud. Pure Math., vol. 6, pp. 525–543. North-Holland, Amsterdam (1985)
39. Terasoma, T.: Complete intersections with middle Picard number 1 defined over $\mathbb{Q}$. Math. Z. **189**(2), 289–296 (1985)
40. Thompson, J.G.: Rational rigidity of $G_2(5)$. In: Proceedings of the Rutgers Group Theory Year, 1983–1984, New Brunswick, N.J., 1983–1984, pp. 321–322. Cambridge University Press, Cambridge (1985)
41. Yun, Z.: Global Springer theory. Adv. Math. **228**, 266–328 (2011)