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ENDOSCOPY FOR HECKE CATEGORIES, CHARACTER SHEAVES AND REPRESENTATIONS

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Abstract

For a reductive group G over a finite field, we show that the neutral block of its mixed Hecke category with a fixed monodromy under the torus action is monoidally equivalent to the mixed Hecke category of the corresponding endoscopic group H with trivial monodromy. We also extend this equivalence to all blocks. We give two applications. One is a relationship between character sheaves on G with a fixed semisimple parameter and unipotent character sheaves on the endoscopic group H, after passing to asymptotic versions. The other is a similar relationship between representations of $G(\mathbb{F}_q)$ with a fixed semisimple parameter and unipotent representations of $H(\mathbb{F}_q)$.

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1. Introduction

1.1. Hecke categories. Let G be a connected split reductive group over a finite field $\mathbb{F}_q$. Let B be a Borel subgroup of G. The (mixed) Hecke category of G is the B-equivariant derived category of complexes of sheaves with $\overline{\mathbb{Q}}_\ell$-coefficients on the flag variety G/B of G whose cohomology sheaves are mixed in the sense of [8, 1.2.2]. We denote this category by $D^b_m(B\backslash G/B)$. The Hecke category $D^b_m(B\backslash G/B)$ carries a monoidal structure under convolution. It gives a categorification of the Hecke algebra $H_q(W)$ attached to the Weyl group of G.

The Hecke category and its variants play a central role in geometric representation theory. On the one hand, when the base field is $\mathbb{C}$, the category of perverse sheaves $\mathrm{Perv}(B_{\mathbb{C}}\backslash G_{\mathbb{C}}/B_{\mathbb{C}})$ is equivalent to a version of category $\mathcal{O}$ for the Lie algebra $\mathfrak{g}$ of $G_{\mathbb{C}}$, by the Beilinson–Bernstein localization theorem. The Kazhdan–Lusztig conjecture [11] relates the stalks of simple perverse sheaves on $B_{\mathbb{C}}\backslash G_{\mathbb{C}}/B_{\mathbb{C}}$ to characters of simple modules in the category $\mathcal{O}$. On the other hand, by the work of Ben-Zvi–Nadler [2] (characteristic zero), Bezrukavnikov–Finkelberg–Ostrik [5] (characteristic zero) and Lusztig [20] (characteristic $p > 0$), the categorical center of the Hecke category is equivalent to the category of unipotent character sheaves (the exact statement varies in different papers; in particular, [5] and [20] contain statements about the asymptotic versions), which in turn is closely related to irreducible characters of finite groups of Lie type $G(\mathbb{F}_q)$.

1.2. Monodromic Hecke categories. In this paper, we consider the monodromic version of the Hecke category. More precisely, let $B = UT$, where U is the unipotent radical of B and T a maximal torus. For two rank-one character sheaves $\mathcal{L}, \mathcal{L}'$ on the torus T (which is the same as a rank-one local system with finite monodromy together with a rigidification at the origin), we consider the equivariant derived category ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ of mixed $\overline{\mathbb{Q}}_\ell$-complexes on $U\backslash G/U$ under the left and right translation action of T with respect to the character sheaves $\mathcal{L}'$ and $\mathcal{L}$, respectively. When $\mathcal{L}$ is the trivial local system, ${}_{\mathcal{L}}\mathcal{D}_{\mathcal{L}}$ is the usual Hecke category $D^b_m(B\backslash G/B)$.

In [13, Ch. 1], the first author proves that the stalks of the simple perverse sheaves in the monodromic Hecke category ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ are given by Kazhdan–Lusztig polynomials for a smaller Weyl group inside W defined using $\mathcal{L}$ or $\mathcal{L}'$. Our main result is a categorical equivalence, which implies this numerical statement. To state it, we need to introduce, on the one hand, blocks in ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ and, on the other hand, the endoscopic group attached to $\mathcal{L}$.

For simplicity, let us restrict to the case $\mathcal{L}' = \mathcal{L}$. The monoidal category ${}_{\mathcal{L}}\mathcal{D}_{\mathcal{L}}$ can in general be decomposed into a direct sum of subcategories called *blocks*. Let ${}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}} \subset {}_{\mathcal{L}}\mathcal{D}_{\mathcal{L}}$ be the block containing the monoidal unit. The simple perverse sheaves in ${}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}$ up to Frobenius twists are parametrized by a normal subgroup $W^{\circ}_{\mathcal{L}}$ of the stabilizer of $\mathcal{L}$ under W. For details, see Definition 4.10. When the center of G is connected (for example, G is of adjoint type), we have ${}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}} = {}_{\mathcal{L}}\mathcal{D}_{\mathcal{L}}$.

Let $\Phi_{\mathcal{L}}$ be the set of roots α of (G, T) such that the pullback of $\mathcal{L}$ along its coroot $\alpha^\vee : \mathbb{G}_m \to T$ is a trivial local system on $\mathbb{G}_m$. Then $\Phi_{\mathcal{L}}$ is a root system. Let H be a connected reductive group over k with T as a maximal torus and $\Phi_{\mathcal{L}}$ as its roots. This is the *endoscopic group* attached to $\mathcal{L}$. The Weyl group W_H of

H is canonically identified with $W_{\mathcal{L}}^{\circ}$. The choice of the Borel B of G gives a Borel B_H of H. Let $\mathcal{D}_H = D_m^b(B_H\backslash H/B_H)$ be the usual mixed Hecke category for H.

THEOREM 1.3 (For a more precise version, see Theorem 9.2). *There is a canonical monoidal equivalence of triangulated categories*

$$\Psi_{\mathcal{L}}^{\circ} : \mathcal{D}_H \xrightarrow{\sim} {}_{\mathcal{L}}\mathcal{D}_{\mathcal{L}}^{\circ}$$

sending simple perverse sheaves to simple perverse sheaves.

At the level of Grothendieck groups, Theorem 1.3 implies an isomorphism between the Hecke algebra for W_H and a monodromic version of the Hecke algebra defined using W and $\mathcal{L}$ (see Section 3.13) preserving the canonical bases of the two Hecke algebras. Such a statement as well as its extension to all blocks is proved by the first author in [**24**, 1.6] and implicitly in [**17**, Lemma 34.7].

In $\mathcal{D}_H$, there are simple perverse sheaves $\mathrm{IC}(w)_H$ (normalized by a Tate twist to be pure of weight zero) for $w \in W_H$. In contrast, in ${}_{\mathcal{L}}\mathcal{D}_{\mathcal{L}}^{\circ}$, we do not *a priori* have canonical simple perverse sheaves indexed by $w \in W_{\mathcal{L}}^{\circ}$; it always involves a choice of a lifting $\dot{w}$ of w to $N_G(T)$. However, the above theorem gives canonical simple perverse sheaves $\Psi_{\mathcal{L}}^{\circ}(\mathrm{IC}(w)_H) \in {}_{\mathcal{L}}\mathcal{D}_{\mathcal{L}}^{\circ}$. These canonical objects, denoted by $\mathrm{IC}(w)_{\mathcal{L}}^{\dagger}$, are defined in Definition 6.7 using the constructions in Section 6.5.

As a consequence of our theorem, we prove that the stalks of $\mathrm{IC}(w)_{\mathcal{L}}^{\dagger}$ are semisimple as Frobenius modules (Proposition 9.10), and similarly Frobenius semisimplicity holds for the convolution (Proposition 9.11).

We also have a version of the theorem covering all blocks of ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ for $\mathcal{L}$ and $\mathcal{L}'$ in the same W-orbit, but it is more complicated to state. See Theorem 10.12. It involves a groupoid $\mathfrak{H}$ whose components are torsors of endoscopic groups, and a subtle modification of the convolution structure by a 3-cocycle of the finite Abelian group $\Omega_{\mathcal{L}} = W_{\mathcal{L}}/W_{\mathcal{L}}^{\circ}$ (see Sections 10.9 and 5.6).

1.4. Remarks on the proof. The initial difficulty in proving Theorem 1.3 lies in the fact that there is no nontrivial homomorphism between H and G in general. For example, when $G = \mathrm{Sp}_{2n}$ and $\mathcal{L}$ of order 2 and fixed by the Weyl group of G, we have $H \cong \mathrm{SO}_{2n}$.

The strategy to prove Theorem 1.3 is to relate both categories to Soergel bimodules for the Coxeter group $W_{\mathcal{L}}^{\circ} = W_H$. For $\mathcal{D}_H$, this is by now well known, following the insight of Soergel [**27**]: taking global sections of simple perverse sheaves on $B_H\backslash H/B_H$ preserves the graded Hom spaces (see [**27**, Erweiterungssatz]). In this paper, we develop an analogue of Soergel's theory for the monodromic Hecke categories ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$. To do this, we replace the global sections functor in the nonmonodromic case by the functor corepresented by the simple perverse sheaf with the largest support in each block of ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$. We show that the resulting functor carries a monoidal structure (Corollary 7.8) and preserves graded Hom spaces between simple perverse sheaves (Theorem 7.9). Using this, we show that ${}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}$ is equivalent to a certain derived category of Soergel bimodules (Theorem 9.6).

The results in Sections 2–7 hold with the same proofs when the mixed category ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ is replaced with the constructible equivariant derived category for the situation over an arbitrary algebraically closed field k. We expect Theorems 1.3 and 10.12 to hold as well over any algebraically closed base field. It is likely that the argument in [**4**, Section 6.5] would allow one to deduce such results from our main results over finite fields.

1.5. Application to representations. Let G_1 be a form of G_k over $\mathbb{F}_q$. Let us recall a rough statement of the classification of irreducible characters of $G_1(\mathbb{F}_q)$. By [**7**, 10.1], one can assign to each irreducible $G_1(\mathbb{F}_q)$-representation over $\overline{\mathbb{Q}}_\ell$ a semisimple geometric conjugacy class s in G_1^* defined over $\mathbb{F}_q$ (here G_1^* is a reductive group over $\mathbb{F}_q$ whose root system is dual to that of G), called the *semisimple parameter* of the irreducible representation. This assignment requires a choice of an isomorphism $\mathrm{Hom}_{\mathrm{cont}}(\varprojlim_n \mu_n(\overline{\mathbb{F}}_q), \overline{\mathbb{Q}}_\ell^\times) \xrightarrow{\sim} \overline{\mathbb{F}}_q^\times$. We may alternatively think of a semisimple parameter of G_1 as a W-orbit $\mathfrak{o}$ of character sheaves on T_k that are stable under the Frobenius map for G_1.

Let $I_{\mathfrak{o}}(G(\mathbb{F}_q))$ be the set of irreducible representations of $G_1(\mathbb{F}_q)$ with semisimple parameter $\mathfrak{o}$. Let $I_u(G_1(\mathbb{F}_q))$ be the set of unipotent irreducible representations of $G_1(\mathbb{F}_q)$ (that is, the case $\mathfrak{o}$ consists of the unit element in $\mathrm{Ch}(T_k)$). It is shown by the first author in [**13**, Theorem 4.23] that the parametrization of $I_{\mathfrak{o}}(G(\mathbb{F}_q))$ is closely related to that of $I_u(H_1(\mathbb{F}_q))$, where H_1 is the endoscopic group attached to $\mathfrak{o}$, under the assumption that the center of G is connected. An extension of such a relationship to all reductive groups G_1 is announced in [**12**, 2.1] and proved in [**16**] and [**18**].

As an application of Theorem 1.3, using results from [**21**] and [**23**] relating representations of $G_1(\mathbb{F}_q)$ to twisted categorical centers of the Hecke categories, we prove a relationship between representations of $G_1(\mathbb{F}_q)$ with a fixed semisimple parameter and unipotent representations of its endoscopic group, without appealing to the classifications mentioned above. We state it under the simplifying assumption that G_k has a connected center, and the general case is in Corollary 12.7.

THEOREM 1.6. *Assume G_k has a connected center. Let G_1 be a form G_k over $\mathbb{F}_q$ and let $\mathfrak{o} \subset \mathrm{Ch}(T_k)$ be a W-orbit that is a semisimple parameter for G_1. Let $\mathcal{L} \in \mathfrak{o}$ and let H_k be the endoscopic group of G_k attached to $\mathcal{L} \in \mathfrak{o}$. Then there is a form H_1 of the endoscopic group H over $\mathbb{F}_q$ and an equivalence of categories*

$$\mathrm{Rep}^{\mathbf{c}}_{\mathfrak{o}}(G_1(\mathbb{F}_q)) \cong \mathrm{Rep}^{\mathbf{c}}_{u}(H_1(\mathbb{F}_q))$$

for each two-sided cell $\mathbf{c}$ of $(G_1, \mathfrak{o})$ (which determines a two-sided cell, also denoted by $\mathbf{c}$, for unipotent representations of H_1).

1.7. Application to character sheaves. Character sheaves on G_k ($k = \overline{\mathbb{F}}_q$) are certain simple perverse sheaves equivariant under the conjugation action of G_k. Each character sheaf has a semisimple parameter, which is a W-orbit $\mathfrak{o} \subset \mathrm{Ch}(T_k)$. Unipotent character sheaves on G_k are those with a trivial semisimple parameter. From the classification of character sheaves in [15, 23.1], there is a close relationship between character sheaves on G_k with a semisimple parameter $\mathfrak{o}$ and unipotent character sheaves on H_k, the endoscopic group attached to some $\mathcal{L} \in \mathfrak{o}$.

As another application of Theorem 1.3, using results from [22], we derive a relationship between the asymptotic versions of character sheaves on G_k with a fixed semisimple parameter and unipotent character sheaves on its endoscopic group. Again we state it under the simplifying assumption that G_k has a connected center, and the general case is Theorem 11.10.

THEOREM 1.8. *Assume G_k has a connected center. Let $\mathfrak{o} \subset \mathrm{Ch}(T_k)$ be a W-orbit. Let $\mathcal{L} \in \mathfrak{o}$ and let H_k be the endoscopic group of G_k attached to $\mathcal{L}$. Then there is a canonical equivalence of braided monoidal categories*

$$\underline{\mathcal{CS}}^{\mathbf{c}}_{\mathfrak{o}}(G_k) \cong \underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H_k)$$

for each two-sided cell $\mathbf{c}$ of $W_H = W^{\circ}_{\mathcal{L}}$.

For definitions of $\underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H_k)$ and $\underline{\mathcal{CS}}^{\mathbf{c}}_{\mathfrak{o}}(G_k)$, see Sections 11.2 and 11.5.

In Theorem 12.6, we also prove a generalization of the above equivalence for character sheaves on disconnected groups.

1.9. Connection to Soergel's work. In this subsection, the base field is $\mathbb{C}$, and we use the same notations G, B and T but now they are understood to be algebraic groups over $\mathbb{C}$. Let $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{t}$ be the Lie algebras of G, B and T. In [27, Theorem 11], Soergel proves the following result. For a dominant but not necessarily integral character $\lambda \in \mathfrak{t}^*$, let $\mathcal{O}_\lambda^\circ$ be the block of the category $\mathcal{O}_\lambda$ (category $\mathcal{O}$ of $\mathfrak{g}$ with infinitesimal character corresponding to λ under the Harish-Chandra isomorphism) containing the simple module $L(\lambda)$ with the highest weight λ. Then up to equivalence, $\mathcal{O}_\lambda^\circ$ only depends on the Coxeter group $(W(\lambda), S(\lambda))$, which is the Weyl group attached to the based root system $\Phi_\lambda = \{\alpha \in \Phi(\mathfrak{g}, \mathfrak{t}) | \langle \alpha^\vee, \lambda\rangle \in \mathbb{Z}\}$ with positive roots $\Phi_\lambda^+ = \Phi_\lambda \cap \Phi^+(\mathfrak{g}, \mathfrak{t})$.

From λ, we get a character $\mathbb{X}_*(T) \subset \mathfrak{t} \xrightarrow{\langle -,\lambda\rangle} \mathbb{C} \xrightarrow{\exp(2\pi i(-))} \mathbb{C}^\times$, giving a rank-one character sheaf $\mathcal{L}_\lambda$ on $T(\mathbb{C})$. By the localization theorem of Beilinson–Bernstein and the Riemann–Hilbert correspondence, $\mathcal{O}_\lambda^\circ$ can be identified with a block ${}_\lambda P^\circ$ in the category $\mathrm{Perv}_{(T,\mathcal{L}_\lambda)}(U\backslash G/U)$ (with the T-action on the left). Soergel's result can then be formulated as an equivalence of Abelian categories ${}_\lambda P^\circ \cong P_H$, where H is the endoscopic group attached to $\mathcal{L}_\lambda$ and $P_H = \mathrm{Perv}(B_H\backslash H/U_H)$. We expect the method used in Soergel's paper can be extended to prove the Koszul dual version of the characteristic zero analogue of Theorem 1.3, with equivariance replaced by weak equivariance (or monodromicity) as in [6].

1.10. Notation and conventions

1.10.1. Frobenius Throughout the article, let $k = \overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$. Let ℓ be a prime different from $p = \mathrm{char}(k)$.

Let $\mathrm{Fr} \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ be the geometric Frobenius. An Fr-module M is a $\overline{\mathbb{Q}}_\ell$-vector space with a $\overline{\mathbb{Q}}_\ell$-linear automorphism $\mathrm{Fr}_M : M \to M$ such that each $v \in M$ is contained in a finite-dimensional subspace stable under Fr_M. The Fr-module M is called *pure of weight n* if for any eigenvalue λ of Fr_M, λ is algebraic over $\mathbb{Q}$ with all conjugates in $\mathbb{C}$ of absolute value $q^{n/2}$.

Fix a square root $q^{1/2}$ of q in $\overline{\mathbb{Q}}_\ell$. We denote by $\overline{\mathbb{Q}}_\ell(1/2)$ the one-dimensional Fr-module $M = \overline{\mathbb{Q}}_\ell$ equipped with the automorphism Fr_M by scalar multiplication by $q^{-1/2}$.

1.10.2. Geometry. We denote

$$\mathrm{pt} := \mathrm{Spec}\,\mathbb{F}_q.$$

For a scheme X over $\mathbb{F}_q$, let $D_m^b(X)$ be the derived category of étale $\overline{\mathbb{Q}}_\ell$-complexes on X whose cohomology sheaves are mixed; see [1, 5.1.5]. If X is equipped with an action of an algebraic group H over $\mathbb{F}_q$, one can follow the method of [3] to define the H-equivariant derived category of mixed $\overline{\mathbb{Q}}_\ell$-complexes denoted by $D_{H,m}^b(X)$ or $D_m^b(H\backslash X)$ (that is, working with Cartesian complexes on the standard simplicial scheme resolving the stack $H\backslash X$).

Similarly, we have the (constructible, $\overline{\mathbb{Q}}_\ell$-coefficient) equivariant derived category $D^b(H_k\backslash X_k)$. We have a pullback functor $\omega : D^b_m(H\backslash X) \to D^b(H_k\backslash X_k)$. For $\mathcal{F}, \mathcal{F}' \in D^b_m(H\backslash X)$, we define

$$\operatorname{Hom}(\mathcal{F}, \mathcal{F}') := \operatorname{Hom}_{D^b(H_k\backslash X_k)}(\omega\mathcal{F}, \omega\mathcal{F}'). \tag{1.1}$$

In other words, the Hom space between two mixed complexes on $H\backslash X$ is taken to be the Hom space of their pullback to $H_k\backslash X_k$. Similarly, $\operatorname{Ext}^i(\mathcal{F}, \mathcal{F}')$ means $\operatorname{Hom}(\mathcal{F}, \mathcal{F}'[i])$ calculated again in $D^b(H_k\backslash X_k)$, and $\mathbf{R}\operatorname{Hom}(\mathcal{F}, \mathcal{F}')$ means $\mathbf{R}\operatorname{Hom}(\omega\mathcal{F}, \omega\mathcal{F}')$, which is an object in $D^b_m(\mathrm{pt})$. The actual morphisms inside the category $D^b_m(H\backslash X)$ will be denoted as

$$\hom(\mathcal{F}, \mathcal{F}') := \operatorname{Hom}_{D^b_m(H\backslash X)}(\mathcal{F}, \mathcal{F}').$$

A *semisimple complex* in $D^b(H_k\backslash X_k)$ means an object isomorphic to a finite direct sum of shifted simple perverse sheaves. A *semisimple complex* in $D^b_m(H\backslash X)$ is that whose image in $D^b(H_k\backslash X_k)$ is semisimple.

For any mixed complex $\mathcal{F} \in D^b_m(H\backslash X)$ and $n \in \mathbb{Z}$, we denote $\mathcal{F}(n/2) := \mathcal{F} \otimes \pi^*\overline{\mathbb{Q}}_\ell(1/2)^{\otimes n}$, where $\pi : H\backslash X \to \mathrm{pt}$ is the natural projection. Then $\mathcal{F}(1)$ is the usual Tate twist. Also, we define

$$\mathcal{F}\langle n\rangle := \mathcal{F}[n](n/2), \quad n \in \mathbb{Z}. \tag{1.2}$$

For an algebraic group H over $\mathbb{F}_q$ acting on a scheme X on the right and on another scheme Y on the left, we denote by $X \overset{H}{\times} Y$ the quotient stack $(X \times Y)/H$, where $h \in H$ acts by $h \cdot (x, y) = (xh^{-1}, hy)$.

1.10.3. Group theory. Let G be a connected split reductive group over $\mathbb{F}_q$. Fix a Borel subgroup B of G with unipotent radical U and a maximal torus $T \subset B$. Let $\Phi(G, T)$ (respectively, $\Phi^\vee(G, T)$) be the set of roots (respectively, coroots) of G with respect to T. The choice of B gives the set of positive roots $\Phi^+ := \Phi^+(G, B, T)$, negative roots $\Phi^- := \Phi^-(G, B, T)$, and a set of simple roots.

Let $W = N_G(T)/T$ be the Weyl group of G, with simple reflections coming from simple roots. For a simple reflection $s \in W$, let α_s and $\alpha_s^\vee$ be the corresponding simple root and simple coroot.

We use e to denote the identity element of W. We use $\dot{e}$ to denote the identity element of G.

For each $w \in W$, we use $\dot{w}$ to denote a lifting of w in $N_G(T)(\mathbb{F}_q)$. For $w = e$, we always lift it to $\dot{e}$, the identity element of G. The equivalence in Theorem 1.3 will not depend on the choice of such liftings, while its extension Theorem 10.12 will depend on choices of liftings on a subset of W.

1.10.4. Other. For a triangulated category $\mathcal{D}$ and $\{X_\alpha\}_{\alpha\in I}$ a collection of objects in $\mathcal{D}$, we denote by $\langle X_\alpha; \alpha \in I\rangle$ the full subcategory of $\mathcal{D}$ whose objects are successive extensions of objects that are isomorphic to X_α, $\alpha \in I$.

Let S be a set with a left action of H_1 and a right action of H_2. We say S is an (H_1, H_2)-*bitorsor* if S is a torsor under the H_1-action and a torsor under the H_2-action. Similarly, we define the notion of bitorsors for schemes with left and right actions of group schemes.

For a category $\mathcal{C}$, let $\mathrm{Ob}(\mathcal{C})$ be the collection of objects in $\mathcal{C}$, and $|\mathcal{C}|$ be the set of isomorphism classes of objects in $\mathcal{C}$.

2. Monodromic Hecke categories

In Sections 2–10, we work over a fixed finite field $\mathbb{F}_q$. In this section, we introduce the main players of the paper: the monodromic Hecke categories.

2.1. Rank-one character sheaves.

2.1. Rank-one character sheaves. For an algebraic group H over $\mathbb{F}_q$, there is the notion of *rank-one character sheaves* on H. These are rank-one $\overline{\mathbb{Q}}_\ell$-local systems $\mathcal{L}$ on H equipped with an isomorphism $m^*\mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L}$ over $H \times H$ (where $m : H \times H \to H$ is the multiplication map) and a trivialization of the stalk $\mathcal{L}_e$ ($e \in H$ is the identity element) satisfying the associativity and unital axioms. We refer to [31, Appendix A] for a systematic treatment of rank-one character sheaves. Let $\mathrm{Ch}(H)$ denote the group of isomorphism classes of rank-one character sheaves. When H is connected, the automorphisms of a rank-one character sheaf reduce to identity.

Recall $k = \overline{\mathbb{F}}_q$. We define

$$\mathrm{Ch}(H_k) = \varinjlim_n \mathrm{Ch}(H_{\mathbb{F}_{q^n}})$$

with transition maps given by the pullback.

Let $\nu : \widetilde{H} \to H$ be a finite étale central isogeny (where $\widetilde{H}$ is a connected algebraic group) with discrete kernel $\ker(\nu)$ (discrete as a group scheme over $\mathbb{F}_q$, that is, a finite Abelian group). Let $\chi : \ker(\nu) \to \overline{\mathbb{Q}}_\ell^\times$ be a homomorphism. Then $\mathcal{L} := \nu_*\overline{\mathbb{Q}}_\ell[\chi]$, the sublocal system of $\nu_*\overline{\mathbb{Q}}_\ell$ on which $\ker(\nu)$ acts via χ, is a rank-one character sheaf on H of finite order. It is shown in [31, A.2] that any element in $\mathrm{Ch}(H)$ arises in this way.

2.2. The case of a torus. When $H = T$ is a torus, the Lang map $\lambda_T : T \to T$ given by $t \mapsto \mathrm{Fr}_{T/\mathbb{F}_q}(t)t^{-1}$ is a finite étale isogeny with kernel $T(\mathbb{F}_q)$. The above construction gives a homomorphism

$$\mathrm{Hom}(T(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell^\times) \to \mathrm{Ch}(T). \tag{2.1}$$

This is in fact a bijection with inverse given by taking the Frobenius trace function of character sheaves; see [31, A.3.3].

LEMMA 2.3. *Let H be a connected reductive group over $\mathbb{F}_q$ with maximal torus T. Let $\mathcal{L} \in \mathrm{Ch}(T)$. Then $\mathcal{L}$ extends to a rank-one character sheaf $\widetilde{\mathcal{L}} \in \mathrm{Ch}(H)$ (necessarily unique) if and only if for every coroot $\alpha^\vee : \mathbb{G}_{m,k} \to T_k$ of H_k, the pullback $(\alpha^\vee)^*\mathcal{L}$ is the trivial rank-one character sheaf on $\mathbb{G}_{m,k}$.*

Proof. First, suppose $\mathcal{L}$ extends to $\widetilde{\mathcal{L}} \in \mathrm{Ch}(H)$, and denote its pullback to H_k again by $\widetilde{\mathcal{L}}$. Let $\alpha^\vee$ be a coroot of H_k and $\varphi_\alpha : \mathrm{SL}_{2,k} \to H_k$ be the homomorphism whose image is the rank-one subgroup of H_k containing the roots $\pm\alpha$. Let $\mathbb{G}_{m,k} \subset \mathrm{SL}_{2,k}$ be the diagonal torus. Then $\varphi_\alpha|_{\mathbb{G}_{m,k}} = \alpha^\vee$. Since $\mathrm{SL}_{2,k}$ does not admit any nontrivial finite central isogeny, $\varphi_\alpha^*\widetilde{\mathcal{L}}$ is trivial; hence $(\alpha^\vee)^*\mathcal{L} = (\varphi_\alpha^*\widetilde{\mathcal{L}})|_{\mathbb{G}_{m,k}}$ is trivial.

Conversely, suppose $\mathcal{L}$ is trivial after pullback along each coroot. The restriction map $\mathrm{Ch}(H) \to \mathrm{Ch}(T)$ is injective by [31, A.2.2]. Let $k'/\mathbb{F}_q$ be a finite extension and $\sigma \in \mathrm{Gal}(k'/\mathbb{F}_q)$ be the Frobenius element. By [31, A.1.2(4)], $\mathrm{Ch}(H) = \mathrm{Ch}(H_{k'})^\sigma$, therefore $\mathrm{Ch}(H) = \mathrm{Ch}(H_{k'}) \cap \mathrm{Ch}(T) \subset \mathrm{Ch}(T_{k'})$. In other words, it suffices to show that $\mathcal{L}_{k'} \in \mathrm{Ch}(T_{k'})$ extends to $H_{k'}$ for some finite extension $k'/\mathbb{F}_q$. Therefore we may base-change the situation by a finite extension of $\mathbb{F}_q$ so that T is split. Below we assume T is split.

Let $\chi : T(\mathbb{F}_q) = \mathbb{X}_*(T) \otimes_{\mathbb{Z}} \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$ be the character corresponding to $\mathcal{L}$ under bijection (2.1). We view χ as a homomorphism $\widetilde{\chi} : \mathbb{X}_*(T) \to \mathrm{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}}_\ell^\times)$. Let $\Lambda = \ker(\widetilde{\chi})$. The assumption on $\mathcal{L}$ implies that Λ contains the coroot lattice of H. By the structure theory of reductive groups, there is a connected split reductive group $\widetilde{H}$ over $\mathbb{F}_q$ with maximal torus $\widetilde{T}$ such that $\Lambda = \mathbb{X}_*(\widetilde{T})$ with coroots $\Phi^\vee(H, T)$. The embedding $\Lambda \subset \mathbb{X}_*(T)$ gives a homomorphism $\nu : \widetilde{H} \to H$ such that $\nu^{-1}(T) = \widetilde{T}$. By construction, the Lang map $\lambda_T : T \to T$ factors as $T \xrightarrow{\beta} \widetilde{T} \xrightarrow{\nu|_{\widetilde{T}}} T$ such that $\ker(\beta) = \ker(\chi)$ and $\ker(\nu) = \ker(\nu|_{\widetilde{T}}) = T(\mathbb{F}_q)/\ker(\chi)$. Hence χ factors through a character $\overline{\chi} : \ker(\nu) \to \overline{\mathbb{Q}}_\ell^\times$. Let $\mathcal{L}' = \nu_*\overline{\mathbb{Q}}_\ell[\overline{\chi}] \in \mathrm{Ch}(H)$. Then $\mathcal{L}'_T = (\nu|_{\widetilde{T}})_*\overline{\mathbb{Q}}_\ell[\overline{\chi}]$, which is isomorphic to $\mathcal{L}$ by construction. Therefore $\mathcal{L}$ extends to $\mathcal{L}' \in \mathrm{Ch}(H)$. □

2.4. Root system attached to $\mathcal{L}$. The action of W on T induces an action of W on $\mathrm{Ch}(T)$: for $w \in W$ and $\mathcal{L} \in \mathrm{Ch}(T)$, define $w\mathcal{L} = (w^{-1})^*\mathcal{L}$. For a root

$\alpha \in \Phi(G, T)$, the action of r_α on $\mathrm{Ch}(T)$ is given by $\mathcal{L} \mapsto \mathcal{L} \otimes (\alpha^\vee \circ \alpha)^* \mathcal{L}^{-1}$ (the map $\alpha^\vee \circ \alpha : T \to \mathbb{G}_m \to T$). For $\mathcal{L} \in \mathrm{Ch}(T)$, let $W_\mathcal{L}$ be its stabilizer under W.

For $\mathcal{L} \in \mathrm{Ch}(T)$, we have a subset of the coroots

$$\Phi^\vee_\mathcal{L} = \{\alpha^\vee \in \Phi^\vee(G, T) | (\alpha^\vee)^* \mathcal{L} \text{ is trivial on } \mathbb{G}_m, \text{ where } \alpha^\vee : \mathbb{G}_m \to T\}.$$

Let $\Phi_\mathcal{L}$ be the subset of $\Phi(G, T)$ corresponding to $\Phi^\vee_\mathcal{L}$; it is a subroot system of $\Phi(G, T)$. Let $W^\circ_\mathcal{L}$ be the Weyl group of $\Phi_\mathcal{L}$: it is the subgroup of W generated by the reflections r_α for $\alpha \in \Phi_\mathcal{L}$. Then $W^\circ_\mathcal{L}$ is a normal subgroup of $W_\mathcal{L}$. (In [22] and [24], $W^\circ_\mathcal{L}$ and $W_\mathcal{L}$ are denoted by W_λ and W'_λ, respectively.) In fact, $\Phi^\vee_\mathcal{L}$ is stable under $W_\mathcal{L}$; hence $W_\mathcal{L}$ normalizes $W^\circ_\mathcal{L}$. We will see in Section 9.1 that if G has a connected center, then $W^\circ_\mathcal{L} = W_\mathcal{L}$.

The subset $\Phi^+_\mathcal{L} = \Phi^+ \cap \Phi_\mathcal{L}$ (where $\Phi^+ \subset \Phi(G, T)$ is the set of positive roots defined by B) gives a notion of positive roots in $\Phi_\mathcal{L}$. This defines a Coxeter group structure on $W^\circ_\mathcal{L}$, where the simple reflections are the reflections given by indecomposable roots in $\Phi^+_\mathcal{L}$. We denote the length function of the Coxeter group $W^\circ_\mathcal{L}$ by

$$\ell_\mathcal{L} : W^\circ_\mathcal{L} \to \mathbb{Z}_{\geqslant 0}. \tag{2.2}$$

For $w \in W$ and $\mathcal{L} \in \mathrm{Ch}(T)$, we have

$$\Phi_{w\mathcal{L}} = w(\Phi_\mathcal{L}) \subset \Phi(G, T). \tag{2.3}$$

For $\mathcal{L}, \mathcal{L}' \in \mathrm{Ch}(T)$, let ${}_{\mathcal{L}'}W_\mathcal{L} = \{w \in W | w\mathcal{L} = \mathcal{L}'\}$. This is nonempty only when $\mathcal{L}$ and $\mathcal{L}'$ are in the same W-orbit. When $\mathcal{L}$ and $\mathcal{L}'$ are in the same W-orbit, ${}_{\mathcal{L}'}W_\mathcal{L}$ is a $(W_{\mathcal{L}'}, W_\mathcal{L})$-bitorsor. Since $W^\circ_\mathcal{L}$ is normal in $W_\mathcal{L}$, for any $x \in {}_{\mathcal{L}'}W_\mathcal{L}$, we have $W^\circ_{\mathcal{L}'} x = W^\circ_{\mathcal{L}'} x W^\circ_\mathcal{L} = x W^\circ_\mathcal{L}$.

2.5. Monodromic complexes. Let H be a connected algebraic group over $\mathbb{F}_q$ acting on a scheme X of finite type over $\mathbb{F}_q$. Let $\mathcal{L} \in \mathrm{Ch}(H)$. We will define a triangulated category $D^b_{(H,\mathcal{L}),m}(X)$ of mixed $\overline{\mathbb{Q}}_\ell$-complexes on X equivariant with respect to $(H, \mathcal{L})$. The case where $\mathcal{L}$ is trivial corresponds to the usual equivariant derived category $D^b_{H,m}(X) = D^b_m(H\backslash X)$ as defined by Bernstein–Lunts in [3].

By the discussion in Section 2.1, there are a finite étale central isogeny $\nu : \widetilde{H} \to H$ and a character $\chi : \ker(\nu) \to \overline{\mathbb{Q}}_\ell^\times$ such that $\mathcal{L}$ appears as the direct summand of $\nu_* \overline{\mathbb{Q}}_\ell$, where $\ker(\nu)$ acts through χ. Consider the equivariant derived category $D^b_m(\widetilde{H}\backslash X)$, where the action of $\widetilde{H}$ is through H via ν. Since the finite Abelian group $\ker(\nu)$ acts trivially on X, it acts on the identity functor of the $\overline{\mathbb{Q}}_\ell$-linear category $D^b_m(\widetilde{H}\backslash X)$. This allows us to decompose $D^b_m(\widetilde{H}\backslash X)$ into a direct sum of full triangulated subcategories according to characters of $\ker(\nu)$.

Let $D^b_{(H,\mathcal{L}),m}(X)$ be the direct summand of $D^b_m(\widetilde{H}\backslash X)$ corresponding to χ. It can be checked that, up to canonical equivalence, the category $D^b_{(H,\mathcal{L}),m}(X)$ does not depend on the choice of $(\widetilde{H}, \chi)$ attached to $\mathcal{L}$ (the essential point is that $\mathrm{H}^*_A(\mathrm{pt}_k, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell$ for a finite group A).

Similarly, one defines the constructible derived category $D^b_{(H_k,\mathcal{L})}(X_k)$ for the spaces base-changed to $k = \overline{\mathbb{F}}_q$. We use convention (1.1) for the Hom spaces in $D^b_m(\widetilde{H}\backslash X)$.

2.6. Functoriality. Let $\nu : H' \to H$ be a homomorphism of algebraic groups and let H act on X. Let $\mathcal{L} \in \mathrm{Ch}(H)$ and $\mathcal{L}' = \nu^*\mathcal{L} \in \mathrm{Ch}(H')$. Let $\pi : H'\backslash X \to H\backslash X$ be the natural map of quotient stacks. Then we have a pair of adjoint functors

$$\pi^* : D^b_{(H,\mathcal{L}),m}(X) \rightleftarrows D^b_{(H',\mathcal{L}'),m}(X) : \pi_*$$

defined as follows. For $\mathcal{F} \in D^b_{(H,\mathcal{L}),m}(X)$, $\pi^*\mathcal{F}$ has the same underlying sheaf on X as $\mathcal{F}$, with the $(H', \mathcal{L}')$-equivariant structure obtained by pulling back the $(H, \mathcal{L})$-equivariant structure on $\mathcal{F}$.

For $\mathcal{F}' \in D^b_{(H',\mathcal{L}'),m}(X)$, consider the action map $a : H \overset{H'}{\times} X \to X$. The complex $\mathcal{L} \boxtimes \mathcal{F}'$ on $H \times X$ carries a natural H'-equivariant structure with respect to the action $h'(h, x) = (hh'^{-1}, h'x)$ (because $\mathcal{L}$ is $(H', \mathcal{L}')$-equivariant with respect to the right translation action of H'). Let $\mathcal{L} \widetilde{\boxtimes} \mathcal{F}'$ be the descent of $\mathcal{L} \boxtimes \mathcal{F}'$ to $H \overset{H'}{\times} X$ and define $\pi_*\mathcal{F}' = a_*(\mathcal{L} \widetilde{\boxtimes} \mathcal{F}')$. Then $\pi_*\mathcal{F}'$ carries a natural $(H, \mathcal{L})$-equivariant structure coming from the left $(H, \mathcal{L})$-equivariant structure on $\mathcal{L}$ itself, and hence defines an object in $D^b_{(H,\mathcal{L}),m}(X)$.

2.7. Monodromic Hecke categories. Let $\mathcal{L}, \mathcal{L}' \in \mathrm{Ch}(T)$. Applying the construction in Section 2.5 to the $T \times T$-action on $U\backslash G/U$ by $(t_1, t_2) : g \mapsto t_1 g t_2^{-1}$, we get the category

$${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}} = D^b_{(T\times T,\mathcal{L}'\boxtimes\mathcal{L}^{-1}),m}(U\backslash G/U).$$

Note that the inverse $\mathcal{L}^{-1}$ (dual local system) of $\mathcal{L}$ appears in the definition, but we still write $\mathcal{L}$ in our notation ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$.

We denote the nonmixed counterpart of ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ by

$${}_{\mathcal{L}'}\underline{\mathcal{D}}_{\mathcal{L}} = D^b_{(T_k\times T_k,\mathcal{L}'\boxtimes\mathcal{L}^{-1})}(U_k\backslash G_k/U_k).$$

We have the pullback functor

$$\omega : {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}} \to {}_{\mathcal{L}'}\underline{\mathcal{D}}_{\mathcal{L}}.$$

For variants of ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ that we introduce later, we put an underline to denote the corresponding nonmixed version.

Each object $\mathcal{F} \in {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ is equipped with an isomorphism

$$a^*\mathcal{F} \cong \mathcal{L}' \boxtimes \mathcal{F} \boxtimes \mathcal{L},$$

where $a : T \times [U\backslash G/U] \times T \to [U\backslash G/U]$ is the action map given by $(t_1, g, t_2) = t_1 g t_2$, together with compatibility data.

In particular, when $\mathcal{L} = \overline{\mathbb{Q}}_\ell = \mathcal{L}'$, ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ is the usual Hecke category $D^b_m(B\backslash G/B)$.

2.8. Basic operations. Let $\mathcal{L}', \mathcal{L}, \mathcal{K}', \mathcal{K} \in \mathrm{Ch}(T)$. For $\mathcal{F} \in {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ and $\mathcal{G} \in {}_{\mathcal{K}'}\mathcal{D}_{\mathcal{K}}$, the inner Hom $\mathbf{R}\underline{\mathrm{Hom}}_{U\backslash G/U}(\mathcal{F}, \mathcal{G})$, viewed as a complex on $U\backslash G/U$, defines an object in ${}_{\mathcal{L}'^{-1}\otimes\mathcal{K}}\mathcal{D}_{\mathcal{L}^{-1}\otimes\mathcal{K}}$. This gives a bifunctor

$$\mathbf{R}\underline{\mathrm{Hom}} : ({}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}})^{\mathrm{opp}} \times {}_{\mathcal{K}'}\mathcal{D}_{\mathcal{K}} \to {}_{\mathcal{L}'^{-1}\otimes\mathcal{K}'}\, \mathcal{D}_{\mathcal{L}^{-1}\otimes\mathcal{K}}.$$

In particular, when $\mathcal{K} = \mathcal{L}$ and $\mathcal{K}' = \mathcal{L}'$, $\mathbf{R}\underline{\mathrm{Hom}}_{U\backslash G/U}(\mathcal{F}, \mathcal{G})$ descends to $B\backslash G/B$ and defines an object in the usual Hecke category $D^b_m(B\backslash G/B)$.

We define a renormalized version of the Verdier duality on ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$. Let $\mathbb{D}_{G/B}$ be the dualizing complex of G/B, viewed as an object in $D^b_m(B\backslash G/B)$. For $\mathcal{F} \in {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$, by the discussion on the inner Hom above, we define the object

$$\mathbb{D}(\mathcal{F}) = \mathbf{R}\underline{\mathrm{Hom}}(\mathcal{F}, \mathbb{D}_{G/B}) \in {}_{\mathcal{L}'^{-1}}\mathcal{D}_{\mathcal{L}^{-1}}.$$

This defines a functor

$$\mathbb{D} : ({}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}})^{\mathrm{opp}} \to {}_{\mathcal{L}'^{-1}}\mathcal{D}_{\mathcal{L}^{-1}},$$

which is an involutive equivalence of categories. We refer to this functor as the *Verdier duality* on ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$.

We define the perverse t-structure on ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ in the following way. We define a full subcategory ${}_{\mathcal{L}'}\mathcal{D}^{\leqslant 0}_{\mathcal{L}}$ (respectively, ${}_{\mathcal{L}'}\mathcal{D}^{\geqslant 0}_{\mathcal{L}}$) to consist of objects $\mathcal{F} \in {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ such that $\mathcal{F}[\dim T]$, as a complex on G/U, lies in ${}^pD^{\leqslant 0}(G/U)$ (respectively, ${}^pD^{\geqslant 0}(G/U)$). Then $({}_{\mathcal{L}'}\mathcal{D}^{\leqslant 0}_{\mathcal{L}}, {}_{\mathcal{L}'}\mathcal{D}^{\geqslant 0}_{\mathcal{L}})$ defines a t-structure, which we shall call the *perverse t-structure* on ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$. With this definition, the Verdier duality functor $\mathbb{D}$ sends perverse sheaves in ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ to perverse sheaves in ${}_{\mathcal{L}'^{-1}}\mathcal{D}_{\mathcal{L}^{-1}}$.

2.9. Strata. For $w \in W$, let $G_w \subset G$ be the B-double coset $G_w = BwB$. (This is abuse of notation as we should have written $B\dot{w}B$ for some $\dot{w} \in N_G(T)(\mathbb{F}_q)$ lifting w. However, the resulting subscheme is independent of the choice of the lifting. In the sequel, we will use such abuse of notation freely.) Let $G_{\leqslant w}$ be the closure of $G_{\leqslant w}$ and $G_{<w} = G_{\leqslant w} - G_w$.

Let ${}_{\mathcal{L}'}\mathcal{D}(w)_{\mathcal{L}} := D^b_{(T\times T,\mathcal{L}'\boxtimes\mathcal{L}^{-1}),m}(U\backslash G_w/U)$. Similarly, define ${}_{\mathcal{L}'}\mathcal{D}(\leqslant w)_{\mathcal{L}}$ and ${}_{\mathcal{L}'}\mathcal{D}(< w)_{\mathcal{L}}$ by replacing G_w with $G_{\leqslant w}$ and $G_{<w}$.

The inclusion $i_w : U\backslash G_w/U \hookrightarrow U\backslash G/U$ induces adjoint pairs

$$i_{w,!} : {}_{\mathcal{L}'}\mathcal{D}(w)_{\mathcal{L}} \rightleftarrows {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}} : i_w^!$$

$$i_w^* : {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}} \rightleftarrows {}_{\mathcal{L}'}\mathcal{D}(w)_{\mathcal{L}} : i_{w,*} .$$

Let $\Gamma(w) \subset T \times T$ be the graph consisting of (wt, t), $t \in T$. The $T \times T$-action on any point in $U\backslash G_w/U$ has stabilizer $\Gamma(w)$. From the definitions, we have the following lemma.

Lemma 2.10. *The category ${}_{\mathcal{L}'}\mathcal{D}(w)_{\mathcal{L}}$ is zero unless $\mathcal{L}' = w\mathcal{L}$. When $\mathcal{L}' = w\mathcal{L}$, taking a stalk at the lifting $\dot{w} \in N_G(T)(\mathbb{F}_q)$ of w induces an equivalence*

$$i_{\dot{w}}^* : {}_{w\mathcal{L}}\mathcal{D}(w)_{\mathcal{L}} \xrightarrow{\sim} D^b_{\Gamma(w),m}(\{\dot{w}\}).$$

Here we are using that $w^*\mathcal{L}' = w^{-1}(\mathcal{L}') \cong \mathcal{L}$ so that $\mathcal{L}' \boxtimes \mathcal{L}^{-1}$ restricts to the trivial character sheaf on $\Gamma(w)$.

2.11. Some objects. In view of Lemma 2.10, ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}} = 0$ unless $\mathcal{L}$ and $\mathcal{L}'$ are in the same W-orbit of $\mathrm{Ch}(T)$. In the remainder of the paper, we fix a W-orbit $\mathfrak{o} \subset \mathrm{Ch}(T)$.

For $w \in W$ with lifting $\dot{w}$, and $\mathcal{L} \in \mathfrak{o}$, let $C(\dot{w})_{\mathcal{L}} \in {}_{w\mathcal{L}}\mathcal{D}(w)_{\mathcal{L}}$ be the object that corresponds to the constant sheaf $\overline{\mathbb{Q}}_\ell\langle\ell(w)\rangle$ under the equivalence $i_{\dot{w}}^*$ in Lemma 2.10. Note that the isomorphism class of $C(\dot{w})_{\mathcal{L}}$ is independent of the lifting $\dot{w}$ while for different liftings, the identifications between the $C(\dot{w})_{\mathcal{L}}$'s are only unique up to scalars.

Define the following perverse sheaves in ${}_{w\mathcal{L}}\mathcal{D}_{\mathcal{L}}$:

$$\Delta(\dot{w})_{\mathcal{L}} = i_{w,!}C(\dot{w})_{\mathcal{L}}, \quad \nabla(\dot{w})_{\mathcal{L}} = i_{w,*}C(\dot{w})_{\mathcal{L}}, \tag{2.4}$$

$$\mathrm{IC}(\dot{w})_{\mathcal{L}} = i_{w,!*}C(\dot{w})_{\mathcal{L}} := \mathrm{Im}(\Delta(\dot{w})_{\mathcal{L}} \to \nabla(\dot{w})_{\mathcal{L}}). \tag{2.5}$$

Remark 2.12. The isomorphism classes of $\omega C(\dot{w})_{\mathcal{L}}$, $\omega\Delta(\dot{w})_{\mathcal{L}}$, $\omega\nabla(\dot{w})_{\mathcal{L}}$ and $\omega\mathrm{IC}(\dot{w})_{\mathcal{L}}$ in ${}_{\mathcal{L}'}\underline{\mathcal{D}}_{\mathcal{L}}$ are independent of the lifting $\dot{w}$. For this reason, we denote these isomorphism classes in ${}_{\mathcal{L}'}\underline{\mathcal{D}}_{\mathcal{L}}$ by

$$\underline{C}(w)_{\mathcal{L}} \in {}_{w\mathcal{L}}\underline{\mathcal{D}}(w)_{\mathcal{L}}, \quad \underline{\Delta}(w)_{\mathcal{L}}, \quad \underline{\nabla}(w)_{\mathcal{L}} \quad \text{and} \quad \underline{\mathrm{IC}}(w)_{\mathcal{L}} \in {}_{w\mathcal{L}}\underline{\mathcal{D}}_{\mathcal{L}}.$$

However, in the mixed category ${}_{w\mathcal{L}}\mathcal{D}_{\mathcal{L}}$, if we change the lifting $\dot{w}$ to another lifting $\ddot{w} = \dot{w}t^{-1}$ ($t \in T(\mathbb{F}_q)$), then we have a canonical isomorphism in ${}_{w\mathcal{L}}\mathcal{D}_{\mathcal{L}}$:

$$\mathrm{IC}(\dot{w})_{\mathcal{L}} \cong \mathrm{IC}(\ddot{w})_{\mathcal{L}} \otimes \mathcal{L}_t, \tag{2.6}$$

where $\mathcal{L}_t$ (the stalk of $\mathcal{L}$ at t) is viewed as a one-dimensional Fr-module. Similar isomorphisms hold for $C(\dot{w})_{\mathcal{L}}$, $\Delta(\dot{w})_{\mathcal{L}}$ and $\nabla(\dot{w})_{\mathcal{L}}$.

3. Convolution

In this section, we define and study properties of the convolution functor on the monodromic Hecke categories. We also use convolution to prove the parity and purity properties of $\mathrm{IC}(\dot{w})_{\mathcal{L}}$ in Proposition 3.11.

3.1. Convolution.

Recall that we fix a W-orbit $\mathfrak{o} \subset \mathrm{Ch}(T)$. Let $\mathcal{L}, \mathcal{L}', \mathcal{L}'' \in \mathfrak{o}$. Consider the diagram

$$\begin{tikzcd}
 & U\backslash G \overset{U}{\times} G/U \arrow[dl, "\pi"'] \arrow[d] & \\
U\backslash G/U \times U\backslash G/U & U\backslash G \overset{B}{\times} G/U \arrow[r, "m"] & U\backslash G/U
\end{tikzcd}$$

For $\mathcal{F} \in {}_{\mathcal{L}''}\mathcal{D}_{\mathcal{L}'}$ and $\mathcal{G} \in {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$, $\pi^*(\mathcal{F} \boxtimes \mathcal{G})$ carries an equivariant structure under the T-action on $U\backslash G \overset{U}{\times} G/U$ given by $T \ni t : (g_1, g_2) \mapsto (g_1 t^{-1}, tg_2)$ (using that $\mathcal{F}$ is $(T, \mathcal{L}'^{-1})$-equivariant for the second T-action and $\mathcal{G}$ is $(T, \mathcal{L}')$-equivariant for the first T-action). Therefore, it descends to a complex $\mathcal{F} \widetilde{\boxtimes} \mathcal{G} \in D^b_{(T\times T, \mathcal{L}''\boxtimes\mathcal{L}^{-1}),m}(U\backslash G \overset{B}{\times} G/U)$. Define

$$\mathcal{F} \star \mathcal{G} = m_*(\mathcal{F} \widetilde{\boxtimes} \mathcal{G}) \in {}_{\mathcal{L}''}\mathcal{D}_{\mathcal{L}}.$$

This construction gives a convolution bifunctor

$$(-) \star (-) : {}_{\mathcal{L}''}\mathcal{D}_{\mathcal{L}'} \times {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}} \to {}_{\mathcal{L}''}\mathcal{D}_{\mathcal{L}}.$$

It is easy to see that convolution carries a natural associativity structure in the obvious sense. Under convolution, ${}_{\mathcal{L}}\mathcal{D}_{\mathcal{L}}$ becomes a monoidal category with the unit object

$$\delta_{\mathcal{L}} := \Delta(\dot{e})_{\mathcal{L}} \cong \mathrm{IC}(\dot{e})_{\mathcal{L}} \cong \nabla(\dot{e})_{\mathcal{L}}.$$

The properness of the multiplication map $m : G \overset{B}{\times} G \to G$ implies the following.

LEMMA 3.2. *There is a natural isomorphism functorial in* $\mathcal{F} \in {}_{\mathcal{L}''}\mathcal{D}_{\mathcal{L}'}$ *and* $\mathcal{G} \in {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$*:*

$$\mathbb{D}(\mathcal{F} \star \mathcal{G}) \cong \mathbb{D}(\mathcal{F}) \star \mathbb{D}(\mathcal{G}).$$

LEMMA 3.3. (1) *If* $\mathcal{F} \in {}_{\mathcal{L}''}\mathcal{D}_{\mathcal{L}'}$ *and* $\mathcal{G} \in {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ *are semisimple complexes, so is* $\mathcal{F} \star \mathcal{G}$.

(2) *If* $\mathcal{F} \in {}_{\mathcal{L}''}\mathcal{D}_{\mathcal{L}'}$ *and* $\mathcal{G} \in {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ *are pure of weight zero, so is* $\mathcal{F} \star \mathcal{G}$.

Proof. (2) follows from the properness of the multiplication map $m : G \overset{B}{\times} G \to G$ and Deligne's weight estimates [1, 5.1.14]. (1) follows from the properness of the multiplication map $m : G \overset{B}{\times} G \to G$ and the decomposition theorem [1]. □

LEMMA 3.4. *Suppose* $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$*; then there are canonical isomorphisms*

$$\Delta(\dot{w}_1)_{w_2\mathcal{L}} \star \Delta(\dot{w}_2)_{\mathcal{L}} \cong \Delta(\dot{w}_1\dot{w}_2)_{\mathcal{L}}, \quad \nabla(\dot{w}_1)_{w_2\mathcal{L}} \star \nabla(\dot{w}_2)_{\mathcal{L}} \cong \nabla(\dot{w}_1\dot{w}_2)_{\mathcal{L}}.$$

Proof. Both isomorphisms follow directly from the fact that the multiplication map $G_{w_1} \overset{B}{\times} G_{w_2} \to G_{w_1 w_2}$ is an isomorphism if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. □

LEMMA 3.5. *Let* $w \in W$*. Then there are isomorphisms*

$$\Delta(\dot{w}^{-1})_{w\mathcal{L}} \star \nabla(\dot{w})_{\mathcal{L}} \cong \Delta(e)_{\mathcal{L}} \cong \nabla(\dot{w}^{-1})_{w\mathcal{L}} \star \Delta(\dot{w})_{\mathcal{L}}.$$

In particular, the functor

$$(-) \star \Delta(\dot{w})_{\mathcal{L}} : {}_{\mathcal{L}'}\mathcal{D}_{w\mathcal{L}} \to {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$$

is an equivalence of categories with inverse given by $(-) \star \nabla(\dot{w}^{-1})_{w\mathcal{L}}$.

Proof. Writing w into a reduced word in simple reflections and using Lemma 3.4, it is enough to prove the statements for $w = s$, a simple reflection. When s is a simple reflection, we may replace G by its Levi subgroup L_s with roots $\pm\alpha_s$. Therefore it suffices to treat the case where G has semisimple rank one. In this case, from the definition of convolution, we have

$$i_{\dot{s}}^{*}(\Delta(\dot{s}^{-1})_{s\mathcal{L}} \star \nabla(\dot{s})_{\mathcal{L}}) = \mathrm{H}^{*}((G/B)_k, \Delta(\dot{s}^{-1})_{s\mathcal{L}} \otimes \mathrm{inv}^{*} R_{\dot{s}}^{*} \nabla(\dot{s})_{\mathcal{L}}).$$

Here $R_{\dot{s}} : U\backslash G \to U\backslash G$ is the right translation by $\dot{s}$ and $\mathrm{inv} : G/U \to U\backslash G$ is given by inversion. Now $\Delta(\dot{s}^{-1})_{s\mathcal{L}}$ is $(T, s\mathcal{L})$-equivariant with respect to the right translation of T on G/U, and $\mathrm{inv}^* R_{\dot{s}}^* \nabla(\dot{s})_{\mathcal{L}}$ is $(T, s\mathcal{L}^{-1})$-equivariant with respect to the right translation. Their tensor product is T-equivariant on the right and hence descends to G/B. We choose an identification $G/B \cong \mathbb{P}^1$ such that the unit coset B corresponds to $0 \in \mathbb{P}^1$, and sB corresponds to $\infty \in \mathbb{P}^1$. Let Y_0, Y_∞ be the preimages of $0, \infty$ in $Y = G/U$, and let $j_0 : Y - Y_0 \hookrightarrow Y$, $j_\infty : Y - Y_\infty \hookrightarrow Y$ be open embeddings. Then $\Delta(\dot{s}^{-1})_{s\mathcal{L}} \cong j_{0!}\mathcal{K}$ for some rank-one tame local system $\mathcal{K}$ on $Y - Y_0$, and $\mathrm{inv}^* R_{\dot{s}}^* \nabla(\dot{s})_{\mathcal{L}} \cong j_{\infty *}\mathcal{K}'$ for some rank-one tame local system $\mathcal{K}'$ on $Y - Y_\infty$ (tame means the corresponding representation of the fundamental group factors through the tame fundamental group). The tensor product $j_{0!}\mathcal{K} \otimes j_{\infty *}\mathcal{K}'$ descends to a complex $\mathcal{G}$ on $\mathbb{P}^1$, which is a rank-one tame local system $\mathcal{K}''$ (descent of $\mathcal{K} \otimes \mathcal{K}'|_{Y - Y_0 - Y_\infty}$) on $\mathbb{P}^1 - \{0, \infty\}$ with !-extension at 0 and $*$-extension at ∞. Therefore

$$i_{\dot{s}}^*(\Delta(\dot{s}^{-1})_{s\mathcal{L}} \star \nabla(\dot{s})_{\mathcal{L}}) \cong \mathrm{H}^*((G/B)_k, \mathcal{G})$$

is the cone shifted by $[-1]$ of the restriction map

$$\mathrm{H}^*(\mathbb{P}^1_k - \{0, \infty\}, \mathcal{K}'') \to i_\infty^* j_* \mathcal{K}'',$$

where $j : \mathbb{P}^1 - \{0, \infty\} \to \mathbb{P}^1$ and $i_\infty : \{\infty\} \hookrightarrow \mathbb{P}^1$ are the inclusions. Since $\mathcal{K}''$ is a tame local system on $\mathbb{P}^1 - \{0, \infty\} \cong \mathbb{G}_m$, the above restriction map is an isomorphism. This shows that the stalk of $\Delta(\dot{s}^{-1})_{s\mathcal{L}} \star \nabla(\dot{s})_{\mathcal{L}}$ vanishes at any lifting $\dot{s}$ of s. Hence $\Delta(\dot{s}^{-1})_{s\mathcal{L}} \star \nabla(\dot{s})_{\mathcal{L}}$ is concentrated in the closed stratum $U\backslash G_e/U$.

We calculate the stalk of $\Delta(\dot{s}^{-1})_{s\mathcal{L}} \star \nabla(\dot{s})_{\mathcal{L}}$ at e by the same method

$$i_e^*(\Delta(\dot{s}^{-1})_{s\mathcal{L}} \star \nabla(\dot{s})_{\mathcal{L}}) \cong \mathrm{H}^*((G/B)_k, \Delta(\dot{s}^{-1})_{s\mathcal{L}} \otimes \mathrm{inv}^*\nabla(\dot{s})_{\mathcal{L}}).$$

Now $\Delta(\dot{s}^{-1})_{s\mathcal{L}} \otimes \mathrm{inv}^*\nabla(\dot{s})_{\mathcal{L}}$ is the extension by zero of the trivial local system on $\mathbb{P}^1 - \{0\}$ whose stalk at ∞ (image of $\dot{s}$ under $G/U \to \mathbb{P}^1$) is canonically identified with $\overline{\mathbb{Q}}_\ell\langle 2\rangle$. Therefore its cohomology is canonically isomorphic to $\mathrm{H}_c^*(\mathbb{P}^1_k - \{0\}, \overline{\mathbb{Q}}_\ell\langle 2\rangle) \cong \overline{\mathbb{Q}}_\ell$. This gives the canonical isomorphism $\Delta(\dot{s}^{-1})_{s\mathcal{L}} \star \nabla(\dot{s})_{\mathcal{L}} \cong \Delta(e)_{\mathcal{L}}$. The second isomorphism follows from the first one by applying the Verdier duality and Lemma 3.2. □

LEMMA 3.6. *Let $s \in W$ be a simple reflection and $s \notin W_{\mathcal{L}}^\circ$.*

(1) *The natural maps $\Delta(\dot{s})_{\mathcal{L}} \to \mathrm{IC}(\dot{s})_{\mathcal{L}} \to \nabla(\dot{s})_{\mathcal{L}}$ are isomorphisms.*

(2) *The functor*

$$(-) \star \mathrm{IC}(\dot{s})_{\mathcal{L}} : {}_{\mathcal{L}'}\mathcal{D}_{s\mathcal{L}} \to {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$$

is an equivalence of categories with inverse given by $(-) \star \mathrm{IC}(\dot{s}^{-1})_{s\mathcal{L}}$.

(3) *The equivalence* $(-) \star \mathrm{IC}(\dot{s})_{\mathcal{L}}$ *sends* $\Delta(\dot{w})_{s\mathcal{L}}$, $\nabla(\dot{w})_{s\mathcal{L}}$ *and* $\mathrm{IC}(\dot{w})_{s\mathcal{L}} \in {}_{ws\mathcal{L}}\mathcal{D}_{s\mathcal{L}}$ *to* $\Delta(\dot{w}\dot{s})_{\mathcal{L}}$, $\nabla(\dot{w}\dot{s})_{\mathcal{L}}$ *and* $\mathrm{IC}(\dot{w}\dot{s})_{\mathcal{L}} \in {}_{ws\mathcal{L}}\mathcal{D}_{\mathcal{L}}$*, respectively, for any* $w \in {}_{\mathcal{L}'}W_{s\mathcal{L}}$.

Proof. (1) We need to show that $i_e^*\mathrm{IC}(\dot{s})_{\mathcal{L}} = 0$ and $i_e^!\mathrm{IC}(\dot{s})_{\mathcal{L}} = 0$. Replacing G by its Levi subgroup L_s containing T and with roots $\pm\alpha_s$, we reduce to the case where G has semisimple rank one. In this case, there is a central isogeny $\nu : Z^\circ \times \mathrm{SL}_2 \to G$, where Z° is the neutral component of the center of G. Let $\mathcal{L}_1 = (\alpha^\vee)^*\mathcal{L} \in \mathrm{Ch}(\mathbb{G}_m)$. The condition $s \notin W_{\mathcal{L}}^\circ$ is equivalent to $\mathcal{L}_1$ being nontrivial. Identifying $\mathbb{G}_m$ with the diagonal torus $T_1 \subset \mathrm{SL}_2$, $\mathrm{IC}(\dot{s})_{\mathcal{L}_1} \in D^b_{(\mathbb{G}_m\times\mathbb{G}_m, s\mathcal{L}_1\boxtimes\mathcal{L}_1^{-1}),m}(U_1\backslash\mathrm{SL}_2/U_1)$ is defined ($U_1 \subset \mathrm{SL}_2$ is the unipotent upper triangular subgroup). Let $\mathcal{L}_0 = \mathcal{L}|_{Z^\circ}$. Then $\nu^*C(\dot{s})_{\mathcal{L}} \cong \mathcal{L}_0 \boxtimes C(\dot{s})_{\mathcal{L}_1}$ on the open stratum of $Z^\circ \times U_1\backslash\mathrm{SL}_2/U_1$. Since ν is finite, $\mathrm{IC}(\dot{s})_{\mathcal{L}}$ is a direct summand of $\nu_*(\mathcal{L}_0 \boxtimes \mathrm{IC}(\dot{s})_{\mathcal{L}_1})$. By proper base change, it suffices to show that the stalks and costalks of $\mathrm{IC}(\dot{s})_{\mathcal{L}_1}$ vanish along the identity coset of $U_1\backslash\mathrm{SL}_2/U_1$. We identify SL_2/U_1 with $\mathbb{A}^2 - \{0\}$ with SL_2 acting as the standard representation on $\mathbb{A}^2$. The right T_1-translation on SL_2/U_1 is the scaling action of $\mathbb{G}_m$ on $\mathbb{A}^2 - \{0\}$. The open stratum $(\mathrm{SL}_2 - B_1)/U_1 \subset \mathrm{SL}_2/U_1$ is $j : \mathbb{A}^1 \times \mathbb{G}_m \hookrightarrow \mathbb{A}^2 - \{0\}$. A direct calculation shows that $C(\dot{s})_{\mathcal{L}_1} \cong \mathrm{pr}_2^*\mathcal{L}_1$, where $\mathrm{pr}_2 : \mathbb{A}^1 \times \mathbb{G}_m \to \mathbb{G}_m$ is the projection to the second factor. Since $\mathcal{L}_1$ is nontrivial, $\mathrm{IC}(\dot{s})_{\mathcal{L}_1} = j_{!*}C(\dot{s})_{\mathcal{L}_1}$ has a zero stalk and a costalk along the closed stratum $\mathbb{G}_m \times \{0\} \subset \mathbb{A}^2 - \{0\}$. This proves (1).

(2) This follows from (1) and Lemma 3.5.

(3) If $\ell(ws) > \ell(w)$, then by (1), $\Delta(\dot{w})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}} \cong \Delta(\dot{w})_{s\mathcal{L}} \star \Delta(\dot{s})_{\mathcal{L}}$, which is isomorphic to $\Delta(\dot{w}\dot{s})_{\mathcal{L}}$ by Lemma 3.4. If $\ell(ws) < \ell(w)$, then by (1), $\Delta(\dot{w})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}} \cong \Delta(\dot{w})_{s\mathcal{L}} \star \nabla(\dot{s})_{\mathcal{L}}$. By Lemma 3.4, we have $\Delta(\dot{w})_{s\mathcal{L}} \cong \Delta(\dot{w}\dot{s})_{\mathcal{L}} \star \Delta(\dot{s}^{-1})_{s\mathcal{L}}$. Therefore $\Delta(\dot{w})_{s\mathcal{L}} \star \nabla(\dot{s})_{\mathcal{L}} \cong \Delta(\dot{w}\dot{s})_{\mathcal{L}} \star \Delta(\dot{s}^{-1})_{s\mathcal{L}} \star \nabla(\dot{s})_{\mathcal{L}} \cong \Delta(\dot{w}\dot{s})_{\mathcal{L}} \star \Delta(\dot{e})_{\mathcal{L}} \cong \Delta(\dot{w}\dot{s})_{\mathcal{L}}$ by Lemma 3.5. In any case, we have $\Delta(\dot{w})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}} \cong \Delta(\dot{w}\dot{s})_{\mathcal{L}}$.

The proof of $\nabla(\dot{w})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}} \cong \nabla(\dot{w}\dot{s})_{\mathcal{L}}$ is similar.

The equivalence $(-) \star \mathrm{IC}(\dot{s})_{\mathcal{L}}$ then preserves the standard objects and costandard objects. Hence it is t-exact for the perverse t-structure, and sends simple perverse sheaves to simple perverse sheaves. Now for $w \in W$, $\mathrm{IC}(\dot{w})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}}$ is a simple perverse sheaf in ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$. Since $\mathrm{IC}(\dot{w})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}}$ receives a nonzero map from $\Delta(\dot{w})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}} \cong \Delta(\dot{w}\dot{s})_{\mathcal{L}}$, it must be isomorphic to $\mathrm{IC}(\dot{w}\dot{s})_{\mathcal{L}}$. □

3.7. The object $\mathrm{IC}(s)_{\mathcal{L}}$ when $s \in W_{\mathcal{L}}^\circ$.

Suppose $s \in W_{\mathcal{L}}^\circ$. Let α_s be the simple root corresponding to s. Let P_s be the standard parabolic subgroup whose Levi subgroup L_s has roots $\{\pm\alpha_s\}$. Let U^s be the unipotent radical of P_s. Since $(\alpha_s^\vee)^*\mathcal{L}$

is trivial, the local system $\mathcal{L}$ extends to a rank-one character sheaf $\widetilde{\mathcal{L}}$ on L_s by Lemma 2.3. In particular, the stalk of $\widetilde{\mathcal{L}}$ at $e \in L_s$ has a canonical trivialization. We use the same notation $\widetilde{\mathcal{L}}$ to denote its pullback to P_s. The object $\widetilde{\mathcal{L}}\langle 1\rangle \in \mathcal{D}^b_{(T\times T,\mathcal{L}\boxtimes\mathcal{L}^{-1}),m}(U\backslash P_s/U)$, extended by zero, can be viewed as an object of ${}_{\mathcal{L}}\mathcal{D}_{\mathcal{L}}$, and as such, it is isomorphic to $\mathrm{IC}(\dot{s})_{\mathcal{L}}$.

In other words, when $s \in W^\circ_{\mathcal{L}}$, we have a *canonical* object $\mathrm{IC}(s)_{\mathcal{L}} := i_{\leqslant s*}\widetilde{\mathcal{L}}\langle 1\rangle \in {}_{\mathcal{L}}\mathcal{D}_{\mathcal{L}}$ equipped with an isomorphism of its stalk at the identity e with $\overline{\mathbb{Q}}_\ell\langle 1\rangle$. We have $\omega\mathrm{IC}(s)_{\mathcal{L}} \cong \omega\mathrm{IC}(\dot{s})_{\mathcal{L}}$.

Let ${}_{\mathcal{L}'}\mathcal{D}_{\widetilde{\mathcal{L}}} = \mathcal{D}^b_{(T\times L_s,\mathcal{L}'\boxtimes\widetilde{\mathcal{L}}^{-1}),m}(U\backslash G/U^s)$, where the action of $T\times L_s$ on $U\backslash G/U^s$ is given by $(t,h)\cdot g = tgh^{-1}$, $t\in T, h\in L_s, g\in G$. Applying the constructions in Section 2.6, we get an adjoint pair

$$\pi_s^* : {}_{\mathcal{L}'}\mathcal{D}_{\widetilde{\mathcal{L}}} \rightleftarrows {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}} : \pi_{s*}.$$

Since P_s/B is proper, π_{s*} also admits a right adjoint

$$\pi_{s*} : {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}} \rightleftarrows {}_{\mathcal{L}'}\mathcal{D}_{\widetilde{\mathcal{L}}} : \pi_s^!$$

and $\pi_s^! \cong \pi_s^*\langle 2\rangle$.

LEMMA 3.8. *Let $\mathcal{L},\mathcal{L}'\in\mathfrak{o}$ and s be a simple reflection in W such that $s\in W^\circ_{\mathcal{L}}$. Then there is a canonical isomorphism of endofunctors*

$$(-)\star\mathrm{IC}(s)_{\mathcal{L}} \cong \pi_s^*\pi_{s*}(-)\langle 1\rangle \cong \pi_s^!\pi_{s*}(-)\langle -1\rangle \in \mathrm{End}({}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}).$$

Proof. Let $a : U\backslash G \overset{B}{\times} P_s \to U\backslash G$ be a map given by the right action of P_s on G. By the definition of convolution and $\mathrm{IC}(s)_{\mathcal{L}} = i_{\leqslant s*}\widetilde{\mathcal{L}}\langle 1\rangle$, we have for $\mathcal{F}\in {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$

$$\mathcal{F}\star\mathrm{IC}(s)_{\mathcal{L}} \cong a_*(\mathcal{F}\,\widetilde{\boxtimes}\,\widetilde{\mathcal{L}}\langle 1\rangle),$$

where $\mathcal{F}\,\widetilde{\boxtimes}\,\widetilde{\mathcal{L}}\langle 1\rangle$ is the descent of $\mathcal{F}\boxtimes\widetilde{\mathcal{L}}\langle 1\rangle$ to $U\backslash G \overset{B}{\times} P_s$. Comparing with the definition of π_{s*}, we see that $a_*(\mathcal{F}\,\widetilde{\boxtimes}\,\widetilde{\mathcal{L}})$ is exactly the underlying complex of $\pi_{s*}\mathcal{F}$. If we only remember the $(T,\mathcal{L})$-equivariance of $a_*(\mathcal{F}\,\widetilde{\boxtimes}\,\widetilde{\mathcal{L}}\langle 1\rangle)$ (by right translation), it is the same as $\pi_s^*\pi_{s*}\mathcal{F}\langle 1\rangle$. □

COROLLARY 3.9. *Let $s\in W$ be a simple reflection such that $s\in W^\circ_{\mathcal{L}}$. Then the functor*

$$(-)\star\mathrm{IC}(s)_{\mathcal{L}} : {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}} \to {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$$

has a right adjoint also given by $(-)\star\mathrm{IC}(s)_{\mathcal{L}}$.

Proof. Let $\mathcal{F} \in {}_{\mathcal{L}'}\mathcal{D}_{s\mathcal{L}}$ and $\mathcal{G} \in {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$. We have natural isomorphisms by Lemma 3.8:

$$\begin{aligned}\mathrm{Hom}(\mathcal{F} \star \mathrm{IC}(s)_{\mathcal{L}}, \mathcal{G}) &\cong \mathrm{Hom}(\pi_s^* \pi_{s*}\mathcal{F}\langle 1\rangle, \mathcal{G}) \cong \mathrm{Hom}(\pi_{s*}\mathcal{F}\langle 1\rangle, \pi_{s*}\mathcal{G}) \\ &\cong \mathrm{Hom}(\mathcal{F}, \pi_s^! \pi_{s*}\mathcal{G}\langle -1\rangle) \cong \mathrm{Hom}(\mathcal{F}, \mathcal{G} \star \mathrm{IC}(s)_{\mathcal{L}}). \qquad \square\end{aligned}$$

Let s be a simple reflection in W and $s \in W_{\mathcal{L}}^{\circ}$. Recall the rank-one character sheaf $\widetilde{\mathcal{L}}$ on L_s, the category ${}_{\mathcal{L}'}\mathcal{D}_{\widetilde{\mathcal{L}}}$ and the functors (π_s^*, π_{s*}) from Section 3.7.

The Bruhat decomposition gives $G = \sqcup_{\overline{w} \in W/\langle s\rangle} B\overline{w}P_s$. For a lifting $\dot{w} \in N_G(T)$ of $\overline{w} \in W/\langle s\rangle$, we have an isomorphism

$$U\backslash B\overline{w}P_s/U^s \cong \dot{w} \cdot (\mathrm{Ad}(\dot{w}^{-1})U \cap L_s)\backslash L_s.$$

The left translation by $t \in T$ on the left side becomes the left translation of $\mathrm{Ad}(\dot{w}^{-1})t$ on $(\mathrm{Ad}(\dot{w}^{-1})U \cap L_s)\backslash L_s$. From this, we get an equivalence

$$i_{\dot{w}}^* : {}_{\mathcal{L}'}\mathcal{D}(\overline{w})_{\widetilde{\mathcal{L}}} := D^b_{(T\times L_s, \mathcal{L}'\boxtimes\widetilde{\mathcal{L}}^{-1}),m}(U\backslash B\overline{w}P_s/U^s) \cong D^b_{(T,\mathcal{L}'\otimes\overline{w}\mathcal{L}^{-1}),m}(\{\dot{w}\}).$$

Therefore ${}_{\mathcal{L}'}\mathcal{D}(\overline{w})_{\widetilde{\mathcal{L}}} = 0$ unless $\mathcal{L}' = \overline{w}\mathcal{L}$, in which case it is equivalent to $D^b_{T,m}(\{\dot{w}\})$. Let $C(\dot{w})_{\widetilde{\mathcal{L}}} \in {}_{\mathcal{L}'}\mathcal{D}(\overline{w})_{\widetilde{\mathcal{L}}}$ correspond to $\overline{\mathbb{Q}}_\ell\langle \ell'(\overline{w})\rangle$ under $i_{\dot{w}}^*$ (here $\ell'(\overline{w})$ is the maximal length of elements in the coset $\overline{w} \in W/\langle s\rangle$). Let $\mathrm{IC}(\dot{w})_{\widetilde{\mathcal{L}}} \in {}_{\mathcal{L}'}\mathcal{D}_{\widetilde{\mathcal{L}}}$ be the middle extension of $C(\dot{w})_{\widetilde{\mathcal{L}}}$. The isomorphism classes of $\omega C(\dot{w})_{\widetilde{\mathcal{L}}}$ and $\omega\mathrm{IC}(\dot{w})_{\widetilde{\mathcal{L}}}$ depend only on $\overline{w}$. Similarly, one defines the !- and $*$-extensions $\Delta(\dot{w})_{\widetilde{\mathcal{L}}}$ and $\nabla(\dot{w})_{\widetilde{\mathcal{L}}}$ of $C(\dot{w})_{\widetilde{\mathcal{L}}}$.

LEMMA 3.10. *Let s be a simple reflection in W and $s \in W_{\mathcal{L}}^{\circ}$. Suppose $\ell(w) > \ell(ws)$; then*

$$\pi_s^*\mathrm{IC}(\dot{w})_{\widetilde{\mathcal{L}}} \cong \mathrm{IC}(\dot{w})_{\mathcal{L}}.$$

Proof. Unwinding the definition of π_s^*, it is the pullback along the smooth map $\widetilde{\pi}_s : G/\widetilde{B} \to G/\widetilde{P}_s$. Here we take a finite étale central isogeny $\nu : \widetilde{L}_s \to L_s$ such that $\widetilde{\mathcal{L}}$ is defined in terms of a character of $\ker(\nu)$ as in Section 2.1; $\widetilde{P}_s = P_s \times_{L_s} \widetilde{L}_s$ and $\widetilde{B} = B \times_{L_s} \widetilde{L}_s$. Since $\widetilde{\pi}_s$ is a smooth $\mathbb{P}^1$-fibration, $\widetilde{\pi}_s^*$ sends simple perverse sheaves to simple perverse sheaves up to a shift. In particular, $\widetilde{\pi}_s^*\mathrm{IC}(\dot{w})_{\widetilde{\mathcal{L}}}$ is the middle extension of $\widetilde{\pi}_s^* C(\dot{w})_{\widetilde{\mathcal{L}}}\langle 1\rangle$, a shifted local system on $\widetilde{\pi}_s^{-1}(B\overline{w}P_s/\widetilde{P}_s) = (G_w \cup G_{ws})/\widetilde{B}$. By looking at stalks at $\dot{w}$, we have $\widetilde{\pi}_s^* C(\dot{w})_{\widetilde{\mathcal{L}}}|_{G_w/\widetilde{B}} \cong C(\dot{w})_{\mathcal{L}}$. Therefore their middle extensions agree, that is, $\widetilde{\pi}_s^*\mathrm{IC}(\dot{w})_{\widetilde{\mathcal{L}}} \cong \mathrm{IC}(\dot{w})_{\mathcal{L}}$. $\square$

The next proposition shows that the stalks and costalks of $\mathrm{IC}(\dot{w})_{\mathcal{L}}$ have the same parity and purity properties as their nonmonodromic counterparts.

PROPOSITION 3.11. *Let $w \in W$, $\mathcal{L} \in \mathfrak{o}$ and $v \in {}_{w\mathcal{L}}W_{\mathcal{L}}$.*

(1) *The complexes $i_v^*\mathrm{IC}(\dot{w})_{\mathcal{L}}$ and $i_v^!\mathrm{IC}(\dot{w})_{\mathcal{L}}$ are pure of weight zero as objects in ${}_{w\mathcal{L}}\mathcal{D}(v)_{\mathcal{L}}$.*

(2) *The (nonmixed) complexes $i_v^*\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ and $i_v^!\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ are isomorphic to direct sums of $\underline{C}(v)_{\mathcal{L}}[n]$ for $n \equiv \ell(w) - \ell(v) \mod 2$.*

Proof. (1) It is enough to show the statement for the stalks; the costalk statement follows by the Verdier duality. We will prove the stalk statement in (1) together with a weak version of the stalk statement in (2) simultaneously by induction on $\ell(w)$.

Denote by Fr-mod$_0$ the category of finite-dimensional Fr-modules pure of weight zero (see Section 1.10.1). We show by induction on $\ell(w)$ that for any $x \in G$, the stalk

$$i_x^*\mathrm{IC}(\dot{w})_{\mathcal{L}} \in \langle \overline{\mathbb{Q}}_\ell\langle n\rangle \otimes V; n \equiv \ell(w) - \ell(v) \mod 2, V \in \text{Fr-mod}_0\rangle. \tag{3.1}$$

For notation $\langle\cdots\rangle$, see Section 1.10.4. The truth of this statement is independent of the lifting $\dot{w}$ of w.

For $w = e$, this is clear. Suppose it is proven for $\ell(w) < N$. For $\ell(w) = N$, let s be a simple reflection such that $\ell(w) = \ell(ws) + 1$. Over k, by Lemma 3.3, $\mathrm{IC}(\dot{w})_{\mathcal{L}}$ is a direct summand of $\mathrm{IC}(\dot{w}\dot{s}^{-1})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}}$. When the situation is over $\mathbb{F}_q$, although we do not know *a priori* that $\mathrm{IC}(\dot{w})_{\mathcal{L}}$ is a direct summand of $\mathrm{IC}(\dot{w}\dot{s}^{-1})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}}$ over $\mathbb{F}_q$, its stalks are subquotients of stalks of $\mathrm{IC}(\dot{w}\dot{s}^{-1})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}}$ as Fr-modules (as the perverse Leray spectral sequence for $\mathrm{IC}(\dot{w}\dot{s}^{-1})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}}$ degenerates at E_2 by the decomposition theorem). Therefore it suffices to show that the stalks of $\mathrm{IC}(\dot{w}\dot{s}^{-1})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}}$ lie in $\langle \overline{\mathbb{Q}}_\ell\langle n\rangle \otimes V; n \equiv \ell(w) - \ell(v) \mod 2, V \in \text{Fr-mod}_0\rangle$.

By inductive hypothesis for ws, $\mathrm{IC}(\dot{w}\dot{s}^{-1})_{s\mathcal{L}}$ lies in

$$\in \langle \Delta(\dot{v})_{s\mathcal{L}}\langle n\rangle \otimes V; v \in {}_{w\mathcal{L}}W_{s\mathcal{L}}, n \equiv \ell(w) - \ell(v) \mod 2, V \in \text{Fr-mod}_0\rangle.$$

Therefore $\mathrm{IC}(\dot{w}\dot{s}^{-1})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}}$ lies in

$$\langle \Delta(\dot{v})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}}\langle n\rangle \otimes V, v \in {}_{w\mathcal{L}}W_{s\mathcal{L}}, n \equiv \ell(w) - \ell(v) \mod 2, V \in \text{Fr-mod}_0\rangle.$$

We will show that the stalks of $\Delta(\dot{v})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}}$ are either zero or of the form $\overline{\mathbb{Q}}_\ell\langle \ell(vs)\rangle \otimes V$ for some one-dimensional $V \in \text{Fr-mod}_0$, which would finish the induction step.

If $s \notin W_{\mathcal{L}}^\circ$, by Lemma 3.6(3), we have $\Delta(\dot{v})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}} \cong \Delta(\dot{v}\dot{s})_{\mathcal{L}}$ and obviously satisfies the desired stalk property.

If $s \in W^{\circ}_{\mathcal{L}}$, by Lemma 3.8, we have $\Delta(\dot{v})_{\mathcal{L}} \star \mathrm{IC}(s)_{\mathcal{L}} \cong \pi_s^*\pi_{s*}\Delta(\dot{v})_{\mathcal{L}}\langle 1\rangle$. We use the notation from Section 3.7. We first consider the case where $\ell(vs) > \ell(v)$. In this case, BvB/B maps isomorphically to $B\overline{v}P_s/P_s$; therefore $\pi_{s*}\Delta(\dot{v})_{\mathcal{L}} \cong \Delta(\dot{v})_{\tilde{\mathcal{L}}}\langle -1\rangle$, whose nonzero stalks are of the form $\overline{\mathbb{Q}}_\ell\langle \ell(v)\rangle \otimes V$ for $V \in \text{Fr-mod}_0$ of dimensional one. Therefore, the nonzero stalks of $\pi_s^*\pi_{s*}\Delta(\dot{v})_{\mathcal{L}}\langle 1\rangle$ are of the form $\overline{\mathbb{Q}}_\ell\langle \ell(v)+1\rangle \otimes V = \overline{\mathbb{Q}}_\ell\langle \ell(vs)\rangle \otimes V$ for one-dimensional $V \in \text{Fr-mod}_0$.

Finally, we arrive at the case $\ell(vs) < \ell(v)$. We have $\Delta(\dot{v})_{\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}} \cong \Delta(\dot{v}\dot{s}^{-1})_{\mathcal{L}} \star \Delta(\dot{s})_{\mathcal{L}} \star \mathrm{IC}(s)_{\mathcal{L}}$. A calculation inside of L_s gives $\Delta(\dot{s})_{\mathcal{L}} \star \mathrm{IC}(s)_{\mathcal{L}} \cong \pi_s^*\pi_{s*}\Delta(\dot{s})_{\mathcal{L}}\langle 1\rangle \cong \mathrm{IC}(\dot{s})_{\mathcal{L}}\langle -1\rangle$. Therefore $\Delta(\dot{v})_{\mathcal{L}} \star \mathrm{IC}(s)_{\mathcal{L}} \cong \Delta(\dot{v}\dot{s}^{-1})_{\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}}\langle -1\rangle$. We are back to the previous case (applied to vs in place of v) to conclude that the nonzero stalks of $\Delta(\dot{v}\dot{s}^{-1})_{\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}}\langle -1\rangle$ are of the form $\overline{\mathbb{Q}}_\ell\langle \ell(v)\rangle\langle -1\rangle \otimes V = \overline{\mathbb{Q}}_\ell\langle \ell(vs)\rangle \otimes V$ for one-dimensional $V \in \text{Fr-mod}_0$. This completes the induction step for proving (3.1).

(2) By (1), $i_v^*\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ and $i_v^!\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ are successive extensions of $\underline{C}(v)_{\mathcal{L}}[n]$ for $n \equiv \ell(w) - \ell(v) \mod 2$. However, there are no nontrivial extensions between $\underline{C}(v)_{\mathcal{L}}$ and $\underline{C}(v)_{\mathcal{L}}[2m]$ ($m \in \mathbb{Z}$) in ${}_{v\mathcal{L}}\underline{\mathcal{D}}(v)_{\mathcal{L}} \cong D^b_{\Gamma(v)_k}(\mathrm{pt}_k)$, because $\mathrm{H}^{\mathrm{odd}}_{\Gamma(v)_k}(\mathrm{pt}_k) = 0$. Therefore $i_v^*\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ and $i_v^!\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ are direct sums of $\underline{C}(v)_{\mathcal{L}}[n]$ for $n \equiv \ell(w) - \ell(v) \mod 2$. ☐

The above proposition will be strengthened in Proposition 9.10 to include Frobenius semisimplicity of stalks and costalks of $\mathrm{IC}(\dot{w})_{\mathcal{L}}$.

Corollary 3.12. *Let $\mathcal{F}, \mathcal{G} \in {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ be semisimple complexes. Then*

$$\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G}) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(\mathcal{F}, \mathcal{G}[n])$$

admits an increasing filtration $F_{\leqslant v}$ by Fr-*submodules indexed by $v \in {}_{\mathcal{L}'}W_{\mathcal{L}}$ (with its partial order inherited from W) such that the associated graded satisfies*

$$\mathrm{Gr}^F_v \mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}^\bullet(i_v^*\mathcal{F}, i_v^!\mathcal{G}). \tag{3.2}$$

Moreover, this filtration is functorial in $\mathcal{F}$ and $\mathcal{G}$.

Proof. The Schubert stratification gives a filtration on $\mathcal{G}$ with $\mathrm{Gr}_v\mathcal{G} \cong i_{v*}i_v^!\mathcal{G}$. We get a corresponding filtered complex structure on $\mathbf{R}\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ with associated graded pieces quasi-isomorphic to $\mathbf{R}\mathrm{Hom}(i_v^*\mathcal{F}, i_v^!\mathcal{G})$. We need to show that the spectral sequence converging to $\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G})$ corresponding to this filtration degenerates at E_1. For this, it suffices to work in the nonmixed category ${}_{\mathcal{L}'}\underline{\mathcal{D}}_{\mathcal{L}}$, and we may assume $\mathcal{F} = \underline{\mathrm{IC}}(w)_{\mathcal{L}}$ and $\mathcal{G} = \underline{\mathrm{IC}}(w')_{\mathcal{L}}$.

Proposition 3.11(2) implies that $i_v^*\mathcal{F}$ is a direct sum of $\underline{C}(v)_{\mathcal{L}}[n]$ for $n \equiv \ell(w) - \ell(v) \mod 2$ and $i_v^!\underline{\mathcal{G}}$ is a direct sum of $\underline{C}(v)_{\mathcal{L}}[n]$ for $n \equiv \ell(w') - \ell(v)$

mod 2. Therefore $\mathbf{R}\mathrm{Hom}(i_v^*\mathcal{F}, i_v^!\mathcal{G})$ is isomorphic to a direct sum of even shifts of $\mathbf{R}\mathrm{Hom}(\underline{C}(v)_{\mathcal{L}}, \underline{C}(v)_{\mathcal{L}})[\ell(w')-\ell(w)] \cong \mathrm{H}^*_{\Gamma(v)_k}(\mathrm{pt}_k)[\ell(w')-\ell(w)]$. Therefore $\mathrm{Hom}^\bullet(i_v^*\mathcal{F}, i_v^!\mathcal{G})$ is concentrated in degrees of a fixed parity independent of v and hence the degeneration at E_1. □

3.13. Monodromic version of the Hecke algebra. Recall the monodromic version of the Hecke algebra for W with monodromy in $\mathfrak{o}$ defined in [24, 1.4]. Let $\mathbf{H}_\mathfrak{o}$ be the unital associative $\mathbb{Z}[v, v^{-1}]$-algebra with generators $T_w (w \in W)$ and $1_\mathcal{L} (\mathcal{L} \in \mathfrak{o})$ and relations

$$\begin{aligned} 1_{\mathcal{L}}1_{\mathcal{L}'} &= \delta_{\mathcal{L},\mathcal{L}'}1_{\mathcal{L}}, \quad \text{for } \mathcal{L}, \mathcal{L}' \in \mathfrak{o};\\ T_wT_{w'} &= T_{ww'}, \quad \text{if } w, w' \in W \text{ and } \ell(ww') = \ell(w) + \ell(w'); \qquad (3.3)\\ T_w1_{\mathcal{L}} &= 1_{w\mathcal{L}}T_w, \quad \text{for } w \in W, \mathcal{L} \in \mathfrak{o};\\ T_s^2 &= v^2T_1 + (v^2-1)\sum_{\mathcal{L}; s\in W_{\mathcal{L}}^\circ} T_s1_{\mathcal{L}}, \ \text{for simple reflections } s \in W; \qquad (3.4)\\ T_1 &= 1 = \sum_{\mathcal{L}\in\mathfrak{o}} 1_{\mathcal{L}}. \end{aligned}$$

The algebra $\mathbf{H}_\mathfrak{o}$ is closely related to the algebra introduced by Yokonuma [30], as explained in [17, 35.3, 35.4]. Note that $\{T_w1_\mathcal{L}; (w, \mathcal{L}) \in W \times \mathfrak{o}\}$ is a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathbf{H}_\mathfrak{o}$. For $w \in W$, we set $\widetilde{T}_w = v^{-\ell(w)}T_w \in \mathbf{H}_\mathfrak{o}$. There is a unique involution$^-$: $\mathbf{H}_\mathfrak{o} \to \mathbf{H}_\mathfrak{o}$ defined by $\overline{v^mT_w1_\mathcal{L}} = v^{-m}T_{w^{-1}}^{-1}1_\mathcal{L}$ for any $(w, \mathcal{L}) \in W \times \mathfrak{o}$ and any $m \in \mathbb{Z}$.

For any $(w, \mathcal{L}) \in W \times \mathfrak{o}$, there is a unique element $c_{w,\mathcal{L}} \in \mathbf{H}_\mathfrak{o}$ such that

- $\overline{c_{w,\mathcal{L}}} = c_{w,\mathcal{L}}$;
- $c_{w,\mathcal{L}} = \sum_{y\in W} p_{y,\mathcal{L};w,\mathcal{L}}\widetilde{T}_y1_\mathcal{L}$, where $p_{y,\mathcal{L};w,\mathcal{L}} \in v^{-1}\mathbb{Z}[v^{-1}]$ if $y \neq w$, and $p_{w,\mathcal{L};w,\mathcal{L}} = 1$.

The elements $\{c_{w,\mathcal{L}}\}_{(w,\mathcal{L})\in W\times\mathfrak{o}}$ form a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathbf{H}_\mathfrak{o}$ called the *canonical basis*. This is analogous to the basis $\{C_w\}$ introduced in [11, Theorem 1.1].

Let $\mathcal{D}_\mathfrak{o} = \oplus_{\mathcal{L}',\mathcal{L}\in\mathfrak{o}}({}_{\mathcal{L}'}\mathcal{D}_\mathcal{L})$. The Grothendieck group $K_0(\mathcal{D}_\mathfrak{o})$ is a $\mathbb{Z}[v, v^{-1}]$-module, where the action of v is given by $\langle -1\rangle$. As in [22, 2.9], there is a unique $\mathbb{Z}[v, v^{-1}]$-linear map

$$\gamma : K_0(\mathcal{D}_\mathfrak{o}) \to \mathbf{H}_\mathfrak{o}$$

sending $\mathcal{F} \in {}_{\mathcal{L}'}\mathcal{D}_\mathcal{L}$ to the element $\sum_{w\in {}_{\mathcal{L}'}W_\mathcal{L}} A_{w,\mathcal{F}}(v)T_w1_\mathcal{L}$, where $A_{w,\mathcal{F}}(v) \in \mathbb{Z}[v, v^{-1}]$ is the virtual Poincaré polynomial of the stalk $i_{\dot{w}}^*\mathcal{F}$, that is, $A_{w,\mathcal{F}}(v) = \sum_{i,j\in\mathbb{Z}}(-1)^i(\dim \mathrm{Gr}_j^W \mathrm{H}^i i_{\dot{w}}^*\mathcal{F})(-v)^j$.

By construction, $\gamma(\Delta(\dot{w})_{\mathcal{L}}) = \widetilde{T}_w 1_{\mathcal{L}}$. Under γ, $\mathrm{IC}(\dot{w})_{\mathcal{L}}$ is sent to $c_{w,\mathcal{L}}$ for $(w, \mathcal{L}) \in W \times \mathfrak{o}$. Indeed, the purity property proved in Proposition 3.11 gives the degree bound for $p_{y,\mathcal{L};w,\mathcal{L}}$ needed to characterize $c_{w,\mathcal{L}}$.

By [22, 2.10], γ is a ring homomorphism.

4. Blocks

In this section, we give a decomposition of ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ into a direct sum of full triangulated subcategories called blocks. First, we need some preparation on Weyl groups.

4.1. Blocks in ${}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$.

Let $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$. Denote

$$ {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}} = {}_{\mathcal{L}'}W_{\mathcal{L}}/W^{\circ}_{\mathcal{L}} = W^{\circ}_{\mathcal{L}'}\backslash {}_{\mathcal{L}'}W_{\mathcal{L}}. $$

Each element $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ is called a *block* of ${}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, and it inherits a partial order restricted from the Bruhat order in W.

Let $\mathcal{L}, \mathcal{L}'$ and $\mathcal{L}'' \in \mathfrak{o}$. Let $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $\gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$. Then the set $\gamma \cdot \beta := \{w_1 w_2 | w_1 \in \gamma, w_2 \in \beta\}$ is equal to $W^{\circ}_{\mathcal{L}''} w_1 w_2 = w_1 W^{\circ}_{\mathcal{L}'} w_2 = w_1 w_2 W^{\circ}_{\mathcal{L}}$ (for any $w_1 \in \gamma$, $w_2 \in \beta$), which defines an element in ${}_{\mathcal{L}''}\underline{W}_{\mathcal{L}}$. This defines a map

$$ (-) \cdot (-) : {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'} \times {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}} \to {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}}, $$

which is associative in the obvious sense.

LEMMA 4.2. *Each block $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ contains a unique minimal element w^{β} and a unique maximal element w_{β} under its partial order. The minimal element w^{β} (respectively, maximal element w_{β}) is characterized by the property that $w^{\beta}(\Phi^{+}_{\mathcal{L}}) \subset \Phi^{+}$ (respectively, $w_{\beta}(\Phi^{+}_{\mathcal{L}}) \subset \Phi^{-}$).*

Proof. In [13, Lemma 1.9(i)], it is shown that each β contains a unique minimal length (hence minimal) element w^{β} characterized by the stated property. Let $\leqslant_{W^{\circ}_{\mathcal{L}}}$ be the Bruhat order on $W^{\circ}_{\mathcal{L}}$ induced by the positive roots $\Phi^{+}_{\mathcal{L}}$; see Section 2.4. By [13, Lemma 1.9(ii)], if $v \leqslant_{W^{\circ}_{\mathcal{L}}} v'$, then $w^{\beta} v \leqslant w^{\beta} v'$. Therefore, if we write $w_{\mathcal{L},0}$ for the longest (and maximal) element in $W^{\circ}_{\mathcal{L}}$, $w^{\beta} w_{\mathcal{L},0}$ is the unique maximal element in β. Clearly, it is characterized by the stated property. □

COROLLARY 4.3. *For $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $\gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$, we have the equalities in W*

$$ w^{\gamma} w^{\beta} = w^{\gamma\beta}, \quad w^{\gamma} w_{\beta} = w_{\gamma\beta}, \quad w_{\gamma} w^{\beta} = w_{\gamma\beta}. $$

Proof. Let us prove the first equality and the proof of the rest is similar. To show $w^\gamma w^\beta$ is the minimal element in the block $\gamma\beta$, by the criterion in Lemma 4.2, it suffices to show that $w^\gamma w^\beta(\Phi^+_{\mathcal{L}}) \subset \Phi^+$. Since w^β is minimal in β, $w^\beta(\Phi^+_{\mathcal{L}}) \subset \Phi^+ \cap \Phi_{\mathcal{L}'} = \Phi^+_{\mathcal{L}'}$ (we are using (2.3)), and indeed equality holds. Then by the same argument, $w^\gamma w^\beta(\Phi^+_{\mathcal{L}}) \subset w^\gamma(\Phi^+_{\mathcal{L}'}) \subset \Phi^+$, which shows that $w^\gamma w^\beta$ is minimal in the block $\gamma\beta$. ☐

COROLLARY 4.4. *Let $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$. Then $w \mapsto w^\beta w w^{\beta,-1}$ gives an isomorphism of* Coxeter groups $W^\circ_{\mathcal{L}} \xrightarrow{\sim} W^\circ_{\mathcal{L}'}$.

Proof. Since $w^\beta \Phi^+_{\mathcal{L}} \subset \Phi^+ \cap \Phi_{\mathcal{L}'} = \Phi^+_{\mathcal{L}'}$, w^β sends simple roots in $\Phi_{\mathcal{L}}$ to simple roots in $\Phi_{\mathcal{L}'}$. Hence conjugation by w^β sends simple reflections in $W^\circ_{\mathcal{L}}$ to simple reflections in $W^\circ_{\mathcal{L}'}$. ☐

4.5. The groupoid Ξ. Let Ξ be the groupoid whose object set is $\mathfrak{o}$, and the morphism set ${}_{\mathcal{L}'}\Xi_{\mathcal{L}} := \mathrm{Hom}_\Xi(\mathcal{L}, \mathcal{L}') = \{w^\beta | \beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}\}$. Clearly, ${}_{\mathcal{L}'}\Xi_{\mathcal{L}}$ is in bijection with ${}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, and we often make the identification ${}_{\mathcal{L}'}\Xi_{\mathcal{L}} \xrightarrow{\sim} {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$. The composition map is defined by the multiplication in W since $w^\gamma w^\beta = w^{\gamma\beta}$ by Corollary 4.3.

Let $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$. For $w \in \beta$, there is a unique $v \in W^\circ_{\mathcal{L}}$ such that $w = w^\beta v$. Define

$$\ell_\beta(w) = \ell_{\mathcal{L}}(v), \tag{4.1}$$

where $\ell_{\mathcal{L}}$ is the length function of the Coxeter group $W^\circ_{\mathcal{L}}$, as defined in (2.2).

The following lemma is a slight generalization of [**14**, Lemma 5.3].

LEMMA 4.6. *Let $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $w \in \beta$.*

(1) $\ell_\beta(w) = \#\{\alpha \in \Phi^+_{\mathcal{L}} | w\alpha < 0\}$.

(2) *For $\gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$, we have*

$$\ell_{\gamma\beta}(w^\gamma w) = \ell_\beta(w).$$

(3) *Write w into a product of simple reflections in W (not necessarily reduced), $w = s_{i_N} \cdots s_{i_2} s_{i_1}$. Let $\mathcal{L}_0 := \mathcal{L}$ and $\mathcal{L}_j = s_{i_j} \cdots s_{i_1}(\mathcal{L})$ for $j \geqslant 1$. Then*

$$\ell_\beta(w) \leqslant \#\{1 \leqslant j \leqslant N | s_{i_j} \in W^\circ_{\mathcal{L}_{j-1}}\}. \tag{4.2}$$

(4) *If in (3) $s_{i_N} \cdots s_{i_2} s_{i_1}$ is reduced, then equality in (4.2) holds.*

Proof. (1) Write $w = w^\beta v$ for $v \in W^\circ_{\mathcal{L}}$. Since w^β sends $\Phi^+_{\mathcal{L}}$ (respectively $\Phi^-_{\mathcal{L}}$) to positive (respectively negative) roots, for $\alpha \in \Phi^+_{\mathcal{L}}$, $w^\beta v\alpha < 0$ if and only if $v\alpha < 0$. Therefore, $\#\{\alpha \in \Phi^+_{\mathcal{L}} | w\alpha < 0\} = \#\{\alpha \in \Phi^+_{\mathcal{L}} | v\alpha < 0\} = \ell_{\mathcal{L}}(v) = \ell_\beta(w)$.

(2) Write $w = w^\beta v$ for $v \in W^\circ_{\mathcal{L}}$. Then $w^\gamma w = (w^\gamma w^\beta)v$. By Corollary 4.3, $w^\gamma w^\beta = w^{\gamma\beta}$; we have $\ell_{\gamma\beta}(w^\gamma w) = \ell_{\mathcal{L}}(v) = \ell_\beta(w)$.

(3) Let $\underline{w}$ be the sequence of simple reflections $(s_{i_N}, \dots, s_{i_2}, s_{i_1})$. Denote the right side of (4.2) by $L(\underline{w})$. We argue by induction on the length N of the sequence $\underline{w}$.

For $N = 0$, the statement is clear. Suppose the statement is proved for all $\underline{w}$ of length $< N$. Let $s = s_{i_N}$ and $\underline{w}' = (s_{i_{N-1}}, \dots, s_{i_2}, s_{i_1})$, $w' = s_{i_{N-1}} \cdots s_{i_2} s_{i_1}$ so that $w = sw'$. Note that $\mathcal{L}_{N-1} = w'\mathcal{L} = s\mathcal{L}'$. Let $\beta' \in {}_{s\mathcal{L}'}\underline{W}_{\mathcal{L}}$ be the block containing w'. By inductive hypothesis, we have $\ell_{\beta'}(w') \leqslant L(\underline{w}')$. We have two cases:

Case 1: $s \notin W^\circ_{s\mathcal{L}'}$. In this case, $L(\underline{w}) = L(\underline{w}')$. On the other hand, s is minimal in its block $\gamma \in {}_{\mathcal{L}'}\underline{W}_{s\mathcal{L}'}$. By part (2), $\ell_\beta(w) = \ell_\beta(sw') = \ell_{\beta'}(w')$. Therefore $\ell_\beta(w) = \ell_{\beta'}(w') \leqslant L(\underline{w}') = L(\underline{w})$.

Case 2: $s \in W^\circ_{s\mathcal{L}'}$. In this case, $L(\underline{w}) = L(\underline{w}') + 1$. On the other hand, we have $w\Phi^+_{\mathcal{L}} \cap \Phi^- = sw'\Phi^+_{\mathcal{L}} \cap \Phi^- = s(w'\Phi^+_{\mathcal{L}} \cap s\Phi^-)$. Since the only difference between Φ^- and $s\Phi^-$ is that $-\alpha_s$ has been changed to α_s, $\#(w'\Phi^+_{\mathcal{L}} \cap s\Phi^-) \leqslant \#(w'\Phi^+_{\mathcal{L}} \cap \Phi^-) + 1$. By part (1), $\ell_\beta(w) = \#(w\Phi^+_{\mathcal{L}} \cap \Phi^-) = \#(w'\Phi^+_{\mathcal{L}} \cap s\Phi^-) \leqslant \#(w'\Phi^+_{\mathcal{L}} \cap \Phi^-) + 1 = \ell_{\beta'}(w') + 1$. Therefore $\ell_\beta(w) \leqslant \ell_{\beta'}(w') + 1 \leqslant L(\underline{w}') + 1 = L(\underline{w})$.

(4) To prove the equality in the case where $\underline{w}$ is a reduced word, one uses the same inductive argument. The only point that needs modification is in Case 2. Since $\ell(w) = \ell(w') + 1$ in this case, we have $w'^{-1}\alpha_s \in \Phi^+$, or $\alpha_s \in w'\Phi^+$. But since $s \in W^\circ_{w'\mathcal{L}}$, we also have $\alpha_s \in \Phi_{w'\mathcal{L}} = w'\Phi_{\mathcal{L}}$; therefore $\alpha_s \in w'\Phi_{\mathcal{L}} \cap w'\Phi^+ \cap s\Phi^- = w'\Phi^+_{\mathcal{L}} \cap s\Phi^-$. Hence $w\Phi^+_{\mathcal{L}} \cap \Phi^- = s(w'\Phi^+_{\mathcal{L}} \cap s\Phi^-) = s(w'\Phi^+_{\mathcal{L}} \cap \Phi^-) \sqcup \{-\alpha_s\}$. By part (1), $\ell_\beta(w) = \ell_{\beta'}(w') + 1$. Therefore $\ell_\beta(w) = \ell_{\beta'}(w') + 1 = L(\underline{w}') + 1 = L(\underline{w})$. □

4.7. Partial order on a block. For a block $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, we define a partial order $\leqslant_\beta$ on elements of β as follows. Every element in β can be written uniquely as $w^\beta w$ for some $w \in W^\circ_{\mathcal{L}}$. Then we define $w^\beta w' \leqslant_\beta w^\beta w$ if and only if $w' \leqslant_{W^\circ_{\mathcal{L}}} w$ (under the Bruhat order of $W^\circ_{\mathcal{L}}$). For $w', w \in W^\circ_{\mathcal{L}'}$, $w'w^\beta \leqslant_\beta ww^\beta$ if and only if $w' \leqslant_{W^\circ_{\mathcal{L}'}} w$ (using Corollary 4.4).

Later we will need the following result comparing the partial order $\leqslant_\beta$ with the partial order restricted from the Bruhat order of W.

LEMMA 4.8. *Let* $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$.

(1) *If* $\gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$, *left multiplication by* w^{γ} *gives an isomorphism of posets* $(\beta, \leqslant_{\beta}) \xrightarrow{\sim} (\gamma\beta, \leqslant_{\gamma\beta})$.

(2) *If* $\delta \in {}_{\mathcal{L}}\underline{W}_{\mathcal{L}''}$, *right multiplication by* w^{δ} *gives an isomorphism of posets* $(\beta, \leqslant_{\beta}) \xrightarrow{\sim} (\beta\delta, \leqslant_{\beta\delta})$.

(3) *For* $w, w' \in \beta$, *if* $w' \leqslant_{\beta} w$, *then* $w' \leqslant w$.

Proof. (1) follows directly from the definition and Corollary 4.3.

(2) It suffices to show that $x \leqslant_{\beta} y$ implies $xw^{\delta} \leqslant_{\beta\delta} yw^{\delta}$, for the reverse implication can be obtained by inverting δ. Left multiplying by $w^{\beta,-1}$, we reduce to the case $\beta = W^{\circ}_{\mathcal{L}}$, the neutral block. Let $x, y \in W^{\circ}_{\mathcal{L}}$ and $x \leqslant_{W^{\circ}_{\mathcal{L}}} y$, and we show $xw^{\delta} \leqslant_{\delta} yw^{\delta}$. Let $x' = w^{\delta,-1}xw^{\delta}$, $y' = w^{\delta,-1}yw^{\delta} \in W^{\circ}_{\mathcal{L}''}$. By Corollary 4.4, $x \leqslant_{W^{\circ}_{\mathcal{L}}} y$ implies $x' \leqslant_{W^{\circ}_{\mathcal{L}''}} y'$; hence $w^{\delta}x' \leqslant_{\delta} w^{\delta}y'$ by definition. Therefore $xw^{\delta} = w^{\delta}x' \leqslant_{\delta} w^{\delta}y' = yw^{\delta}$.

(3) Induction on $\ell(w)$. The statement is clear for $w = e$. Assume the statement is true for all w with $\ell(w) < N$ (for varying β). Now suppose $\ell(w) = N$. Write $w = w_1 s$ for some simple reflection s such that $\ell(w) = \ell(w_1) + 1$.

If $s \notin W^{\circ}_{\mathcal{L}}$, write $\beta' = \beta s$ and $w' = w'_1 s$. Then $w'_1, w_1 \in \beta'$, and $w'_1 \leqslant_{\beta'} w_1$ since right multiplication by s is an isomorphism of posets $\beta' \xrightarrow{\sim} \beta$ by (2). Applying the inductive hypothesis to w_1, we get $w'_1 \leqslant w_1$. Hence $w' = w'_1 s \leqslant \max\{w_1, w_1 s\} = w$.

If $s \in W^{\circ}_{\mathcal{L}}$, then s is a simple reflection in $W^{\circ}_{\mathcal{L}}$. Since $w' \leqslant_{\beta} w_1 s$, either $w' \leqslant_{\beta} w_1$ or $w' = w'_1 s$ and $w'_1 \leqslant_{\beta} w_1$. In the former case, applying the inductive hypothesis to w_1, we see $w' \leqslant w_1 \leqslant w$. In the latter case, applying the inductive hypothesis to w_1, we get $w'_1 \leqslant w_1$; hence $w' = w'_1 s \leqslant \max\{w_1, w_1 s\} = w$. □

REMARK 4.9. The converse of Lemma 4.8(3) is not true in general. In particular, if $w, w' \in W^{\circ}_{\mathcal{L}}$, then $w' \leqslant w$ does not necessarily imply $w' \leqslant_{W^{\circ}_{\mathcal{L}}} w$.

DEFINITION 4.10. (1) For each $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, let ${}_{\mathcal{L}'}\underline{\mathcal{D}}^{\beta}_{\mathcal{L}}$ be the full triangulated subcategory of ${}_{\mathcal{L}'}\underline{\mathcal{D}}_{\mathcal{L}}$ generated by $\{\underline{\Delta}(w)_{\mathcal{L}}\}_{w\in\beta}$. Let ${}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}} \subset {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ be the preimage of ${}_{\mathcal{L}'}\underline{\mathcal{D}}^{\beta}_{\mathcal{L}}$ under ω. We call ${}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}}$ (respectively, ${}_{\mathcal{L}}\underline{\mathcal{D}}^{\beta}_{\mathcal{L}}$) a *block* of ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$ (respectively, ${}_{\mathcal{L}}\underline{\mathcal{D}}_{\mathcal{L}}$).

(2) When β is the unit coset $W^{\circ}_{\mathcal{L}}$, we denote the block ${}_{\mathcal{L}}\mathcal{D}^{\beta}_{\mathcal{L}}$ (respectively, ${}_{\mathcal{L}}\underline{\mathcal{D}}^{\beta}_{\mathcal{L}}$) by ${}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}$ (respectively, ${}_{\mathcal{L}}\underline{\mathcal{D}}^{\circ}_{\mathcal{L}}$), and call it the *neutral block* .

The terminology ‘block’ is justified by the next proposition.

PROPOSITION 4.11 (Block decomposition). *We have direct sum decompositions of the triangulated categories*

$$ {}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}} = \bigoplus_{\beta\in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}} {}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}}, \quad {}_{\mathcal{L}'}\underline{\mathcal{D}}_{\mathcal{L}} = \bigoplus_{\beta\in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}} {}_{\mathcal{L}'}\underline{\mathcal{D}}^{\beta}_{\mathcal{L}}. \tag{4.3} $$

Proof. We prove the nonmixed statement, and the mixed version follows. Clearly, the subcategories $\{{}_{\mathcal{L}'}\underline{\mathcal{D}}^{\beta}_{\mathcal{L}}\}_{\beta\in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}}$ generate ${}_{\mathcal{L}'}\underline{\mathcal{D}}_{\mathcal{L}}$. It remains to show that if $w_1, w_2 \in {}_{\mathcal{L}'}W_{\mathcal{L}}$ are not in the same right $W^{\circ}_{\mathcal{L}}$ coset, then

$$ \mathbf{R}\mathrm{Hom}(\underline{\Delta}(w_1)_{\mathcal{L}}, \underline{\Delta}(w_2)_{\mathcal{L}}) = 0. \tag{4.4} $$

We prove (4.4) by induction on $\ell(w_2)$. For $\ell(w_2) = 0$, that is, $w_2 = e$, by adjunction, $\mathbf{R}\mathrm{Hom}(\underline{\Delta}(w_1)_{\mathcal{L}}, \underline{\Delta}(e)_{\mathcal{L}}) \cong \mathbf{R}\mathrm{Hom}(\underline{C}(w_1), i^{!}_{w_1}\underline{\Delta}(e)_{\mathcal{L}})$, which vanishes whenever $w_1 \neq e$. This verifies (4.4) for $\ell(w_2) = 0$.

Suppose (4.4) is proved for $\ell(w_2) < n$ $(n > 0)$. Consider the case $\ell(w_2) = n$ and $w_1 \in {}_{\mathcal{L}'}W_{\mathcal{L}} - w_2 W^{\circ}_{\mathcal{L}}$. Let s be a simple reflection such that $\ell(w_2) = \ell(w_2 s) + 1$. By Lemma 3.5, we have $\underline{\Delta}(w_2)_{\mathcal{L}} \star \underline{\nabla}(s)_{s\mathcal{L}} \cong \underline{\Delta}(w_2 s)_{s\mathcal{L}}$. Since $\star\underline{\nabla}(s)_{s\mathcal{L}}$ is an equivalence, we have

$$ \begin{aligned} \mathbf{R}\mathrm{Hom}(\underline{\Delta}(w_1)_{\mathcal{L}}, \underline{\Delta}(w_2)_{\mathcal{L}}) &\cong \mathbf{R}\mathrm{Hom}(\underline{\Delta}(w_1)_{\mathcal{L}} \star \underline{\nabla}(s)_{s\mathcal{L}}, \underline{\Delta}(w_2)_{\mathcal{L}} \star \underline{\nabla}(s)_{s\mathcal{L}}) \\ &\cong \mathbf{R}\mathrm{Hom}(\underline{\Delta}(w_1)_{\mathcal{L}} \star \underline{\nabla}(s)_{s\mathcal{L}}, \underline{\Delta}(w_2 s)_{\mathcal{L}}). \end{aligned} $$

If either $\ell(w_1) = \ell(w_1 s) + 1$ or $s \notin W^{\circ}_{\mathcal{L}}$, then by either Lemma 3.5 or Lemma 3.6(1), we similarly have $\underline{\Delta}(w_1)_{\mathcal{L}} \star \underline{\nabla}(s)_{s\mathcal{L}} \cong \underline{\Delta}(w_1 s)_{s\mathcal{L}}$. Hence $\mathbf{R}\mathrm{Hom}(\underline{\Delta}(w_1)_{\mathcal{L}} \star \underline{\nabla}(s)_{s\mathcal{L}}, \underline{\Delta}(w_2 s)_{\mathcal{L}}) = \mathbf{R}\mathrm{Hom}(\underline{\Delta}(w_1 s)_{s\mathcal{L}}, \underline{\Delta}(w_2 s)_{s\mathcal{L}})$, which vanishes by inductive hypothesis since $\ell(w_2 s) < n$.

It remains to treat the case $s \in W^{\circ}_{\mathcal{L}}$ and $\ell(w_1) = \ell(w_1 s) - 1$. Since $\underline{\nabla}(s)_{\mathcal{L}}$ is in the triangulated subcategory generated by $\underline{\Delta}(s)_{\mathcal{L}}$ and $\underline{\Delta}(e)_{\mathcal{L}}$, $\underline{\Delta}(w_1)_{\mathcal{L}} \star \underline{\nabla}(s)_{\mathcal{L}}$ is in the triangulated subcategory generated by $\underline{\Delta}(w_1)_{\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}} \cong \underline{\Delta}(w_1 s)_{\mathcal{L}}$ and $\underline{\Delta}(w_1)_{\mathcal{L}} \star \underline{\Delta}(e)_{\mathcal{L}} = \underline{\Delta}(w_1)_{\mathcal{L}}$, and we are done again by inductive hypothesis applied to $w_2 s$. □

COROLLARY 4.12. *Let* $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ *and* $w \in \beta$. *Then* $\nabla(\dot{w})_{\mathcal{L}}$ *and* $\mathrm{IC}(\dot{w})_{\mathcal{L}} \in {}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}}$. *In particular,* ${}_{\mathcal{L}'}\underline{\mathcal{D}}^{\beta}_{\mathcal{L}}$ *is also the full triangulated subcategory of* ${}_{\mathcal{L}'}\underline{\mathcal{D}}_{\mathcal{L}}$ *generated either by the collection* $\{\underline{\mathrm{IC}}(w)_{\mathcal{L}}\}_{w\in\beta}$ *or by the collection* $\{\underline{\nabla}(w)_{\mathcal{L}}\}_{w\in\beta}$.

Proof. Since $\underline{\nabla}(w)_{\mathcal{L}}$ and $\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ are indecomposable objects and they admit nonzero maps from $\underline{\Delta}(w)_{\mathcal{L}}$, they must lie in the same summand as $\underline{\Delta}(w)_{\mathcal{L}}$ in decomposition (4.3) for ${}_{\mathcal{L}'}\underline{\mathcal{D}}_{\mathcal{L}}$. □

PROPOSITION 4.13 (Convolution preserves blocks). *Let $\mathcal{L}, \mathcal{L}'$ and $\mathcal{L}'' \in \mathfrak{o}$. Let $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $\gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$. Then*

$${}_{\mathcal{L}''}\mathcal{D}^{\gamma}_{\mathcal{L}'} \star {}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}} \subset {}_{\mathcal{L}''}\mathcal{D}^{\gamma\cdot\beta}_{\mathcal{L}}.$$

Proof. It suffices to show the same statement for the nonmixed categories. By definition, it suffices to show that for any w_1 in a block $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and any other block $\gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$,

$${}_{\mathcal{L}''}\mathcal{D}^{\gamma}_{\mathcal{L}'} \star \underline{\Delta}(w_1)_{\mathcal{L}} \subset {}_{\mathcal{L}''}\underline{\mathcal{D}}^{\gamma\cdot\beta}_{\mathcal{L}}. \tag{4.5}$$

We prove this by induction on $\ell(w_1)$. When $\ell(w_1) = 0$, $w_1 = e$, the statement is clear since $(-) \star \underline{\Delta}(e)_{\mathcal{L}}$ is the identity functor.

Next we consider the case $\ell(w_1) = 1$, that is, $w_1 = s$ is a simple reflection. If $s \notin W^{\circ}_{\mathcal{L}}$, then by Lemma 3.6(3), for any $w_2 \in \gamma$, $\underline{\Delta}(w_2)_{s\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}} \cong \underline{\Delta}(w_2 s)_{\mathcal{L}}$, which implies (4.5).

If $s \in W^{\circ}_{\mathcal{L}}$, it suffices to show that

$$\underline{\Delta}(w_2)_{\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}} \in {}_{\mathcal{L}''}\underline{\mathcal{D}}^{\gamma}_{\mathcal{L}} \tag{4.6}$$

(now w_2 and $w_2 s$ are in the same block denoted by γ). If $\ell(w_2 s) = \ell(w_2)+1$, then by Lemma 3.4, $\underline{\Delta}(w_2)_{\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}} \cong \underline{\Delta}(w_2 s)_{\mathcal{L}}$, which verifies (4.6). If $\ell(w_2 s) = \ell(w_2) - 1$, then by 3.4, we have $\underline{\Delta}(w_2)_{\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}} \cong \underline{\Delta}(w_2 s)_{\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}}$. Since $\underline{\Delta}(s)_{\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}} \in {}_{\mathcal{L}}\underline{\mathcal{D}}(\leqslant s)_{\mathcal{L}}$, which is generated by $\underline{\Delta}(s)_{\mathcal{L}}$ and $\underline{\Delta}(e)_{\mathcal{L}}$, we have $\underline{\Delta}(w_2)_{\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}} \in \langle \underline{\Delta}(w_2 s)_{\mathcal{L}}, \underline{\Delta}(w_2 s)_{\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}}) \rangle = \langle \underline{\Delta}(w_2 s)_{\mathcal{L}}, \underline{\Delta}(w_2)_{\mathcal{L}} \rangle$, which verifies (4.6) in this case. This completes the proof when $\ell(w_1) = 1$.

Now consider the case $\ell(w_1) \geqslant 2$. Write $w_1 = w_1' s$, where s is a simple reflection in W and $\ell(w_1) = \ell(w_1') + 1$. Then $\underline{\Delta}(w_1)_{\mathcal{L}} \cong \underline{\Delta}(w_1')_{s\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}}$. By inductive hypothesis applied to $\ell(w_1')$, we have ${}_{\mathcal{L}''}\mathcal{D}^{\gamma}_{\mathcal{L}'} \star \underline{\Delta}(w_1')_{s\mathcal{L}} \subset {}_{\mathcal{L}''}\underline{\mathcal{D}}^{\gamma\cdot\beta'}_{s\mathcal{L}}$, where $\beta' \in {}_{\mathcal{L}'}\underline{W}_{s\mathcal{L}}$ is the block containing w_1'. By the proven case for simple reflections, ${}_{\mathcal{L}''}\underline{\mathcal{D}}^{\gamma\cdot\beta'}_{s\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}} \subset {}_{\mathcal{L}''}\underline{\mathcal{D}}^{\gamma\cdot\beta}_{\mathcal{L}}$. Combining these two facts, we get (4.5) for w_1. ☐

We will also need the following statement about stalks of $\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ later.

LEMMA 4.14. *Let $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $w \in \beta$. Then $i_v^*\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ and $i_v^!\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ vanish unless $v \in \beta$ and $v \leqslant_{\beta} w$.*

Proof. It is enough to prove the stalk statement, and the costalk statement follows by the Verdier duality.

Induction on $\ell(w)$. The statement is clear for $w = e$. Suppose it is proved for $\ell(w) < N$ (for varying β), and we now prove it for $\ell(w) = N$. Write $w = w's$ for some simple reflection s such that $\ell(w) = \ell(w') + 1$.

If $s \notin W^{\circ}_{\mathcal{L}}$, then $\underline{\mathrm{IC}}(w)_{\mathcal{L}} \cong \underline{\mathrm{IC}}(w')_{s\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}}$ by Lemma 3.6(3). Let $\beta' = \beta s$. Applying inductive hypothesis to $\underline{\mathrm{IC}}(w')_{s\mathcal{L}}$, we see that $\underline{\mathrm{IC}}(w')_{s\mathcal{L}}$ lies in $\langle \underline{\Delta}(v)_{s\mathcal{L}}[n]; v \leqslant_{\beta'} w', n \in \mathbb{Z}\rangle$. Therefore, $\underline{\mathrm{IC}}(w')_{s\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}}$ lies in

$$\langle \underline{\Delta}(v)_{s\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}}[n]; v \leqslant_{\beta'} w', n \in \mathbb{Z}\rangle.$$

Note that $\underline{\Delta}(v)_{s\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}} = \underline{\Delta}(vs)_{\mathcal{L}}$, and $v \leqslant_{\beta'} w'$ implies $vs \leqslant_{\beta} w's = w$. We see that $\underline{\mathrm{IC}}(w)_{\mathcal{L}} \in \langle \underline{\Delta}(v)_{\mathcal{L}}[n]; v \leqslant_{\beta} w, n \in \mathbb{Z}\rangle$; hence $i^{*}_{v}\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ is zero unless $v \leqslant_{\beta} w$.

If $s \in W^{\circ}_{\mathcal{L}}$, then $\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ is a direct summand of $\underline{\mathrm{IC}}(w')_{\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}}$, and we shall prove the stalk statement for the latter. Applying inductive hypothesis to $\underline{\mathrm{IC}}(w')_{\mathcal{L}}$, we see that $\underline{\mathrm{IC}}(w')_{s\mathcal{L}} \in \langle \underline{\Delta}(v)_{s\mathcal{L}}[n]; v \leqslant_{\beta'} w', n \in \mathbb{Z}\rangle$. Therefore, $\underline{\mathrm{IC}}(w')_{\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}} \in \langle \underline{\Delta}(v)_{\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}}[n]; v \leqslant_{\beta'} w', n \in \mathbb{Z}\rangle$. Since $\underline{\Delta}(v)_{\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}}$ only has stalks along G_v and G_{vs}, we see that $i^{*}_{v}(\underline{\mathrm{IC}}(w')_{\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}})$ is nonzero only if either $v \leqslant_{\beta} w'$ or $vs \leqslant_{\beta} w'$. In the former case, $v \leqslant_{\beta} w' \leqslant_{\beta} w$ and in the latter, $vs \leqslant_{\beta} \max_{\beta}\{w', w's\} = w$ (using that s is a simple reflection in $W^{\circ}_{\mathcal{L}}$). In either case, $i^{*}_{v}(\underline{\mathrm{IC}}(w')_{\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}})$ is zero unless $v \leqslant_{\beta} w$. This completes the induction step. □

5. Minimal IC sheaves

In this section, we study the simple perverse sheaves with minimal support in each block, and use them to prove categorical equivalences among different blocks.

5.1. Minimal IC sheaves.

For $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, any object $\xi \in {}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}}$ is called a *minimal IC sheaf* if $\omega\xi \cong \underline{\mathrm{IC}}(w^{\beta})_{\mathcal{L}}$. We denote by ${}_{\mathcal{L}'}\mathfrak{P}^{\beta}_{\mathcal{L}}$ the groupoid of minimal IC sheaves in ${}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}}$. The automorphism group of objects in ${}_{\mathcal{L}'}\mathfrak{P}^{\beta}_{\mathcal{L}}$ is $\overline{\mathbb{Q}}^{\times}_{\ell}$.

PROPOSITION 5.2. *Let $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $\dot{w}^{\beta}$ be a lifting of w^{β}.*

(1) *The natural maps $\Delta(\dot{w}^{\beta})_{\mathcal{L}} \to \mathrm{IC}(\dot{w}^{\beta}) \to \nabla(\dot{w}^{\beta})_{\mathcal{L}}$ are isomorphisms.*

(2) *Let $\mathcal{L}'' \in \mathfrak{o}$ and $\gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$. Then the functor*

$$(-) \star \mathrm{IC}(\dot{w}^{\beta}) : {}_{\mathcal{L}''}\mathcal{D}^{\gamma}_{\mathcal{L}'} \to {}_{\mathcal{L}''}\mathcal{D}^{\gamma\beta}_{\mathcal{L}}$$

is an equivalence with inverse $(-) \star \mathrm{IC}(\dot{w}^{\beta,-1})$. A similar statement is true for left convolution with $\mathrm{IC}(\dot{w}^{\beta})$.

(3) *The equivalence $(-) \star \mathrm{IC}(\dot{w}^{\beta})$ sends $\Delta(\dot{w})_{\mathcal{L}}, \nabla(\dot{w})_{\mathcal{L}}$ and $\mathrm{IC}(\dot{w})_{\mathcal{L}}$ to $\Delta(\dot{w}\dot{w}^{\beta})_{\mathcal{L}}$, $\nabla(\dot{w}\dot{w}^{\beta})_{\mathcal{L}}$ and $\mathrm{IC}(\dot{w}\dot{w}^{\beta})_{\mathcal{L}}$, for all $w \in W$.*

Proof. We prove all the statements by induction on $\ell(w^\beta)$. For $\ell(w^\beta) = 0$, the statements are clear. Suppose the statements are true for $\ell(w^\beta) < n$. Let β be such that $\ell(w^\beta) = n$. Write $w^\beta = w's$ for some simple reflection s such that $\ell(w') = n - 1$. We have $s \notin W^\circ_{\mathcal{L}}$ for otherwise $w' \in \beta$ and it is shorter than w. Let $\beta' \in {}_{\mathcal{L}'}\underline{W}_{s\mathcal{L}}$ be the block containing w'. We must have $w' = w^{\beta'}$ for otherwise $\ell(w^{\beta'}s) \leqslant \ell(w^{\beta'}) + 1 \leqslant \ell(w')$ and $w^{\beta'}s \in \beta$ would be shorter than w^β. Hence $w^\beta = w^{\beta'}s$.

For part (1), it suffices to show its nonmixed version. By Lemma 3.6(3), $\underline{\mathrm{IC}}(w^{\beta'})_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}} \cong \underline{\mathrm{IC}}(w^{\beta'}s)_{\mathcal{L}} \cong \underline{\mathrm{IC}}(w^\beta)_{\mathcal{L}}$. By inductive hypothesis, $\underline{\Delta}(w^{\beta'})_{s\mathcal{L}} \xrightarrow{\sim} \underline{\nabla}(w^{\beta'})_{s\mathcal{L}}$. By Lemma 3.6(1), $\underline{\Delta}(s)_{\mathcal{L}} \xrightarrow{\sim} \underline{\mathrm{IC}}(s)_{\mathcal{L}} \xrightarrow{\sim} \underline{\nabla}(s)_{\mathcal{L}}$. Hence the natural map $\underline{\Delta}(w^\beta)_{\mathcal{L}} \to \underline{\mathrm{IC}}(w^\beta)_{\mathcal{L}}$ can be factorized into isomorphisms $\underline{\Delta}(w^\beta)_{\mathcal{L}} = \underline{\Delta}(w^{\beta'}s)_{\mathcal{L}} \cong \underline{\Delta}(w^{\beta'})_{s\mathcal{L}} \star \underline{\Delta}(s)_{\mathcal{L}} \cong \underline{\mathrm{IC}}(w^{\beta'})_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}} \cong \underline{\mathrm{IC}}(w^{\beta'}s)_{\mathcal{L}} = \underline{\mathrm{IC}}(w^\beta)_{\mathcal{L}}$. By the Verdier duality, the natural map $\underline{\mathrm{IC}}(w^\beta)_{\mathcal{L}} \to \underline{\nabla}(w^\beta)_{\mathcal{L}}$ is also an isomorphism. This proves part (1) for $\underline{\mathrm{IC}}(w^\beta)_{\mathcal{L}}$.

Part (2) follows from (1) together with Lemma 3.5.

Finally, we show part (3). By inductive hypothesis, $\Delta(\dot{w})_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^{\beta'})_{s\mathcal{L}} \cong \Delta(\dot{w}\dot{w}^{\beta'})_{s\mathcal{L}}$. Therefore $\Delta(\dot{w})_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^{\beta'}\dot{s})_{\mathcal{L}} \cong \Delta(\dot{w})_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^{\beta'})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}} \cong \Delta(\dot{w}\dot{w}^{\beta'})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}} \cong \Delta(\dot{w}\dot{w}^{\beta'}\dot{s})_{\mathcal{L}}$, where we use Lemma 3.6(3). Write $\dot{w}^\beta = \dot{w}^{\beta'}\dot{s}t$ for $t \in T(\mathbb{F}_q)$. Then by (2.6), $\mathrm{IC}(\dot{w}^\beta)_{\mathcal{L}} = \mathrm{IC}(\dot{w}^{\beta'}\dot{s})_{\mathcal{L}} \otimes \mathcal{L}_t$, and $\Delta(\dot{w}\dot{w}^\beta)_{\mathcal{L}} = \Delta(\dot{w}\dot{w}^{\beta'}\dot{s})_{\mathcal{L}} \otimes \mathcal{L}_t$. Therefore $\Delta(\dot{w})_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^{\beta'}\dot{s})_{\mathcal{L}} \cong \Delta(\dot{w}\dot{w}^{\beta'}\dot{s})_{\mathcal{L}}$ implies $\Delta(\dot{w})_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^\beta)_{\mathcal{L}} \cong \Delta(\dot{w}\dot{w}^\beta)_{\mathcal{L}}$. The arguments for ∇ and IC are similar. ☐

We may strengthen statement (3) in the above proposition to canonical isomorphisms. To do this, we first need a lemma. The rest of this section is only used in Section 10.

LEMMA 5.3. *Let $\dot{w}, \dot{w}' \in N_G(T)$ be any liftings of $w, w' \in W$, respectively. Let $m_{w,w'} : G_w \overset{B}{\times} G_{w'} \to G$ be the multiplication map. Let B^- be the Borel subgroup of G such that $B \cap B^- = T$, and let U^- be the unipotent radical of B^-. We denote* $\mathrm{Ad}(\dot{w})U$ *by* ${}^wU$.

(1) *The following map is an isomorphism*

$$\begin{aligned} U^- \cap {}^{w^{-1}}U \cap {}^{w'}U &\xrightarrow{\sim} m^{-1}_{w,w'}(\dot{w}\dot{w}') \\ u &\mapsto (\dot{w}u, u^{-1}\dot{w}'). \end{aligned}$$

(2) *We have* $\dim(U^- \cap {}^{w^{-1}}U \cap {}^{w'}U) = \frac{1}{2}(\ell(w) + \ell(w') - \ell(ww'))$. *In particular, $m^{-1}_{w,w'}(\dot{w}\dot{w}')$ is isomorphic to an affine space of dimension* $\frac{1}{2}(\ell(w) + \ell(w') - \ell(ww'))$.

Proof. (1) By the Bruhat decomposition, any $g \in G_w$ can be written uniquely as $\dot{w}ub$, where $u \in {}^{w^{-1}}U \cap U^-$ and $b \in B$; any $g' \in G_{w'}$ can be written uniquely as $b'u'\dot{w}'$, where $b' \in B$ and $u' \in {}^{w'}U \cap U^-$. Using these facts, we have an isomorphism

$$\begin{aligned}\varphi : ({}^{w^{-1}}U \cap U^-) \times B \times ({}^{w'}U \cap U^-) &\xrightarrow{\sim} G_w \overset{B}{\times} G_{w'} \\ (u, b, u') &\mapsto (\dot{w}ub, u'\dot{w}'). \end{aligned} \tag{5.1}$$

We write a point $(g, g') \in m^{-1}_{w,w'}(\dot{w}\dot{w}')$ as $\varphi(u, b, u')$ as above; then $gg' = \dot{w}\dot{w}'$ implies $ubu' = 1$, or $b = u^{-1}u'^{-1}$. Since $b \in B$ and $u^{-1}u'^{-1} \in U^-$, we must have $b = 1$ and $u' = u^{-1}$, and the latter implies $u \in U^- \cap {}^{w^{-1}}U \cap {}^{w'}U$. Therefore, restricting φ to triples (u, b, u'), where $b = 1$ and $u' = u^{-1}$, gives an isomorphism

$$\begin{aligned}U^- \cap {}^{w^{-1}}U \cap {}^{w'}U &\xrightarrow{\sim} m^{-1}_{w,w'}(\dot{w}\dot{w}') \\ u &\mapsto \varphi(u, 1, u^{-1}) = (\dot{w}u, u^{-1}\dot{w}'). \end{aligned}$$

(2) Since $\dim(U^- \cap {}^{w^{-1}}U \cap {}^{w'}U) = \#(\Phi^- \cap w^{-1}\Phi^+ \cap w'\Phi^+)$, and $\ell(w) = \#(\Phi^- \cap w\Phi^+)$ for all $w \in W$, the dimension formula is equivalent to

$$2\#(\Phi^-\cap w^{-1}\Phi^+\cap w'\Phi^+) = \#(\Phi^-\cap w^{-1}\Phi^+)+\#(\Phi^-\cap w'\Phi^+)-\#(\Phi^-\cap ww'\Phi^+). \tag{5.2}$$

We have

$$\begin{aligned}\#(\Phi^- \cap w^{-1}\Phi^+) &= \#(\Phi^- \cap w^{-1}\Phi^+ \cap w'\Phi^+) + \#(\Phi^- \cap w^{-1}\Phi^+ \cap w'\Phi^-), \\ \#(\Phi^- \cap w'\Phi^+) &= \#(\Phi^- \cap w^{-1}\Phi^+ \cap w'\Phi^+) + \#(\Phi^- \cap w^{-1}\Phi^- \cap w'\Phi^+), \\ \#(\Phi^- \cap ww'\Phi^+) &= \#(w^{-1}\Phi^- \cap w'\Phi^+).\end{aligned}$$

Thus to prove (5.2), it is enough to prove

$$\#(\Phi^-\cap w^{-1}\Phi^+\cap w'\Phi^-) + \#(\Phi^-\cap w^{-1}\Phi^-\cap w'\Phi^+) = \#(w^{-1}\Phi^-\cap w'\Phi^+). \tag{5.3}$$

By the change of variable $\alpha \mapsto -\alpha$, we see that

$$\#(\Phi^- \cap w^{-1}\Phi^+ \cap w'\Phi^-) = \#(\Phi^+ \cap w^{-1}\Phi^- \cap w'\Phi^+),$$

so that (5.3) is equivalent to

$$\#(\Phi^+ \cap w^{-1}\Phi^- \cap w'\Phi^+) + \#(\Phi^- \cap w^{-1}\Phi^- \cap w'\Phi^+) = \#(w^{-1}\Phi^- \cap w'\Phi^+),$$

which is obvious. □

The next result will not be used in the rest of the paper.

COROLLARY 5.4. *Let* $\mathfrak{B}$ *be the flag variety of* G*, and* $\mathfrak{O}_w \subset \mathfrak{B} \times \mathfrak{B}$ *the* G*-orbit containing* $(1, \dot w)$*,* $w \in W$*. Let* w_1, w_2, w_3 *be elements of* W *such that* $w_1 w_2 w_3 = 1$*. Let*

$$A_{w_1,w_2,w_3} = \{(B_1, B_2, B_3) \in \mathfrak{B}^3 | (B_1, B_2) \in \mathfrak{O}_{w_1}, (B_2, B_3) \in \mathfrak{O}_{w_2}, (B_3, B_1) \in \mathfrak{O}_{w_3}\}.$$

Then A_{w_1,w_2,w_3} *is a single* G*-orbit under the diagonal* G*-action on* $\mathfrak{B}^3$*, and* $\dim(A_{w_1,w_2,w_3}) = \dim \mathfrak{B} + (\ell(w_1) + \ell(w_2) + \ell(w_3))/2$.

Proof. Since G acts transitively (by simultaneous conjugation) on $\mathfrak{O}_{w_3}$, it is enough to show that for fixed $(B_3, B_1) \in \mathfrak{O}_{w_3}$, the conjugation action of $B_1 \cap B_3$ on $A' := \{B_2 \in \mathfrak{B} | (B_1, B_2) \in \mathfrak{O}_{w_1}, (B_2, B_3) \in \mathfrak{O}_{w_2}\}$ is transitive and that $\dim(A') = \dim \mathfrak{B} + (\ell(w_1) + \ell(w_2) + \ell(w_3))/2 - \dim \mathfrak{O}_{w_3} = (\ell(w_1) + \ell(w_2) - \ell(w_3))/2$. We may assume that $B_1 = B$, $B_3 = {}^{w_3^{-1}}B = {}^{w_1 w_2}B$. Then $A' = \{gB \in G/B | g \in G_{w_1}, g^{-1}\dot w_1 \dot w_2 \in G_{w_2}\}$, and it can be identified with the fiber $m^{-1}_{w_1,w_2}(\dot w_1 \dot w_2)$ considered in Lemma 5.3: $gB \in A'$ corresponds to $(g, g^{-1}\dot w_1 \dot w_2) \in m^{-1}_{w_1,w_2}(\dot w_1 \dot w_2)$. By Lemma 5.3(1), the action of $U \cap {}^{w_1 w_2}U$ on $m^{-1}_{w_1,w_2}(\dot w_1 \dot w_2)$ by $u \cdot (g, g^{-1}\dot w_1 \dot w_2) = (ug, g^{-1}u^{-1}\dot w_1 \dot w_2)$ is already transitive; therefore the action of $B \cap {}^{w_1 w_2}B$ on A' by left translation on gB is also transitive. The dimension formula follows from Lemma 5.3(2). ☐

CONSTRUCTION 5.5. Let $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $w \in W$. We will construct *canonical* isomorphisms

$$\Delta(\dot w)_{\mathcal{L}'} \star \mathrm{IC}(\dot w^\beta)_{\mathcal{L}} \cong \Delta(\dot w \dot w^\beta)_{\mathcal{L}}, \tag{5.4}$$

$$\nabla(\dot w)_{\mathcal{L}'} \star \mathrm{IC}(\dot w^\beta)_{\mathcal{L}} \cong \nabla(\dot w \dot w^\beta)_{\mathcal{L}}, \tag{5.5}$$

$$\mathrm{IC}(\dot w)_{\mathcal{L}'} \star \mathrm{IC}(\dot w^\beta)_{\mathcal{L}} \cong \mathrm{IC}(\dot w \dot w^\beta)_{\mathcal{L}}. \tag{5.6}$$

There are similar canonical isomorphisms for left convolution with $\mathrm{IC}(\dot w^\beta)_{\mathcal{L}}$.

By Proposition 5.2(3), we know that the two sides of the above equations are indeed isomorphic, and such isomorphisms are unique up to a scalar (for the endomorphisms of $\Delta(\dot w \dot w^\beta)_{\mathcal{L}}$, $\nabla(\dot w \dot w^\beta)_{\mathcal{L}}$ and $\mathrm{IC}(\dot w \dot w^\beta)_{\mathcal{L}}$ are scalars).

We first construct the canonical isomorphism (5.4). For this, it suffices to construct a canonical isomorphism between the stalks of the two sides at $\dot w \dot w^\beta$. By the definition of convolution, we have

$$i^*_{\dot w \dot w^\beta}(\Delta(\dot w)_{\mathcal{L}'} \star \mathrm{IC}(\dot w^\beta)_{\mathcal{L}}) \cong \mathrm{H}^*_c(m^{-1}_{w,w^\beta}(\dot w \dot w^\beta)_k, C(\dot w)_{\mathcal{L}'} \overset{B}{\boxtimes} C(\dot w^\beta)_{\mathcal{L}}|_{m^{-1}_{w,w^\beta}(\dot w \dot w^\beta)}).$$

Here $C(\dot w)_{\mathcal{L}'} \overset{B}{\boxtimes} C(\dot w^\beta)_{\mathcal{L}}$ is the descent of $C(\dot w)_{\mathcal{L}'} \boxtimes C(\dot w^\beta)_{\mathcal{L}}$ to $G_w \overset{B}{\times} G_{w^\beta}$, and $m_{w,w^\beta} : G_w \overset{B}{\times} G_{w^\beta} \to G$ is the multiplication map. Using Lemma 5.3(1), we may identify $m_{w,w^\beta}^{-1}(\dot{w}\dot{w}^\beta)$ with the unipotent group $U^- \cap {}^{w^{-1}}U \cap {}^{w^\beta}U$, under which the restriction of $C(\dot{w})_{\mathcal{L}'} \overset{B}{\boxtimes} C(\dot{w}^\beta)_{\mathcal{L}}$ is canonically isomorphic to the constant sheaf $\overline{\mathbb{Q}}_\ell\langle\ell(w)+\ell(w^\beta)\rangle$ since the stalk of $C(\dot{w})_{\mathcal{L}'}$ at $\dot{w}$ and the stalk of $C(\dot{w}^\beta)_{\mathcal{L}}$ at $\dot{w}^\beta$ are canonically isomorphic to $\overline{\mathbb{Q}}_\ell\langle\ell(w)\rangle$ and $\overline{\mathbb{Q}}_\ell\langle\ell(w^\beta)\rangle$, respectively, by construction. Therefore we have a canonical isomorphism of Fr-modules

$$\begin{aligned} i^*_{\dot{w}\dot{w}^\beta}(\Delta(\dot{w})_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^\beta)_{\mathcal{L}}) &\cong \mathrm{H}^*_c(U_k^- \cap {}^{w^{-1}}U_k \cap {}^{w^\beta}U_k, \overline{\mathbb{Q}}_\ell\langle\ell(w)+\ell(w^\beta)\rangle) \\ &\cong \overline{\mathbb{Q}}_\ell\langle\ell(w)+\ell(w^\beta)\rangle\langle-\ell(w)-\ell(w^\beta)+\ell(ww^\beta)\rangle \\ &= \overline{\mathbb{Q}}_\ell\langle\ell(ww^\beta)\rangle \cong i^*_{\dot{w}\dot{w}^\beta}\Delta(\dot{w}\dot{w}^\beta)_{\mathcal{L}}, \end{aligned}$$

where we used the dimension formula for $U^- \cap {}^{w^{-1}}U \cap {}^{w^\beta}U$ proved in Lemma 5.3(2). We define the canonical isomorphism (5.4) to be the one that restricts to the above isomorphism after taking stalks at $\dot{w}\dot{w}^\beta$.

To construct the canonical isomorphism (5.6), we consider the following diagram

$$\begin{tikzcd} \Delta(\dot{w})_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^\beta)_{\mathcal{L}} \arrow[r, "(5.4)"] \arrow[d] & \Delta(\dot{w}\dot{w}^\beta)_{\mathcal{L}} \arrow[d] \\ \mathrm{IC}(\dot{w})_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^\beta)_{\mathcal{L}} \arrow[r, dashed, "\lambda"] & \mathrm{IC}(\dot{w}\dot{w}^\beta)_{\mathcal{L}} \end{tikzcd}$$

where the vertical maps are induced from the canonical maps $\Delta(\dot{w})_{\mathcal{L}} \to \mathrm{IC}(\dot{w})_{\mathcal{L}}$, and the upper horizontal map is the one constructed just now. Since $\mathrm{Hom}(\Delta(\dot{w})_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^\beta)_{\mathcal{L}}, \mathrm{IC}(\dot{w}\dot{w}^\beta)_{\mathcal{L}})$ is one-dimensional, an arbitrary choice of the isomorphism λ (dashed arrow) would make the diagram commutative up to a nonzero scalar. Hence there is a unique choice of the isomorphism λ making the above diagram commutative. This constructs the desired map (5.6). The construction of (5.5) is similar.

WARNING 5.6. For two blocks $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $\gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$, Construction 5.5 gives a canonical isomorphism

$$\mathrm{can}_{\dot{w}^\gamma,\dot{w}^\beta} : \mathrm{IC}(\dot{w}^\gamma)_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^\beta)_{\mathcal{L}} \cong \mathrm{IC}(\dot{w}^\gamma\dot{w}^\beta)_{\mathcal{L}}. \tag{5.7}$$

Let $\delta \in {}_{\mathcal{L}'''}\underline{W}_{\mathcal{L}''}$ be yet another block. We have two isomorphisms between $\mathrm{IC}(\dot{w}^\delta)_{\mathcal{L}''} \star \mathrm{IC}(\dot{w}^\gamma)_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^\beta)_{\mathcal{L}}$ and $\mathrm{IC}(\dot{w}^\delta\dot{w}^\gamma\dot{w}^\beta)_{\mathcal{L}}$ given by first doing convolution $\mathrm{IC}(\dot{w}^\gamma)_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^\beta)_{\mathcal{L}}$ or doing $\mathrm{IC}(\dot{w}^\delta)_{\mathcal{L}''} \star \mathrm{IC}(\dot{w}^\gamma)_{\mathcal{L}'}$:

$$\begin{tikzcd}[column sep=huge] \mathrm{IC}(\dot{w}^\delta)_{\mathcal{L}''} \star \mathrm{IC}(\dot{w}^\gamma)_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^\beta)_{\mathcal{L}} \arrow[r, shift left, "\mathrm{can}_{\dot{w}^\delta\dot{w}^\gamma,\dot{w}^\beta}\circ(\mathrm{can}_{\dot{w}^\delta,\dot{w}^\gamma}\star\mathrm{id})"] \arrow[r, shift right, "\mathrm{can}_{\dot{w}^\delta,\dot{w}^\gamma\dot{w}^\beta}\circ(\mathrm{id}\star\mathrm{can}_{\dot{w}^\gamma,\dot{w}^\beta})"'] & \mathrm{IC}(\dot{w}^\delta\dot{w}^\gamma\dot{w}^\beta)_{\mathcal{L}} \end{tikzcd} . \tag{5.8}$$

However, these two maps are not equal in general, as we will see from the following example.

EXAMPLE 5.7. Consider the case $G = \mathrm{SL}_2$, and $\mathcal{L} \in \mathrm{Ch}(T)$ nontrivial. Let $\dot{s} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ be a lifting of the nontrivial element $s \in W$, and $s\mathcal{L} = \mathcal{L}^{-1}$. In this case, both $\mathrm{IC}(\dot{s})_{\mathcal{L}}$ and $\mathrm{IC}(\dot{e})_{\mathcal{L}} = \delta_{\mathcal{L}}$ are minimal IC sheaves. We claim that the two isomorphisms between $\mathrm{IC}(\dot{s})_{\mathcal{L}} \star \mathrm{IC}(\dot{s}^{-1})_{s\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}}$ and $\mathrm{IC}(\dot{s})_{\mathcal{L}}$ given as in (5.8) differ by a sign.

Indeed, the stalk of $\mathcal{F} = \mathrm{IC}(\dot{s})_{\mathcal{L}} \star \mathrm{IC}(\dot{s}^{-1})_{\mathcal{L}} \star \mathrm{IC}(\dot{s})_{\mathcal{L}}$ at $\dot{s}$ can be calculated from the definition of the convolution as follows. We identify G/U with $\mathbb{A}^2 - \{0\}$, where U is the stabilizer of $e_1 = (1, 0)$. The fiber of the three-fold convolution morphism $G \overset{U}{\times} G \overset{U}{\times} G \to G$ over $\dot{s}$ can be identified with pairs of vectors $(v_1, v_2) \in (\mathbb{A}^2 - \{0\})^2$ via the map $(g_1, g_2, g_3) \mapsto (g_1 e_1, g_1 g_2 e_1)$. The open subset $Y = \{v_1 = (x_1, y_1) \in \mathbb{A}^2 - \{0\}, v_2 = (x_2, y_2) \in \mathbb{A}^2 - \{0\} | y_1 \neq 0, x_2 \neq 0, x_1 y_2 - x_2 y_1 \neq 0\}$ of $(\mathbb{A}^2 - \{0\})^2$ is relevant to our calculation. For any invertible function f on Y, we use $\mathcal{L}_f$ to denote the pullback $f^*\mathcal{L}$. We consider the local system $\mathcal{K} = \mathcal{L}_{-y_1} \mathcal{L}^{-1}_{x_1 y_2 - x_2 y_1} \mathcal{L}_{x_2}$. Let $\mathbb{G}_m \times \mathbb{G}_m$ act on Y by scaling the vectors (x_1, y_1) and (x_2, y_2) separately. Then $\mathcal{K}$ is equivariant under the $\mathbb{G}_m^2$ action on Y and hence descends to a local system on $X = Y/\mathbb{G}_m^2$, which we still denote by $\mathcal{K}$. We have a canonical isomorphism

$$i_{\dot{s}}^* \mathcal{F} \cong \mathrm{H}_c^*(X, \mathcal{K})\langle 3 \rangle.$$

Now $X \hookrightarrow \mathbb{A}^2$ by coordinates $u = x_1/y_1$ and $v = y_2/x_2$, and with image $\mathbb{A}^2 - \{uv = 1\}$. The local system $\mathcal{K} = \mathcal{L}^{-1}_{1-uv}$ on X. Therefore we have canonically

$$i_{\dot{s}}^* \mathcal{F} \cong \mathrm{H}_c^*(X, \mathcal{L}^{-1}_{1-uv})\langle 3 \rangle.$$

The isomorphism $\mathrm{can}_{\dot{e},\dot{s}} \circ (\mathrm{can}_{\dot{s},\dot{s}^{-1}} \star \mathrm{id})$ corresponds to the isomorphism by restriction to the line $v = 0$:

$$i_{v=0}^* : \mathrm{H}_c^*(X, \mathcal{L}^{-1}_{1-uv})\langle 3 \rangle \xrightarrow{\sim} \mathrm{H}_c^*(\mathbb{A}^1_{v=0}, \overline{\mathbb{Q}}_\ell)\langle 3 \rangle \cong \overline{\mathbb{Q}}_\ell\langle 1 \rangle.$$

Here we have used the canonical trivialization of the stalk of $\mathcal{L}$ at 1, and the fundamental class of $\mathbb{A}^1$. Similarly, the other isomorphism $\mathrm{can}_{\dot{s},\dot{e}} \circ (\mathrm{id} \star \mathrm{can}_{\dot{s}^{-1},\dot{s}})$ corresponds to the isomorphism by restriction to the line $u = 0$. Let $\sigma : X \to X$ be the involution $(u, v) \mapsto (v, u)$. Then $\mathcal{L}^{-1}_{1-uv}$ has a canonical σ-equivariant structure such that the σ-action on the stalk at $(0, 0)$ is the identity. This induces an involution σ^* on $\mathrm{H}^*_c(X, \mathcal{L}^{-1}_{1-uv})\langle 3\rangle$, and the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{H}^*_c(X, \mathcal{L}^{-1}_{1-uv})\langle 3\rangle \arrow[rr, "\sigma^*"] \arrow[d, "i^*_{v=0}"'] & & \mathrm{H}^*_c(X, \mathcal{L}^{-1}_{1-uv})\langle 3\rangle \arrow[d, "i^*_{u=0}"] \\
\mathrm{H}^*_c(\mathbb{A}^1_{v=0}, \overline{\mathbb{Q}}_\ell)\langle 3\rangle \arrow[r, equal] & \overline{\mathbb{Q}}_\ell\langle 1\rangle \arrow[r, equal] & \mathrm{H}^*_c(\mathbb{A}^1_{u=0}, \overline{\mathbb{Q}}_\ell)
\end{tikzcd}$$

We claim that σ^* acts on the one-dimensional space $\mathrm{H}^*_c(X, \mathcal{L}^{-1}_{1-uv})$ by -1, which would imply our claim in the beginning of this example.

We compare two traces $\mathrm{Tr}_1 = \mathrm{Tr}(\mathrm{Fr}, \mathrm{H}^*_c(X, \mathcal{L}^{-1}_{1-uv}))$ and $\mathrm{Tr}_2 = \mathrm{Tr}(\sigma^* \circ \mathrm{Fr}, \mathrm{H}^*_c(X, \mathcal{L}^{-1}_{1-uv}))$. Let χ be the character of $\mathbb{F}_q^\times$ corresponding to $\mathcal{L}^{-1}$. By the Lefschetz trace formula, $\mathrm{Tr}_1 = \sum_{u,v\in\mathbb{F}_q, uv\neq 1} \chi(1-uv)$. The fiber of $(u, v) \mapsto a = 1-uv$ has $q-1$ elements over $a \neq 1$, and has $2q-1$ elements over $a = 1$. Therefore, $\mathrm{Tr}_1 = (q-1)\sum_{a\neq 0,1} \chi(a) + (2q-1) = q$ since $\chi \neq 1$. On the other hand, $\sigma^* \circ \mathrm{Fr}$ is the Frobenius for the variety $X' \subset \mathrm{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{A}^1 - \{\mathrm{Nm} = 1\}$, which becomes isomorphic to X over $\mathbb{F}_{q^2}$. Using this interpretation, we have $\mathrm{Tr}_2 = \sum_{u\in\mathbb{F}_{q^2}, \mathrm{Nm}(u)\neq 1} \chi(1-\mathrm{Nm}(u))$. The fiber of the map $\mathbb{F}_{q^2} \ni u \mapsto 1 - \mathrm{Nm}(u) = a \in \mathbb{F}_q$ has $q+1$ elements over $a \neq 1$ and 1 element over $a = 1$. Therefore $\mathrm{Tr}_2 = (q+1)\sum_{a\neq 0,1} \chi(a) + 1 = -q$. This shows $\mathrm{Tr}_1 = -\mathrm{Tr}_2$; hence σ^* acts by -1 on the one-dimensional space $\mathrm{H}^*_c(X, \mathcal{L}^{-1}_{1-uv})$.

5.8. The 3-cocycle. For three composable blocks β, γ, δ, let $\sigma(\dot{w}^\delta, \dot{w}^\gamma, \dot{w}^\beta)$ be the ratio of the two isomorphisms in (5.8) (top over bottom). It is easy to see that $\sigma(\dot{w}^\delta, \dot{w}^\gamma, \dot{w}^\beta)$ depends only on β, γ, δ, so we denote it by $\sigma(w^\delta, w^\gamma, w^\beta)$. Recall the groupoid Ξ defined in Section 4.5. By the pentagon axiom for the associativity of the convolution, the assignment $(w^\delta, w^\gamma, w^\beta) \mapsto \sigma(w^\delta, w^\gamma, w^\beta)$ defines a 3-cocycle $\sigma \in Z^3(\Xi, \overline{\mathbb{Q}}_\ell^\times)$. In other words, for four composable morphisms $w^\epsilon, w^\delta, w^\gamma$ and w^β in Ξ,

$$\sigma(w^\delta, w^\gamma, w^\beta)\sigma(w^{\epsilon\delta}, w^\gamma, w^\beta)^{-1}\sigma(w^\epsilon, w^{\delta\gamma}, w^\beta) \\ \sigma(w^\epsilon, w^\delta, w^{\gamma\beta})^{-1}\sigma(w^\epsilon, w^\delta, w^\gamma) = 1.$$

In [32, Section 4], we will show that σ always takes values in $\{\pm 1\}$. In fact, there is a 3-cocycle $\epsilon_3^W \in Z^3(W, \{\pm 1\})$ canonically attached to the Coxeter group (W, S), and σ is the pullback of ϵ_3^W along the natural map $\Xi \to [\mathrm{pt}/W]$. In [32, Section 5], we will also calculate the cohomology class of σ, which often turns out to be nontrivial.

6. Maximal IC sheaves

6.1. Maximal IC sheaves. Let $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$ and $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$. Recall that w_β is the longest element in the block β. An object $\mathcal{F} \in {}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}}$ is called a *maximal IC sheaf* if $\omega\mathcal{F} \cong \underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}}$.

When $\mathcal{L}, \mathcal{L}'$ are trivial, there is only one block β in $D^b_m(B\backslash G/B)$, $w_\beta = w_0$ is the longest element in W and $\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}} \cong \overline{\mathbb{Q}}_\ell[\dim G/B]$ is a shifted constant sheaf on $B\backslash G/B$. The constant sheaf $\overline{\mathbb{Q}}_\ell$ on $B\backslash G/B$ has two remarkable properties: (a) convolution with it always yields a direct sum of constant sheaves; (b) its stalks and costalks are one-dimensional. Below we will prove analogues of these properties for maximal IC sheaves in each block.

PROPOSITION 6.2. *Let $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $\gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$. Let $w \in \beta$.*

1. *For any $w \in \beta$, the convolution $\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \underline{\mathrm{IC}}(w)_{\mathcal{L}}$ is isomorphic to a direct sum of shifts of $\underline{\mathrm{IC}}(w_{\gamma\beta})_{\mathcal{L}}$.*
2. *The perverse cohomology ${}^p\mathrm{H}^i(\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \underline{\mathrm{IC}}(w)_{\mathcal{L}})$ vanishes unless $-\ell_\beta(w) \leqslant i \leqslant \ell_\beta(w)$.*
3. *There are isomorphisms*

$$
\begin{aligned}
{}^p\mathrm{H}^{\ell_\beta(w)}(\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \underline{\mathrm{IC}}(w)_{\mathcal{L}}) &\cong \underline{\mathrm{IC}}(w_{\gamma\beta})_{\mathcal{L}},\\
{}^p\mathrm{H}^{-\ell_\beta(w)}(\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \underline{\mathrm{IC}}(w)_{\mathcal{L}}) &\cong \underline{\mathrm{IC}}(w_{\gamma\beta})_{\mathcal{L}}.
\end{aligned}
$$

Proof. We prove the statements simultaneously by induction on $\ell(w)$. For $w = e$, the statement is clear.

If $\ell(w) = 1$, w is a simple reflection s.

If $s \notin W^\circ_{\mathcal{L}}$, Lemma 3.6(3) implies that $\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}} \cong \underline{\mathrm{IC}}(w_\gamma s)_{\mathcal{L}}$. By Corollary 4.3, $w_\gamma s = w_{\gamma\beta}$ is the maximal element in $\gamma\beta$; hence $\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}} \cong \underline{\mathrm{IC}}(w_{\gamma\beta})_{\mathcal{L}}$. Note that $\ell_\beta(s) = 0$ in this case, and (2)(3) hold trivially.

If $s \in W^\circ_{\mathcal{L}}$ (hence $\mathcal{L}' = s\mathcal{L} = \mathcal{L}$), then by Lemma 3.10, $\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}} \cong \pi_s^*\underline{\mathrm{IC}}(\overline{w}_\gamma)_{\widetilde{\mathcal{L}}}[1]$; here $\underline{\mathrm{IC}}(\overline{w}_\gamma)_{\widetilde{\mathcal{L}}} = \omega\mathrm{IC}(\dot{w}_\gamma)_{\widetilde{\mathcal{L}}} \in {}_{\mathcal{L}}\underline{\mathcal{D}}_{\widetilde{\mathcal{L}}}$. By Lemma 3.8, $\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}} \cong \pi_s^*\pi_{s*}\pi_s^*\underline{\mathrm{IC}}(\overline{w}_\gamma)_{\widetilde{\mathcal{L}}}[1]$. By the projection formula, $\pi_{s*}\pi_s^*\underline{\mathrm{IC}}(\overline{w}_\gamma)_{\widetilde{\mathcal{L}}} \cong \underline{\mathrm{IC}}(\overline{w}_\gamma)_{\widetilde{\mathcal{L}}} \otimes \mathrm{H}^*(\mathbb{P}^1_k)$ because $\pi_s : G/B \to G/P_s$ is a $\mathbb{P}^1$-fibration. Therefore

$$
\begin{aligned}
\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}} &\cong \pi_s^*\underline{\mathrm{IC}}(\overline{w}_\gamma)_{\widetilde{\mathcal{L}}}[1] \oplus \pi_s^*\underline{\mathrm{IC}}(\overline{w}_\gamma)_{\widetilde{\mathcal{L}}}[-1]\\
&= \underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}}[1] \oplus \underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}}[-1].
\end{aligned}
$$

Note that $\ell_\beta(s) = 1$ in this case, and (2)(3) follow from the above isomorphism. This settles the case $\ell(w) = 1$.

For $\ell(w) > 1$, write $w = w's$ for some simple reflection s such that $\ell(w') = \ell(w) - 1$. Let $\beta' = \beta s \in {}_{\mathcal{L}'}\underline{W}_{s\mathcal{L}}$ so $w' \in \beta'$. We shall first prove the analogues of the statements (1)(2) for $\underline{\mathrm{IC}}(w')_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}}$ instead of $\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ (in statement (2), the range for i is still $[-\ell_\beta(w), \ell_\beta(w)]$).

By inductive hypothesis, $\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \underline{\mathrm{IC}}(w')_{s\mathcal{L}} \cong \underline{\mathrm{IC}}(w_{\gamma\beta'})_{s\mathcal{L}} \otimes V'$ for a graded $\overline{\mathbb{Q}}_\ell$-vector space $V' = \oplus_{n\in\mathbb{Z}} V'_n[-n]$ such that $V'_n = 0$ unless $-\ell_{\beta'}(w') \leqslant n \leqslant \ell_{\beta'}(w')$ and $\dim V'_{\pm\ell_{\beta'}(w')} = 1$. Therefore

$$\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star (\underline{\mathrm{IC}}(w')_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}}) \cong \oplus_n V'_n[-n] \otimes (\underline{\mathrm{IC}}(w_{\gamma\beta'})_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}}). \quad (6.1)$$

If $s \notin W^\circ_{\mathcal{L}}$, we have $\ell_\beta(w) = \ell_{\beta'}(w')$ by the formula for ℓ_β given in Lemma 4.6(4). We also have $\underline{\mathrm{IC}}(w_{\gamma\beta'})_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}} \cong \underline{\mathrm{IC}}(w_{\gamma\beta})$ by Lemma 3.6(3). The statements (1)(2)(3) for $\underline{\mathrm{IC}}(w')_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}}$ follow easily from (6.1).

If $s \in W^\circ_{\mathcal{L}}$, we have $\ell_\beta(w) = \ell_{\beta'}(w') + 1$ by the formula for ℓ_β given in Lemma 4.6(4). By the $w = s$ case already treated in the beginning of the proof, we have $\underline{\mathrm{IC}}(w_{\gamma\beta'})_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}} \cong \underline{\mathrm{IC}}(w_{\gamma\beta})_{\mathcal{L}}[1] \oplus \underline{\mathrm{IC}}(w_{\gamma\beta})_{\mathcal{L}}[-1]$. Therefore $\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \underline{\mathrm{IC}}(w')_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}} \cong \oplus_n (V'_n[-n-1] \oplus V'_n[-n+1]) \otimes \underline{\mathrm{IC}}(w_{\gamma\beta})_{\mathcal{L}} \cong \oplus_{n\in\mathbb{Z}} (V'_{n-1} \oplus V'_{n+1}) \otimes \underline{\mathrm{IC}}(w_{\gamma\beta})_{\mathcal{L}}[-n]$. The statements (1)(2)(3) follow from the known properties of V'.

Finally we deduce the statements (1)(2)(3) for $\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ from the proven statements for $\underline{\mathrm{IC}}(w')_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}}$. When $s \notin W^\circ_{\mathcal{L}}$, we have $\underline{\mathrm{IC}}(w')_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}} \cong \underline{\mathrm{IC}}(w)_{\mathcal{L}}$ by Lemma 3.6(3). Therefore the statements are already proven. Below we deal with the case $s \in W^\circ_{\mathcal{L}}$.

By the decomposition theorem, $\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ is a direct summand of $\underline{\mathrm{IC}}(w')_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}}$. By Lemma 6.3, $\underline{\mathrm{IC}}(w')_{\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}}$ is itself perverse. Hence we can write

$$\underline{\mathrm{IC}}(w')_{\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}} \cong \underline{\mathrm{IC}}(w)_{\mathcal{L}} \oplus \mathcal{P}$$

for some semisimple perverse sheaf $\mathcal{P} \in {}_{\mathcal{L}'}\underline{\mathcal{D}}_{\mathcal{L}}$ with support in $U\backslash G_{<w}/U$. We see that

$$\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \underline{\mathrm{IC}}(w')_{\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}} \cong \underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \underline{\mathrm{IC}}(w)_{\mathcal{L}} \oplus \underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \mathcal{P}.$$

By the proven statements (1)(2)(3) for the left side above, we have

$$\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \underline{\mathrm{IC}}(w)_{\mathcal{L}} \oplus \underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \mathcal{P} \cong \underline{\mathrm{IC}}(w_{\gamma\beta})_{\mathcal{L}} \otimes (\oplus_n V_n[-n]), \quad (6.2)$$

where V_n is a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space, $V_n = 0$ unless $-\ell_\beta(w) \leqslant n \leqslant \ell_\beta(w)$, and $\dim V_{\pm\ell_\beta(w)} = 1$.

Part (2) for $\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \underline{\mathrm{IC}}(w)_{\mathcal{L}}$ is now clear from the degree range on the right side of (6.2).

Part (1). In view of (6.2), each perverse cohomology sheaf of $\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \underline{\mathrm{IC}}(w)_{\mathcal{L}}$ is a direct summand of a direct sum of $\underline{\mathrm{IC}}(w_{\gamma\beta})_{\mathcal{L}}$, and hence itself a direct sum of $\underline{\mathrm{IC}}(w_{\gamma\beta})_{\mathcal{L}}$. By the decomposition theorem, $\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \underline{\mathrm{IC}}(w)_{\mathcal{L}}$ is then a direct sum of shifts of $\underline{\mathrm{IC}}(w_{\gamma\beta})_{\mathcal{L}}$.

Part (3). We claim that $\mathcal{P}$ is a direct sum of $\underline{\mathrm{IC}}(v)_{\mathcal{L}}$ for $v \in \beta$ and $v <_\beta w$. In fact, the argument in the third paragraph of Lemma 4.14 shows that $\underline{\mathrm{IC}}(w')_{\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}}$ only has stalks along G_v for $v \leqslant_\beta w$. Now $\mathcal{P}$ is supported on $U\backslash G_{<w}/U$; its direct summands can only be $\underline{\mathrm{IC}}(v)_{\mathcal{L}}$ for $v \in \beta$ and $v <_\beta w$.

By the above claim, we have $\ell_\beta(v) < \ell_\beta(w)$ for any $\underline{\mathrm{IC}}(v)_{\mathcal{L}}$ that shows up in $\mathcal{P}$. By inductive hypothesis applied to these $\underline{\mathrm{IC}}(v)_{\mathcal{L}}$, we have ${}^p\mathrm{H}^{\pm\ell_\beta(w)}(\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \mathcal{P}) = 0$. Therefore ${}^p\mathrm{H}^{\pm\ell_\beta(w)}(\underline{\mathrm{IC}}(w_\gamma)_{\mathcal{L}'} \star \underline{\mathrm{IC}}(w)_{\mathcal{L}}) \cong \underline{\mathrm{IC}}(w_{\gamma\beta})_{\mathcal{L}} \otimes V_{\pm\ell_\beta(w)} \cong \underline{\mathrm{IC}}(w_{\gamma\beta})_{\mathcal{L}}$. □

LEMMA 6.3. *Let $s \in W$ be a simple reflection and $w' \in W$ be such that $\ell(w's) = \ell(w') + 1$. Then $\underline{\mathrm{IC}}(w')_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}}$ is a perverse sheaf.*

Proof. If $s \notin W^\circ_{\mathcal{L}}$, then $\underline{\mathrm{IC}}(w')_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}} \cong \underline{\mathrm{IC}}(w)_{\mathcal{L}}$ by Lemma 3.6(3).

If $s \in W^\circ_{\mathcal{L}}$, by Lemma 3.8, we have $\underline{\mathrm{IC}}(w')_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}} \cong \pi_s^*\pi_{s*}\underline{\mathrm{IC}}(w')_{s\mathcal{L}}[1]$. It suffices to show that $\pi_{s*}\underline{\mathrm{IC}}(w')_{s\mathcal{L}}$ is perverse in the following sense. Let $\nu : \widetilde{L}_s \to L_s$ be a finite étale isogeny such that $\widetilde{\mathcal{L}}$ is defined via a character of $\ker(\nu)$. Let $\widetilde{P}_s = P_s \times_{L_s} \widetilde{L}_s$ and $\widetilde{B} = B \times_{L_s} \widetilde{L}_s$. We have the projection map $\widetilde{\pi}_s : G/\widetilde{B} \to G/\widetilde{P}_s$. Viewing $\underline{\mathrm{IC}}(w')_{s\mathcal{L}}$ as a complex on the stack $G/\widetilde{B}$, then $\pi_{s*}\underline{\mathrm{IC}}(w')_{s\mathcal{L}}$ as a complex on $G/\widetilde{P}_s$ is simply $\widetilde{\pi}_{s*}\underline{\mathrm{IC}}(w')_{s\mathcal{L}}$. We shall show that $\widetilde{\pi}_{s*}\underline{\mathrm{IC}}(w')_{s\mathcal{L}}$ is a perverse sheaf on $G/\widetilde{P}_s$. Since $\widetilde{\pi}_s$ is smooth of relative dimension 1, $\widetilde{\pi}_s^*[1]$ preserves perverse sheaves, which would imply that $\underline{\mathrm{IC}}(w')_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}} \cong \widetilde{\pi}_s^*\widetilde{\pi}_{s*}\underline{\mathrm{IC}}(w')_{s\mathcal{L}}[1]$ is perverse.

For $v \in W$, let $\overline{v}$ be its image in $W/\langle s\rangle$. Then $G/\widetilde{P}_s = \bigsqcup_{\overline{v}\in W/\langle s\rangle} B\overline{v}P_s/\widetilde{P}_s$ is a stratification of $G/\widetilde{P}_s$, and $\dim B\overline{v}P_s/\widetilde{P}_s = \ell(\overline{v}) := \min\{\ell(v), \ell(vs)\}$. By the Verdier duality, it suffices to show that for any $v \leqslant w'$ and $x \in (B\overline{v}P_s)/\widetilde{P}_s = (G_v \cup G_{vs})/\widetilde{P}_s$, the stalk of $\widetilde{\pi}_{s*}\underline{\mathrm{IC}}(w')_{s\mathcal{L}}$ at x, which is $\mathrm{H}^*(\widetilde{\pi}_s^{-1}(x), \underline{\mathrm{IC}}(w')_{s\mathcal{L}}|_{\widetilde{\pi}_s^{-1}(x)})$, lies in degrees $\leqslant -\ell(\overline{v})$. Note that $\widetilde{\pi}_s^{-1}(x) \cong \mathbb{P}^1$ for any $x \in G/\widetilde{P}_s$.

First, consider the case $v < w'$, and we may assume $vs < v$. Then $\underline{\mathrm{IC}}(w')_{s\mathcal{L}}|_{G_v}$ lies in degrees $\leqslant -\ell(v) - 1$ and $\underline{\mathrm{IC}}(w')_{s\mathcal{L}}|_{G_{vs}}$ lies in degrees $\leqslant -\ell(vs) - 1 = -\ell(v)$. We have $\widetilde{\pi}_s^{-1}(x) \cap G_v/\widetilde{B} \cong \mathbb{A}^1$ and $\widetilde{\pi}_s^{-1}(x) \cap G_{vs}/\widetilde{B} \cong \mathrm{pt}$. Therefore $\mathrm{H}^*(\widetilde{\pi}_s^{-1}(x), \underline{\mathrm{IC}}(w')_{s\mathcal{L}}|_{\widetilde{\pi}_s^{-1}(x)})$ lies in degrees $\leqslant -\ell(v) - 1 + 2 = -\ell(\overline{v})$.

If $v = w'$, then the stalk of $\widetilde{\pi}_{s*}\underline{\mathrm{IC}}(w')_{s\mathcal{L}}$ at $x \in (B\overline{w}'P_s)/\widetilde{P}_s$ lies in degree $-\ell(w')$ because $G_{w'}/\widetilde{B} \to (B\overline{w}'P_s)/\widetilde{P}_s$ is an isomorphism. This finishes the stalk degree estimates needed to show that $\widetilde{\pi}_{s*}\underline{\mathrm{IC}}(w')_{s\mathcal{L}}$ is perverse. □

PROPOSITION 6.4. *Let $N_{\mathcal{L}}$ be the length of the longest element in the Coxeter* *group $W^\circ_{\mathcal{L}}$ (with respect to its own simple reflections). For $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $w \in \beta$, we have*

$$i^*_w \underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}} \cong \underline{C}(w)_{\mathcal{L}}[N_{\mathcal{L}} - \ell_\beta(w)]; \quad i^!_w \underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}} \cong \underline{C}(w)_{\mathcal{L}}[-N_{\mathcal{L}} + \ell_\beta(w)].$$

Here ℓ_β is the function defined in (4.1).

Proof. The second isomorphism follows from the first one by the Verdier duality. We prove the first one by backward induction on $\ell(w)$ (we allow $\mathcal{L}$ to vary in $\mathfrak{o}$, and w_β is determined by w and $\mathcal{L}$). If $w = w_0$ is the longest element in W, then $w_0 = w_\beta$ for the block β containing w_0, and $i^*_{w_0}\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}} \cong \underline{C}(w_0)_{\mathcal{L}}$ by definition (and in this case $\ell_\beta(w_0) = N_{\mathcal{L}}$).

Now suppose the isomorphism holds for any $w \in W$ such that $\ell(w) > n$. Let $w \in W$ be such that $\ell(w) = n$, and let $\beta \in {}_{w\mathcal{L}}\underline{W}_{\mathcal{L}}$ be the block containing w. Let s be a simple reflection in W such that $\ell(ws) = \ell(w) + 1$. We denote ws by w'. Let $\beta' = \beta s \in {}_{w\mathcal{L}}\underline{W}_{s\mathcal{L}}$, the block containing w'.

If $s \notin W^\circ_{\mathcal{L}}$, then by Lemma 3.6(1), right convolution with $\underline{\mathrm{IC}}(s)_{s\mathcal{L}}$ gives an equivalence ${}_{w\mathcal{L}}\underline{\mathcal{D}}_{\mathcal{L}} \to {}_{w\mathcal{L}}\underline{\mathcal{D}}_{s\mathcal{L}}$ sending $\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}}$ to $\underline{\mathrm{IC}}(w_\beta s)_{s\mathcal{L}}$ and $\underline{\nabla}(w)_{\mathcal{L}}$ to $\underline{\nabla}(w')_{s\mathcal{L}}$. By Corollary 4.3, $w_\beta s = w_{\beta'}$. Therefore we have an isomorphism of graded $\mathrm{H}^*_{T_k}(\mathrm{pt}_k)$-modules (coming from the left T-action)

$$\mathrm{Hom}(\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}}, \underline{\nabla}(w)_{\mathcal{L}}) \cong \mathrm{Hom}(\underline{\mathrm{IC}}(w_{\beta'})_{s\mathcal{L}}, \underline{\nabla}(w')_{s\mathcal{L}}). \tag{6.3}$$

Applying the inductive hypothesis to $\underline{\mathrm{IC}}(w_{\beta'})_{s\mathcal{L}}$ and w' (which is longer than w), we get

$$\begin{aligned}\mathrm{Hom}(\underline{\mathrm{IC}}(w_{\beta'})_{s\mathcal{L}}, \underline{\nabla}(w')_{s\mathcal{L}}) &\cong \mathrm{Hom}(i^*_{w'}\underline{\mathrm{IC}}(w_{\beta'})_{s\mathcal{L}}, \underline{C}(w')_{s\mathcal{L}})\\ &\cong \mathrm{End}(\underline{C}(w')_{s\mathcal{L}})[-N_{s\mathcal{L}} + \ell_{\beta'}(w')] \cong \mathrm{H}^*_{\Gamma(w')_k}(\mathrm{pt}_k)[-N_{s\mathcal{L}} + \ell_{\beta'}(w')].\end{aligned}$$

The last isomorphism uses Lemma 2.10. Similarly,

$$\begin{aligned}\mathrm{Hom}(\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}}, \underline{\nabla}(w)_{\mathcal{L}}) &\cong \mathrm{Hom}(i^*_w\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}}, \underline{C}(w)_{\mathcal{L}})\\ &\cong \mathrm{Hom}_{[\{\dot{w}\}/\Gamma(w)]}(i^*_{\dot{w}}(\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}}), \overline{\mathbb{Q}}_\ell)[\ell(w)].\end{aligned}$$

In view of (6.3), we have an isomorphism of graded $\mathrm{H}^*_{T_k}(\mathrm{pt}_k)$-modules

$$\mathrm{H}^*_{\Gamma(w')_k}(\{\dot{w}\})[-N_{s\mathcal{L}} + \ell_{\beta'}(w')] \cong \mathrm{Hom}(i^*_{\dot{w}}\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}}, \overline{\mathbb{Q}}_\ell)[\ell(w)].$$

This forces $i^*_{\dot{w}}(\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}}) \cong \overline{\mathbb{Q}}_\ell[\ell(w) + N_{s\mathcal{L}} - \ell_{\beta'}(w')] \in \mathcal{D}^b_{\Gamma(w)_k}(\{\dot{w}\})$, which implies that $i^*_w\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}} \cong \underline{C}(w)_{\mathcal{L}}[N_{s\mathcal{L}} - \ell_{\beta'}(w')]$ by Lemma 2.10. Clearly, $N_{s\mathcal{L}} = N_{\mathcal{L}}$. By Lemma 4.6(4), $\ell_\beta(w) = \ell_{\beta'}(w')$. Therefore $i^*_w\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}} \cong \underline{C}(w)_{\mathcal{L}}[N_{\mathcal{L}} - \ell_\beta(w)]$.

If $s \in W^\circ_{\mathcal{L}}$, then $\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}}$ is in the image of π_s^* by Lemma 3.10, which implies that the stalks of $\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}}$ at $\dot{w}$ and at $\dot{w}'$ are isomorphic to each other. By inductive hypothesis, the stalk $i^*_{\dot{w}'}\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}} \cong \overline{\mathbb{Q}}_\ell[\ell(w') + N_{\mathcal{L}} - \ell_\beta(w')]$ (now $w' \in \beta$). By Lemma 4.6(4), we have $\ell_\beta(w') = \ell_\beta(ws) = \ell_\beta(w) + 1$. Therefore $i^*_{\dot{w}}\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}} \cong \overline{\mathbb{Q}}_\ell[\ell(w') + N_{\mathcal{L}} - \ell_\beta(w')] \cong \overline{\mathbb{Q}}_\ell[\ell(w) + N_{\mathcal{L}} - \ell_\beta(w)]$, and hence $i^*_{\dot{w}}\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}} \cong \underline{C}(w)_{\mathcal{L}}[N_{\mathcal{L}} - \ell_\beta(w)]$. ☐

6.5. Rigidified maximal IC sheaf in the neutral block. Let $\mathcal{L} \in \mathfrak{o}$. Recall that $\delta_{\mathcal{L}} = \mathrm{IC}(\dot{e})_{\mathcal{L}} \in {}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}}$ is the monoidal unit of ${}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}}$ under convolution. Recall that $N_{\mathcal{L}}$ is the length of the longest element $w_{\mathcal{L},0}$ in the Coxeter group $W^\circ_{\mathcal{L}}$ (in terms of simple reflections in $W^\circ_{\mathcal{L}}$).

A *rigidified maximal IC sheaf* in ${}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}}$ is a pair (Θ, ϵ), where $\Theta \in {}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}}$ is such that $\Theta[N_{\mathcal{L}}]$ is a maximal IC sheaf (that is, $\omega\Theta[N_{\mathcal{L}}] \cong \underline{\mathrm{IC}}(w_{\mathcal{L},0})_{\mathcal{L}}$) and $\epsilon : \Theta \to \delta_{\mathcal{L}}$ is a nonzero map in ${}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}}$.

Rigidified maximal IC sheaves exist. Indeed, by Proposition 6.4, $i^*_e\mathrm{IC}(\dot{w}_{\mathcal{L},0})[-N_{\mathcal{L}}] \cong C(\dot{e}) \otimes V$ for a one-dimensional Fr-module V. Therefore, for $\Theta = \mathrm{IC}(\dot{w}_{\mathcal{L},0})[-N_{\mathcal{L}}] \otimes V^*$, we get a nonzero map $\epsilon : \Theta \to \delta_{\mathcal{L}}$ by adjunction.

Let (Θ, ϵ) and (Θ', ϵ') be two rigidified maximal IC sheaves in ${}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}}$. Then there is a unique isomorphism $\underline{\alpha} : \omega\Theta \xrightarrow{\sim} \omega\Theta'$ such that $\epsilon' \circ \underline{\alpha} = \epsilon$ as elements in $\mathrm{Hom}(\Theta, \delta_{\mathcal{L}})$. The uniqueness of $\underline{\alpha}$ implies that $\mathrm{Fr}(\underline{\alpha}) = \underline{\alpha}$; moreover, $\mathrm{Hom}(\Theta, \Theta'[-1]) = 0$, and hence $\underline{\alpha}$ uniquely lifts to an isomorphism $\alpha : \Theta \xrightarrow{\sim} \Theta'$ inside ${}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}}$. Therefore any two rigidified maximal IC sheaves are isomorphic to each other. Moreover, the automorphism group of any rigidified maximal IC sheaf is trivial. Therefore we may identify all the rigidified maximal IC sheaves in ${}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}}$ as a single object and denote it by

$$(\Theta^\circ_{\mathcal{L}}, \epsilon_{\mathcal{L}} : \Theta^\circ_{\mathcal{L}} \to \delta_{\mathcal{L}}).$$

We denote by

$$(\underline{\Theta}^\circ_{\mathcal{L}}, \underline{\epsilon}_{\mathcal{L}}) = \omega(\Theta^\circ_{\mathcal{L}}, \epsilon_{\mathcal{L}})$$

the rigidified maximal IC sheaf in ${}_{\mathcal{L}}\underline{\mathcal{D}}^\circ_{\mathcal{L}}$.

PROPOSITION 6.6. *There is a unique coalgebra structure on $\Theta^\circ_{\mathcal{L}}$ (inside the monoidal category ${}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}}$) with $\epsilon_{\mathcal{L}}$ as the counit map.*

Proof. For each $n \geqslant 2$, let $(\Theta^\circ_{\mathcal{L}})^{\star n}$ be the n-fold convolution of $\Theta^\circ_{\mathcal{L}}$. We will construct a comultiplication map

$$\mu^n_{\mathcal{L}} : \Theta^\circ_{\mathcal{L}} \to (\Theta^\circ_{\mathcal{L}})^{\star n}$$

characterized as the unique map such that the following diagram is commutative

$$\begin{tikzcd}
\Theta^\circ_{\mathcal{L}} \arrow[r, "\mu^n_{\mathcal{L}}"] \arrow[d, "\epsilon_{\mathcal{L}}"] & (\Theta^\circ_{\mathcal{L}})^{\star n} \arrow[d, "\epsilon^{\star n}_{\mathcal{L}}"] \\
\delta_{\mathcal{L}} \arrow[r, "\sim"] & (\delta_{\mathcal{L}})^{\star n}
\end{tikzcd} \tag{6.4}$$

where the bottom arrow is the canonical isomorphism from the monoidal unit structure on $\delta_{\mathcal{L}}$.

Let $\underline{\delta}_{\mathcal{L}} = \omega\delta_{\mathcal{L}}$. By an iterated application of Proposition 6.2(3), we see that ${}^p\mathrm{H}^i((\underline{\Theta}^\circ_{\mathcal{L}})^{\star n}) = 0$ for $i < N_{\mathcal{L}}$, and $\underline{\Theta}^\circ_{\mathcal{L}}[N_{\mathcal{L}}] \cong {}^p\mathrm{H}^{N_{\mathcal{L}}}((\underline{\Theta}^\circ_{\mathcal{L}})^{\star n})$. Therefore there is a nonzero map $\underline{\mu}^n : \underline{\Theta}^\circ_{\mathcal{L}} \to (\underline{\Theta}^\circ_{\mathcal{L}})^{\star n}$, unique up to a scalar. Since nonzero maps $\underline{\Theta}^\circ_{\mathcal{L}} \to \underline{\delta}_{\mathcal{L}}$ are unique up to a scalar, the claim below shows that there is a unique nonzero multiple of $\underline{\mu}^n$ (call it $\underline{\mu}^n_{\mathcal{L}}$) that makes the nonmixed version of diagram (6.4) (that is, the diagram after applying ω to all terms) commutative. The uniqueness of $\underline{\mu}^n_{\mathcal{L}}$ implies that it is invariant under Frobenius; moreover, $\mathrm{Hom}(\underline{\Theta}^\circ_{\mathcal{L}}, (\underline{\Theta}^\circ_{\mathcal{L}})^{\star n}[-1]) = 0$ for perverse degree reasons. Therefore $\underline{\mu}^n_{\mathcal{L}}$ determines uniquely a morphism $\mu^n_{\mathcal{L}} : \Theta^\circ_{\mathcal{L}} \to (\Theta^\circ_{\mathcal{L}})^{\star n}$ in ${}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}}$.

CLAIM. *The composition (for any nonzero choice of* $\underline{\mu}^n$*)*

$$\underline{\Theta}^\circ_{\mathcal{L}} \xrightarrow{\underline{\mu}^n} (\underline{\Theta}^\circ_{\mathcal{L}})^{\star n} \xrightarrow{\underline{\epsilon}^{\star n}_{\mathcal{L}}} (\underline{\delta}_{\mathcal{L}})^{\star n} \cong \underline{\delta}_{\mathcal{L}}$$

is nonzero.

Proof of Claim. We prove the claim by induction on n. For $n = 2$, we take the degree zero stalks of the above maps at the identity element $\dot{e} \in G$. The map becomes (where $i : \{\dot{e}\} \hookrightarrow G$ is the inclusion)

$$\begin{aligned} i^*\underline{\Theta}^\circ_{\mathcal{L}} &\xrightarrow{\mathrm{H}^0 i^* \underline{\mu}^2} \mathrm{H}^0 i^*(\underline{\Theta}^\circ_{\mathcal{L}} \star \underline{\Theta}^\circ_{\mathcal{L}}) \cong \mathrm{H}^0((G/B)_k, \mathrm{inv}^*\underline{\Theta}^\circ_{\mathcal{L}} \otimes \underline{\Theta}^\circ_{\mathcal{L}}) \\ &\xrightarrow{\mathrm{res}} \mathrm{H}^0 i^*(\mathrm{inv}^*\underline{\Theta}^\circ_{\mathcal{L}} \otimes \underline{\Theta}^\circ_{\mathcal{L}}) \\ &\cong i^*\underline{\Theta}^\circ_{\mathcal{L}} \otimes i^*\underline{\Theta}^\circ_{\mathcal{L}} \cong i^*\underline{\delta}_{\mathcal{L}} \otimes i^*\underline{\delta}_{\mathcal{L}} = \overline{\mathbb{Q}}_\ell = i^*\underline{\delta}_{\mathcal{L}}. \end{aligned} \tag{6.5}$$

Here $\mathrm{H}^0 i^*\underline{\mu}^2$ is an isomorphism since $\underline{\Theta}^\circ_{\mathcal{L}} \star \underline{\Theta}^\circ_{\mathcal{L}}$ is a direct sum of $\underline{\Theta}^\circ_{\mathcal{L}}$ and $\underline{\Theta}^\circ_{\mathcal{L}}[-j]$ for $j > 0$ by Proposition 6.2, and $i^*\underline{\Theta}^\circ_{\mathcal{L}}$ is concentrated in degree 0 by Proposition 6.4. The second isomorphism follows from the definition of the convolution, where $\mathrm{inv} : G \to G$ is the inversion map. The map 'res' is the restriction map to $\{\dot{e}\}$. To prove the claim, we show that composition (6.5) is an isomorphism. It suffices to show that res is an isomorphism. Let $\mathcal{F} = \mathrm{inv}^*\underline{\Theta}^\circ_{\mathcal{L}} \otimes \underline{\Theta}^\circ_{\mathcal{L}} \in D^b(B_k\backslash G_k/B_k)$. By Proposition 6.4, the stalk of $\mathcal{F}$

along the cell $X_{w,k} = G_{w,k}/B_k$ vanishes if $w \notin W_{\mathcal{L}}^{\circ}$, and if $w \in W_{\mathcal{L}}^{\circ}$, it lies in degree $2(-\ell(w) + \ell_{\mathcal{L}}(w))$. We compute $\mathrm{H}^*((G/B)_k, \mathcal{F})$ using the stratification $G/B = \sqcup_{w\in W} X_w$; the contribution of X_w is 0 if $w \notin W_{\mathcal{L}}^{\circ}$ and is $\mathrm{H}_c^*(X_{w,k}, \overline{\mathbb{Q}}_\ell[2\ell(w) - 2\ell_{\mathcal{L}}(w)]) \cong \overline{\mathbb{Q}}_\ell[-2\ell_{\mathcal{L}}(w)]$ for $w \in W_{\mathcal{L}}^{\circ}$. This shows that the only contribution to $\mathrm{H}^0((G/B)_k, \mathcal{F})$ is from the point stratum X_e; hence res is an isomorphism. This proves the case $n = 2$.

Suppose the claim is proved for $n - 1$. Up to a nonzero scalar, $\underline{\mu}^n$ is equal to the composition

$$\underline{\Theta}_{\mathcal{L}}^{\circ} \xrightarrow{\underline{\mu}^2} \underline{\Theta}_{\mathcal{L}}^{\circ} \star \underline{\Theta}_{\mathcal{L}}^{\circ} \xrightarrow{\underline{\mu}^{n-1}\star\mathrm{id}} (\underline{\Theta}_{\mathcal{L}}^{\circ})^{\star(n-1)} \star \underline{\Theta}_{\mathcal{L}}^{\circ}.$$

Composing with $\underline{\epsilon}_{\mathcal{L}}^{\star n}$, we see that up to a nonzero scalar, $\underline{\epsilon}_{\mathcal{L}}^{\star n} \circ \underline{\mu}^n$ can be rewritten as the composition

$$\underline{\Theta}_{\mathcal{L}}^{\circ} \xrightarrow{\underline{\mu}^2} \underline{\Theta}_{\mathcal{L}}^{\circ} \star \underline{\Theta}_{\mathcal{L}}^{\circ} \xrightarrow{(\underline{\epsilon}_{\mathcal{L}}^{\star(n-1)}\circ\underline{\mu}^{n-1})\star\mathrm{id}} \underline{\delta}_{\mathcal{L}} \star \underline{\Theta}_{\mathcal{L}}^{\circ} \xrightarrow{\mathrm{id}\star\underline{\epsilon}_{\mathcal{L}}} \underline{\delta}_{\mathcal{L}} \star \underline{\delta}_{\mathcal{L}} \cong \underline{\delta}_{\mathcal{L}}.$$

By inductive hypothesis, $\underline{\epsilon}_{\mathcal{L}}^{\star(n-1)} \circ \underline{\mu}^{n-1}$ is a nonzero multiple of $\underline{\epsilon}_{\mathcal{L}}$. Therefore the above composition is, up to a nonzero scalar, $\underline{\epsilon}_{\mathcal{L}}^{\star 2} \circ \underline{\mu}^2$, which is nonzero by the $n = 2$ case proved above. This completes the induction step. □

We continue with the proof of Proposition 6.6. Co-associativity of $\mu_{\mathcal{L}}^2$ follows by the uniqueness of $\mu_{\mathcal{L}}^3$. It remains to check the counit axioms, that is, the compositions

$$\Theta_{\mathcal{L}}^{\circ} \xrightarrow{\mu_{\mathcal{L}}^2} \Theta_{\mathcal{L}}^{\circ} \star \Theta_{\mathcal{L}}^{\circ} \xrightarrow{\mathrm{id}\star\epsilon_{\mathcal{L}}} \Theta_{\mathcal{L}}^{\circ} \star \delta_{\mathcal{L}} \cong \Theta_{\mathcal{L}}^{\circ}, \tag{6.6}$$

$$\Theta_{\mathcal{L}}^{\circ} \xrightarrow{\mu_{\mathcal{L}}^2} \Theta_{\mathcal{L}}^{\circ} \star \Theta_{\mathcal{L}}^{\circ} \xrightarrow{\epsilon_{\mathcal{L}}\star\mathrm{id}} \delta_{\mathcal{L}} \star \Theta_{\mathcal{L}}^{\circ} \cong \Theta_{\mathcal{L}}^{\circ} \tag{6.7}$$

are the identity maps. Composing (6.6) with $\epsilon_{\mathcal{L}}$, we recover the map $\Theta_{\mathcal{L}}^{\circ} \xrightarrow{\mu_{\mathcal{L}}^2} \Theta_{\mathcal{L}}^{\circ} \star \Theta_{\mathcal{L}}^{\circ} \xrightarrow{\epsilon_{\mathcal{L}}^{\star 2}} \delta_{\mathcal{L}}$, which is equal to $\epsilon_{\mathcal{L}}$ by construction. This forces (6.6) to be the identity because the endomorphisms of $\Theta_{\mathcal{L}}^{\circ}$ are scalars. The same argument works to show that (6.7) is the identity map. □

DEFINITION 6.7. For $w \in W_{\mathcal{L}}^{\circ}$, define

$$C(w)_{\mathcal{L}}^{\dagger} := i_w^* \Theta_{\mathcal{L}}^{\circ}\langle \ell_{\mathcal{L}}(w)\rangle,$$
$$\Delta(w)_{\mathcal{L}}^{\dagger} := i_{w!}C(w)_{\mathcal{L}}^{\dagger}, \quad \nabla(w)_{\mathcal{L}}^{\dagger} := i_{w*}C(w)_{\mathcal{L}}^{\dagger}, \quad \mathrm{IC}(w)_{\mathcal{L}}^{\dagger} := i_{w!*}C(w)_{\mathcal{L}}^{\dagger}.$$

By Proposition 6.4, $\omega C(w)_{\mathcal{L}}^{\dagger} \cong \underline{C}(w)_{\mathcal{L}}$. By Proposition 3.11(1), $C(w)_{\mathcal{L}}^{\dagger}$ is pure of weight zero. Therefore $\omega\mathrm{IC}(w)_{\mathcal{L}}^{\dagger} \cong \underline{\mathrm{IC}}(w)_{\mathcal{L}}$ and $\mathrm{IC}(w)_{\mathcal{L}}^{\dagger}$ is pure of weight zero. We call $\mathrm{IC}(w)_{\mathcal{L}}^{\dagger}$ a *rigidified IC sheaf*.

LEMMA 6.8. *There is a unique map* $\theta_w^\dagger : \Theta_{\mathcal{L}}^\circ \to \mathrm{IC}(w)_{\mathcal{L}}^\dagger\langle -\ell_{\mathcal{L}}(w)\rangle$ *whose restriction under* i_w^* *is the identity map of* $C(w)_{\mathcal{L}}^\dagger\langle -\ell_{\mathcal{L}}(w)\rangle$.

Proof. By Corollary 3.12, there is a filtration on $M = \mathrm{Hom}^\bullet(\Theta_{\mathcal{L}}^\circ, \mathrm{IC}(w)_{\mathcal{L}}^\dagger\langle -\ell_{\mathcal{L}}(w)\rangle)$ indexed by $\{v \in W_{\mathcal{L}}^\circ; v \leqslant w\}$ such that $\mathrm{Gr}_v^F M \cong \mathrm{Hom}^\bullet(i_v^*\Theta_{\mathcal{L}}^\circ, i_v^!\mathrm{IC}(w)_{\mathcal{L}}^\dagger\langle -\ell_{\mathcal{L}}(w)\rangle)$ as graded $R \otimes R$-modules. By Proposition 6.4, we have

$$\omega\mathrm{Gr}_v^F M \cong \mathrm{Hom}^\bullet(\underline{C}(v)_{\mathcal{L}}, i_v^!\underline{\mathrm{IC}}(w)_{\mathcal{L}})[\ell_{\mathcal{L}}(v) - \ell_{\mathcal{L}}(w)].$$

If $v < w$, $i_v^!\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ lies in perverse degrees > 0; moreover, this costalk is zero unless $v <_{W_{\mathcal{L}}^\circ} w$ by Lemma 4.14 (in particular $\ell_{\mathcal{L}}(v) < \ell_{\mathcal{L}}(w)$). These imply that $\mathrm{Gr}_v^F M$ is concentrated in degrees $\geqslant 2$ for $v < w$. Therefore the quotient map $M \to \mathrm{Gr}_w^F M$ is an isomorphism in degrees $\leqslant 1$, and in particular in degree 0. Now $\mathrm{Gr}_w^F M = \mathrm{Hom}^\bullet(i_w^*\Theta_{\mathcal{L}}^\circ, i_w^*\mathrm{IC}(w)_{\mathcal{L}}^\dagger\langle -\ell_{\mathcal{L}}(w)\rangle)$, which is the same as $\mathrm{Hom}^\bullet(C(w)_{\mathcal{L}}^\dagger\langle -\ell_{\mathcal{L}}(w)\rangle, C(w)_{\mathcal{L}}^\dagger\langle -\ell_{\mathcal{L}}(w)\rangle)$, and the quotient map $M \to \mathrm{Gr}_w^F M$ is induced by i_w^*. Therefore there is a unique $\theta_w^\dagger \in M^0$ mapping to $\mathrm{id} \in \mathrm{End}(C(w)_{\mathcal{L}}^\dagger\langle -\ell_{\mathcal{L}}(w)\rangle) = (\mathrm{Gr}_w^F M)^0$. ☐

LEMMA 6.9. (1) *There is a unique isomorphism* $\iota_e : \mathrm{IC}(e)_{\mathcal{L}}^\dagger \cong \delta_{\mathcal{L}}$ *such that* $\iota_e \circ \theta_e^\dagger = \epsilon_{\mathcal{L}}$.

(2) *Let* $s \in W$ *be a simple reflection and* $s \in W_{\mathcal{L}}^\circ$. *Recall the object* $\mathrm{IC}(s)_{\mathcal{L}}$ *introduced in Section 3.7. Then there is a unique isomorphism* $\iota_s : \mathrm{IC}(s)_{\mathcal{L}}^\dagger \cong \mathrm{IC}(s)_{\mathcal{L}}$ *such that the composition* $\iota_s \circ \theta_s^\dagger : \Theta_{\mathcal{L}}^\circ \to \mathrm{IC}(s)_{\mathcal{L}}\langle -1\rangle$ *restricts to the identity map on the stalks at* $e \in G$. *(Recall the stalks of both* $\Theta_{\mathcal{L}}^\circ$ *and* $\mathrm{IC}(s)_{\mathcal{L}}\langle -1\rangle$ *are equipped with an isomorphism with the trivial* Fr*-module* $\overline{\mathbb{Q}}_\ell$.*)*

Proof. (1) The rigidification $\epsilon_{\mathcal{L}} : \Theta_{\mathcal{L}}^\circ \to \delta_{\mathcal{L}}$ gives by adjunction a nonzero map $C(e)_{\mathcal{L}}^\dagger = i_e^*\Theta_{\mathcal{L}}^\circ \to C(\dot{e})_{\mathcal{L}}$, which has to be an isomorphism. This induces the desired isomorphism ι_e. The uniqueness part is clear.

(2) By Lemma 3.10, we can write $\Theta_{\mathcal{L}}^\circ = \pi_s^*\overline{\Theta}$ for some shifted perverse sheaf $\overline{\Theta} \in {}_{\mathcal{L}}\mathcal{D}_{\widetilde{\mathcal{L}}}$. Since the stalk of $\Theta_{\mathcal{L}}^\circ$ at $\dot{e}$ is the trivial Fr-module by the rigidification $\epsilon_{\mathcal{L}}$, we have $\overline{\Theta}|_{P_s} \cong \widetilde{\mathcal{L}}$, and hence $i_{\leqslant s}^*\Theta_{\mathcal{L}}^\circ \cong \widetilde{\mathcal{L}} \in {}_{\mathcal{L}}\mathcal{D}(\leqslant s)_{\mathcal{L}}$. By adjunction $\theta_s^\dagger$ gives a nonzero map $\widetilde{\mathcal{L}} \cong i_{\leqslant s}^*\Theta_{\mathcal{L}}^\circ \to i_{\leqslant s}^*\mathrm{IC}(s)_{\mathcal{L}}^\dagger\langle -1\rangle$, which has to be an isomorphism. This induces an isomorphism $\mathrm{IC}(s)_{\mathcal{L}}^\dagger \cong i_{\leqslant s*}\widetilde{\mathcal{L}}\langle 1\rangle = \mathrm{IC}(s)_{\mathcal{L}}$. The uniqueness of ι_s is clear. ☐

6.10. Rigidified maximal IC sheaves in general. Let $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$ and $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$. Let $\xi \in {}_{\mathcal{L}'}\mathfrak{P}^{\beta}_{\mathcal{L}}$ be a minimal IC sheaf in the block β (see Section 5.1). Let

$$\Theta(\xi) := \Theta^{\circ}_{\mathcal{L}} \star \xi.$$

Then $\Theta(\xi)$ is equipped with a nonzero map

$$\epsilon(\xi) := \epsilon_{\mathcal{L}'} \star \mathrm{id}_{\xi} : \Theta(\xi) = \Theta^{\circ}_{\mathcal{L}'} \star \xi \to \delta_{\mathcal{L}'} \star \xi \cong \xi.$$

The pair $(\Theta(\xi), \epsilon(\xi))$ has a trivial automorphism group. We denote

$$(\underline{\Theta}(\xi), \underline{\epsilon}(\xi)) := \omega(\Theta(\xi), \epsilon(\xi)) \in {}_{\mathcal{L}'}\underline{\mathcal{D}}^{\beta}_{\mathcal{L}}.$$

By Proposition 5.2(3), $\underline{\Theta}(\xi) \cong \underline{\mathrm{IC}}(w_{\beta})_{\mathcal{L}}[-N_{\mathcal{L}}]$.

Lemma 6.11. *For $\xi \in {}_{\mathcal{L}'}\mathfrak{P}^{\beta}_{\mathcal{L}}$, there is a unique isomorphism*

$$\tau(\xi) : \xi \star \Theta^{\circ}_{\mathcal{L}} \xrightarrow{\sim} \Theta^{\circ}_{\mathcal{L}'} \star \xi$$

making the following diagram commutative

$$\begin{tikzcd}
\xi \star \Theta^{\circ}_{\mathcal{L}} \arrow[rr, "\tau(\xi)"] \arrow[d, "\mathrm{id}\star\epsilon_{\mathcal{L}}"] & & \Theta^{\circ}_{\mathcal{L}'} \star \xi \arrow[d, "\epsilon_{\mathcal{L}'}\star\mathrm{id}"] \\
\xi \star \delta_{\mathcal{L}} \arrow[r, "\sim"] & \xi \arrow[r, "\sim"] & \delta_{\mathcal{L}'} \star \xi
\end{tikzcd} \tag{6.8}$$

Proof. By Proposition 5.2, both $\omega(\xi \star \Theta^{\circ}_{\mathcal{L}})$ and $\omega(\Theta^{\circ}_{\mathcal{L}'} \star \xi)$ are isomorphic to $\underline{\mathrm{IC}}(w_{\beta})_{\mathcal{L}}[-N_{\mathcal{L}}]$. Therefore isomorphisms $\underline{\tau} : \omega(\xi \star \Theta^{\circ}_{\mathcal{L}}) \xrightarrow{\sim} \omega(\Theta^{\circ}_{\mathcal{L}'} \star \xi)$ are unique up to a nonzero scalar. Moreover, since $\xi \star (-)$ is an equivalence, $\mathrm{Hom}(\xi \star \Theta^{\circ}_{\mathcal{L}}, \xi) \cong \mathrm{Hom}(\Theta^{\circ}_{\mathcal{L}}, \delta_{\mathcal{L}}) \cong \overline{\mathbb{Q}}_{\ell}$. Therefore there is a unique $\underline{\tau}$ making the nonmixed version of diagram (6.8) commutative. Uniqueness of $\underline{\tau}$ implies that it is Fr-invariant and lifts to a unique isomorphism $\tau(\xi)$ in ${}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}}$. □

Let $\mathcal{L}'' \in \mathfrak{o}$, $\gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$ and $\eta \in {}_{\mathcal{L}''}\mathfrak{P}^{\gamma}_{\mathcal{L}'}$. To save notation, we will abbreviate $\eta \star \xi$ by $\eta\xi$. Consider the composition

$$\begin{aligned}
\varphi(\eta, \xi) : \Theta(\eta\xi) &= \Theta^{\circ}_{\mathcal{L}''} \star (\eta\xi) \\
&\xrightarrow{\mu^{2}_{\mathcal{L}''}\star\mathrm{id}} (\Theta^{\circ}_{\mathcal{L}''} \star \Theta^{\circ}_{\mathcal{L}'}) \star (\eta\xi) = \Theta^{\circ}_{\mathcal{L}''} \star (\Theta^{\circ}_{\mathcal{L}'} \star \eta) \star \xi \\
&\xrightarrow{\mathrm{id}\star\tau(\eta)^{-1}\star\mathrm{id}} \Theta^{\circ}_{\mathcal{L}''} \star \eta \star \Theta^{\circ}_{\mathcal{L}'} \star \xi = \Theta(\eta) \star \Theta(\xi).
\end{aligned}$$

Here $\mu^{2}_{\mathcal{L}''} : \Theta^{\circ}_{\mathcal{L}''} \to \Theta^{\circ}_{\mathcal{L}''} \star \Theta^{\circ}_{\mathcal{L}''}$ is the comultiplication constructed in Proposition 6.6.

PROPOSITION 6.12. *The notation is the same as above.*

(1) *The composition*

$$\Theta(\eta\xi) \xrightarrow{\varphi(\eta,\xi)} \Theta(\eta) \star \Theta(\xi) \xrightarrow{\epsilon(\eta)\star\epsilon(\xi)} \eta\xi$$

is the same as $\epsilon(\eta\xi)$.

(2) *The following compositions are the identity maps*

$$\Theta(\xi) \xrightarrow{\varphi(\delta_{\mathcal{L}'},\xi)} \Theta^{\circ}_{\mathcal{L}'} \star \Theta(\xi) \xrightarrow{\epsilon_{\mathcal{L}'}\star \mathrm{id}} \delta_{\mathcal{L}'} \star \Theta(\xi) \cong \Theta(\xi),$$
$$\Theta(\xi) \xrightarrow{\varphi(\xi,\delta_{\mathcal{L}})} \Theta(\xi) \star \Theta^{\circ}_{\mathcal{L}} \xrightarrow{\mathrm{id}\star\epsilon_{\mathcal{L}}} \Theta(\xi) \star \delta_{\mathcal{L}} \cong \Theta(\xi).$$

(3) *For* $\mathcal{L}''' \in \mathfrak{o}$ *and* $\zeta \in {}_{\mathcal{L}'''}\Xi_{\mathcal{L}''}$, *the following diagram is commutative*

$$\begin{tikzcd}
\Theta(\zeta\eta\xi) \arrow[r, "\varphi(\zeta,\eta\xi)"] \arrow[d, "\varphi(\zeta\eta,\xi)"'] & \Theta(\zeta) \star \Theta(\eta\xi) \arrow[d, "\mathrm{id}\star\varphi(\eta,\xi)"] \\
\Theta(\zeta\eta) \star \Theta(\xi) \arrow[r, "\varphi(\zeta,\eta)\star\mathrm{id}"] & \Theta(\zeta) \star \Theta(\eta) \star \Theta(\xi)
\end{tikzcd} \tag{6.9}$$

Proof. Part (1) follows from the definition of $\varphi(\eta,\xi)$ and the characterizing property of the comultiplication $\mu^2_{\mathcal{L}}$ on $\Theta^{\circ}_{\mathcal{L}}$ that $\epsilon^{\star 2}_{\mathcal{L}} \circ \mu^2_{\mathcal{L}} = \epsilon_{\mathcal{L}}$.

The proof of (2) is similar to the verification of the counit axioms in the proof of Proposition 6.6. We omit it here.

To prove (3), we observe that by Proposition 6.2, $\Theta(\zeta\eta\xi)$ is identified with the lowest nonzero perverse cohomology of $\Theta(\zeta) \star \Theta(\eta) \star \Theta(\xi)$. Therefore nonzero maps $\Theta(\zeta\eta\xi) \to \Theta(\zeta) \star \Theta(\eta) \star \Theta(\xi)$ are unique up to a scalar. Therefore it suffices to show that, after composing with $\epsilon(\zeta) \star \epsilon(\eta) \star \epsilon(\xi) : \Theta(\zeta) \star \Theta(\eta) \star \Theta(\xi) \to \zeta\eta\xi$, both compositions in diagram (6.9) are equal to $\epsilon(\zeta\eta\xi)$. But this follows from iterated applications of part (1). ☐

7. Monodromic Soergel functor

In this section, we introduce the Soergel functor between the monodromic Hecke category and the category of graded R-bimodules, construct its monoidal structure and prove an analogue of Soergel's Extension Theorem for this functor.

7.1. ***R*-bimodules with Frobenius actions.** Let

$$R = \mathrm{H}^*_{T_k}(\mathrm{pt}_k, \overline{\mathbb{Q}}_\ell) \cong \mathrm{Sym}(\mathbb{X}^*(T)_{\overline{\mathbb{Q}}_\ell}),$$

with the grading $\deg \mathbb{X}^*(T)_{\overline{\mathbb{Q}}_\ell} = 2$, and the Frobenius action on $\mathbb{X}^*(T)_{\overline{\mathbb{Q}}_\ell}$ by q. Let $R \otimes R$-gmod be the category of $\mathbb{Z}$-graded $R \otimes R$-modules. Let $(R \otimes R, \mathrm{Fr})$-gmod be the category of $\mathbb{Z}$-graded $R \otimes R$-modules $M = \oplus_n M^n$ with a degree-preserving automorphism $\mathrm{Fr} : M \to M$ compatible with the Frobenius action on $R \otimes R$; that is, for homogeneous $a \in R$ and $m \in M$, we have $\mathrm{Fr}((a \otimes 1)m) = q^{\deg(a)/2}(a \otimes 1)\mathrm{Fr}(m)$ and $\mathrm{Fr}((1 \otimes a)m) = q^{\deg(a)/2}(1 \otimes a)\mathrm{Fr}(m)$. Let $\omega : (R \otimes R, \mathrm{Fr})$-gmod $\to R \otimes R$-gmod be the functor forgetting the Frobenius action.

We use [1] for the degree shift for graded $(R \otimes R, \mathrm{Fr})$-modules, that is, if $M = \oplus_{n\in\mathbb{Z}} M^n \in R \otimes R$-gmod, $M[1]$ is the graded $R \otimes R$-module with $(M[1])^n = M^{n+1}$ as Fr-modules. For $M \in (R \otimes R, \mathrm{Fr})$-gmod, $M(n/2)$ is the same graded $R \otimes R$-module as M with the Frobenius action multiplied by $q^{-n/2}$. Let $\langle n\rangle$ be the composition $[n](n/2)$.

For $M_1, M_2 \in (R \otimes R, \mathrm{Fr})$-gmod, we understand $M_1 \otimes_R M_2$ as the tensor product of M_1 and M_2 with respect to the second R-action on M_1 and the first R-action on M_2.

For $M_1, M_2 \in R \otimes R$-gmod, their inner Hom is the graded $R \otimes R$-module

$$\mathrm{Hom}^\bullet(M_1, M_2) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{R\otimes R\text{-gmod}}(M_1, M_2[n]).$$

If $M_1, M_2 \in (R \otimes R, \mathrm{Fr})$-gmod, then $\mathrm{Hom}^\bullet(M_1, M_2)$ is also naturally an object in $(R \otimes R, \mathrm{Fr})$-gmod.

For two objects $\mathcal{F}, \mathcal{G} \in {}_{\mathcal{L}'}\mathcal{D}^\beta_{\mathcal{L}}$, let

$$\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G}) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(\mathcal{F}, \mathcal{G}[n]).$$

Since $\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G}) = \mathrm{H}^*_{T_k\times T_k}((U\backslash G/U)_k, \mathbf{R}\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}))$, it is a graded $(R \otimes R, \mathrm{Fr})$-module, where the $R \otimes R = \mathrm{H}^*_{T_k\times T_k}(\mathrm{pt}_k)$-action comes from the $T \times T$-action on $U\backslash G/U$ given in Section 2.7. The same notation applies to ${}_{\mathcal{L}'}\mathcal{D}(w)_{\mathcal{L}}$ and ${}_{\mathcal{L}'}\mathcal{D}(\leqslant w)_{\mathcal{L}}$.

For each $w \in W$, let $R(w)$ be the graded R-bimodule, which is the quotient of $R \otimes R$ by the ideal generated by $w(a) \otimes 1 - 1 \otimes a$ for all $a \in R$. We have a canonical isomorphism in $(R \otimes R, \mathrm{Fr})$-gmod:

$$R(w) \cong \mathrm{H}^*_{\Gamma(w)_k}(\mathrm{pt}_k) \cong \mathrm{Hom}^\bullet(C(\dot{w})_{\mathcal{L}}, C(\dot{w})_{\mathcal{L}}).$$

Definition 7.2. (1) Let $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $\xi \in {}_{\mathcal{L}'}\mathfrak{P}^\beta_{\mathcal{L}}$ be a minimal IC sheaf in the block β. The *mixed Soergel functor* associated with ξ is the functor

$$\mathbb{M}_\xi := \mathrm{Hom}^\bullet(\Theta(\xi), -) : {}_{\mathcal{L}'}\mathcal{D}^\beta_{\mathcal{L}} \to (R \otimes R, \mathrm{Fr})\text{-gmod}.$$

(2) The *nonmixed Soergel functor* associated with ξ is

$$\underline{\mathbb{M}}_\xi := \mathrm{Hom}^\bullet(\underline{\Theta}(\xi), -) : {}_{\mathcal{L}'}\underline{\mathcal{D}}^\beta_{\mathcal{L}} \to R \otimes R\text{-gmod}.$$

(3) When $\xi = \delta_{\mathcal{L}} \in {}_{\mathcal{L}}\mathfrak{P}^\circ_{\mathcal{L}}$, we denote the corresponding Soergel functors by $\mathbb{M}^\circ = \mathrm{Hom}^\bullet(\Theta^\circ_{\mathcal{L}}, -)$ and $\underline{\mathbb{M}}^\circ = \mathrm{Hom}^\bullet(\underline{\Theta}^\circ_{\mathcal{L}}, -)$.

LEMMA 7.3. *Let $\xi \in {}_{\mathcal{L}'}\mathfrak{P}^\beta_{\mathcal{L}}$. There is a canonical isomorphism in* $(R \otimes R, \mathrm{Fr})$-gmod

$$\mathbb{M}_\xi(\xi) \cong R(w^\beta)$$

under which the canonical map $\epsilon(\xi) : \Theta(\xi) \to \xi$ corresponds to $1 \in R(w^\beta)$.

Proof. By definition, we have

$$\begin{aligned}\mathbf{R}\mathrm{Hom}(\underline{\Theta}(\xi), \omega\xi) &\cong \mathbf{R}\mathrm{Hom}(\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}}[-N_{\mathcal{L}}], \underline{\nabla}(w^\beta)_{\mathcal{L}})\\ &= \mathbf{R}\mathrm{Hom}(i^*_{w^\beta}\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}}, \underline{C}(w^\beta)_{\mathcal{L}})[N_{\mathcal{L}}].\end{aligned}$$

By Proposition 6.4, $i^*_{w^\beta}\underline{\mathrm{IC}}(w_\beta)_{\mathcal{L}} \cong \underline{C}(w^\beta)_{\mathcal{L}}[N_{\mathcal{L}}]$. Therefore,

$$\mathbf{R}\mathrm{Hom}(\underline{\Theta}(\xi), \omega\xi) \cong \mathbf{R}\mathrm{Hom}(\underline{C}(w^\beta)_{\mathcal{L}}, \underline{C}(w^\beta)_{\mathcal{L}}).$$

Taking cohomology, we get an isomorphism of graded $R \otimes R$-modules $\alpha : \underline{\mathbb{M}}_\xi(\xi) \cong \omega R(w^\beta)$, well defined up to a scalar. We normalize this isomorphism by requiring that $\epsilon(\xi)$ go to $1 \in R(w^\beta)$. Since both $\epsilon(\xi)$ and $1 \in R(w^\beta)$ are invariant under Fr, α is also Fr-equivariant. ☐

LEMMA 7.4. *Let $s \in W$ be a simple reflection and $s \in W^\circ_{\mathcal{L}}$. Recall we have a canonical isomorphism $\mathrm{IC}(s)_{\mathcal{L}} \cong \mathrm{IC}(s)^\dagger_{\mathcal{L}}$ given by Lemma 6.9(2).*

(1) *Let $\xi \in {}_{\mathcal{L}'}\mathfrak{P}^\beta_{\mathcal{L}}$ for some block $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $\mathcal{F} \in {}_{\mathcal{L}'}\mathcal{D}^\beta_{\mathcal{L}}$. There is a canonical isomorphism in* $(R \otimes R, \mathrm{Fr})$-gmod

$$\mathbb{M}_\xi(\mathcal{F}) \otimes_{R^s} R\langle 1\rangle \xrightarrow{\sim} \mathbb{M}_\xi(\mathcal{F} \star \mathrm{IC}(s)^\dagger_{\mathcal{L}}) \tag{7.1}$$

such that the composition

$$\begin{tikzcd} \mathbb{M}_\xi(\mathcal{F}) \arrow[r, "(7.1)"] & \mathbb{M}_\xi(\mathcal{F} \star \mathrm{IC}(s)^\dagger_{\mathcal{L}}\langle -1\rangle) \arrow[d, "\wr"', "\text{Lemma 3.8}"] & \\ & \mathbb{M}_\xi(\pi_s^*\pi_{s*}\mathcal{F}) \arrow[r, "\mathrm{adj}"] & \mathbb{M}_\xi(\mathcal{F}) \end{tikzcd} \tag{7.2}$$

is the identity.

(2) *There is a canonical isomorphism in* $(R\otimes R,\mathrm{Fr})$-gmod

$$\mathbb{M}^{\circ}(\mathrm{IC}(s)^{\dagger}_{\mathcal{L}}\langle -1\rangle)\cong R\otimes_{R^s}R \tag{7.3}$$

under which $\theta_s^{\dagger}$ *corresponds to* $1\otimes 1$.

Proof. (1) Let $\widetilde{\mathcal{L}}\in\mathrm{Ch}(L_s)$ be the extension of $\mathcal{L}$. By Lemma 3.10, we can write $\Theta(\xi)=\pi_s^*\overline{\Theta}$ for some shifted perverse sheaf $\overline{\Theta}\in{}_{\mathcal{L}}\mathcal{D}_{\widetilde{\mathcal{L}}}$. By Lemma 3.8, we have

$$\begin{aligned}\mathrm{Hom}^{\bullet}(\Theta(\xi),\mathcal{F})&\cong\mathrm{Hom}^{\bullet}(\pi_s^*\overline{\Theta},\mathcal{F})\cong\mathrm{Hom}^{\bullet}(\overline{\Theta},\pi_{s*}\mathcal{F})\\&=\mathrm{H}^*((B\backslash G/P_s)_k,\mathbf{R}\underline{\mathrm{Hom}}(\overline{\Theta},\pi_{s*}\mathcal{F})).\end{aligned}\tag{7.4}$$

The right side above is naturally a graded $(R\otimes R^s,\mathrm{Fr})$-module, for $R^s=\mathrm{H}^*_{(P_s)_k}(\mathrm{pt}_k)$.

Let π_s also denote the projection $B\backslash G/B\to B\backslash G/P_s$. For any complex $\mathcal{K}\in D^b_m(B\backslash G/P_s)$, the pullback $\mathrm{H}^*((B\backslash G/P_s)_k,\mathcal{K})\to\mathrm{H}^*((B\backslash G/B)_k,\pi_s^*\mathcal{K})$ is right R^s-linear. It then induces a natural map in $(R\otimes R,\mathrm{Fr})$-gmod:

$$\mathrm{H}^*((B\backslash G/P_s)_k,\mathcal{K})\otimes_{R^s}R\to\mathrm{H}^*((B\backslash G/B)_k,\pi_s^*\mathcal{K}).\tag{7.5}$$

This is in fact a bijection because

$$\begin{aligned}\mathrm{H}^*((B\backslash G/B)_k,\pi_s^*\mathcal{K})&\cong\mathrm{H}^*((B\backslash G/P_s)_k,\pi_{s*}\pi_s^*\mathcal{K})\\&\cong\mathrm{H}^*((B\backslash G/P_s)_k,\mathcal{K}\otimes\pi_{s*}\overline{\mathbb{Q}}_\ell)\end{aligned}$$

and $\pi_{s*}\overline{\mathbb{Q}}_\ell\cong\overline{\mathbb{Q}}_\ell\oplus\overline{\mathbb{Q}}_\ell\langle -2\rangle$ (in $D^b_m(B\backslash G/P_s)$) corresponding to the decomposition $R=R^s\oplus\alpha_sR^s$. Applying isomorphism (7.5) to $\mathcal{K}=\mathbf{R}\underline{\mathrm{Hom}}(\overline{\Theta},\pi_{s*}\mathcal{F})$, we get

$$\begin{aligned}&\mathrm{H}^*((B\backslash G/P_s)_k,\mathbf{R}\underline{\mathrm{Hom}}(\overline{\Theta},\pi_{s*}\mathcal{F}))\otimes_{R^s}R\\&\cong\mathrm{H}^*((B\backslash G/B)_k,\pi_s^*\mathbf{R}\underline{\mathrm{Hom}}(\overline{\Theta},\pi_{s*}\mathcal{F}))\\&\cong\mathrm{H}^*((B\backslash G/B)_k,\mathbf{R}\underline{\mathrm{Hom}}(\pi_s^*\overline{\Theta},\pi_s^*\pi_{s*}\mathcal{F}))\\&\cong\mathrm{Hom}^{\bullet}(\Theta(\xi),\mathcal{F}\star\mathrm{IC}(s)^{\dagger}_{\mathcal{L}}\langle -1\rangle).\end{aligned}$$

Here we have used Lemma 3.8. Combining this with (7.4), we get an isomorphism

$$\begin{aligned}\mathbb{M}_\xi(\mathcal{F})\otimes_{R^s}R\langle 1\rangle&=\mathrm{Hom}^{\bullet}(\Theta(\xi),\mathcal{F})\otimes_{R^s}R\langle 1\rangle\\&\cong\mathrm{Hom}^{\bullet}(\Theta(\xi),\mathcal{F}\star\mathrm{IC}(s)^{\dagger}_{\mathcal{L}})=\mathbb{M}_\xi(\mathcal{F}\star\mathrm{IC}(s)^{\dagger}_{\mathcal{L}}).\end{aligned}$$

The construction above shows that composition (7.2) is induced by applying $\mathrm{Hom}^{\bullet}(\overline{\Theta},-)$ to the composition of adjunction maps $\pi_{s*}\mathcal{F}\to\pi_{s*}\pi_s^*\pi_{s*}\mathcal{F}\to\pi_{s*}\mathcal{F}$, which is the identity.

(2) Taking $\mathcal{F}=\delta_{\mathcal{L}}$ in (1), we get the canonical isomorphism (7.3). The fact that $\theta_s^{\dagger}$ corresponds to $1\otimes 1$ follows from the fact that (7.2) is the identity for $\mathcal{F}=\delta_{\mathcal{L}}$. This proves part (2). $\square$

7.5. Monoidal structure. Let $\beta\in{}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, $\gamma\in{}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$, $\mathcal{F}\in{}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}}$, $\mathcal{G}\in{}_{\mathcal{L}''}\mathcal{D}^{\gamma}_{\mathcal{L}'}$, $\xi\in{}_{\mathcal{L}'}\mathfrak{P}^{\beta}_{\mathcal{L}}$ and $\eta\in{}_{\mathcal{L}''}\mathfrak{P}^{\gamma}_{\mathcal{L}'}$. Consider the maps

$$\begin{aligned}\mathrm{Hom}(\Theta(\eta),\mathcal{G}[i])\times\mathrm{Hom}(\Theta(\xi),\mathcal{F}[j]) &\overset{\star}{\to}\mathrm{Hom}(\Theta(\eta)\star\Theta(\xi),\mathcal{G}\star\mathcal{F}[i+j])\\ &\xrightarrow{(-)\circ\varphi(\eta,\xi)}\mathrm{Hom}(\Theta(\eta\xi),\mathcal{G}\star\mathcal{F}[i+j]).\end{aligned}$$

Taking the direct sum over $i,j\in\mathbb{Z}$, we get a pairing

$$(\cdot,\cdot):\mathbb{M}_{\eta}(\mathcal{G})\times\mathbb{M}_{\xi}(\mathcal{F})\to\mathbb{M}_{\eta\xi}(\mathcal{G}\star\mathcal{F})$$

satisfying the following relations for $a\in R$, $f\in\mathbb{M}_{\xi}(\mathcal{F})$ and $g\in\mathbb{M}_{\eta}(\mathcal{G})$:

$$\begin{aligned}&((1\otimes a)\cdot g,f)=(g,(a\otimes 1)\cdot f),\\ &((a\otimes 1)\cdot g,f)=(a\otimes 1)\cdot(g,f),\quad (g,(1\otimes a)\cdot f)=(1\otimes a)\cdot(g,f).\end{aligned}$$

Therefore it induces a map in $(R\otimes R,\mathrm{Fr})$-gmod:

$$c_{\eta,\xi}(\mathcal{G},\mathcal{F}):\mathbb{M}_{\eta}(\mathcal{G})\otimes_R\mathbb{M}_{\xi}(\mathcal{F})\to\mathbb{M}_{\eta\xi}(\mathcal{G}\star\mathcal{F}).\tag{7.6}$$

As $\mathcal{F}$ and $\mathcal{G}$ vary, the above maps form a natural transformation between two bifunctors ${}_{\mathcal{L}''}\mathcal{D}^{\gamma}_{\mathcal{L}'}\times{}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}}\to(R\otimes R,\mathrm{Fr})$-gmod defined by the left and right sides. The co-associativity of $\{\varphi(\eta,\xi)\}$ as shown in Proposition 6.12(3) implies that maps (7.6) are associative for three composable ξ,η,ζ.

LEMMA 7.6. *With the above notation, $c_{\eta,\xi}(\mathcal{G},\xi)$ is an isomorphism in $(R\otimes R,$ Fr)-gmod. In particular (by Lemma 7.3), there is a canonical isomorphism*

$$\mathbb{M}_{\eta}(\mathcal{G})\otimes_R R(w^{\beta})\xrightarrow{\sim}\mathbb{M}_{\eta\xi}(\mathcal{G}\star\xi).$$

A similar statement holds when $\mathcal{G}$ appears in the second factor.

Proof. By definition, we have

$$\begin{aligned}\psi:\mathbb{M}_{\eta\xi}(\mathcal{G}\star\xi)&=\mathrm{Hom}^{\bullet}(\Theta(\eta\xi),\mathcal{G}\star\xi)=\mathrm{Hom}^{\bullet}(\Theta(\eta)\star\xi,\mathcal{G}\star\xi)\\ &\cong\mathrm{Hom}^{\bullet}(\Theta(\eta),\mathcal{G})=\mathbb{M}_{\eta}(\mathcal{G}),\end{aligned}$$

where we used the fact that $\star\xi$ is an equivalence (Proposition 5.2). The composition

$$\mathbb{M}_{\eta}(\mathcal{G})\otimes_R R(w^{\beta})\cong\mathbb{M}_{\eta}(\mathcal{G})\otimes_R\mathbb{M}_{\xi}(\xi)\xrightarrow{c_{\eta,\xi}(\mathcal{G},\xi)}\mathbb{M}_{\eta\xi}(\mathcal{G}\star\xi)\overset{\psi}{\to}\mathbb{M}_{\eta}(\mathcal{G})$$

sends $f \otimes 1$ to f, and hence it is an isomorphism. This implies that $c_{\eta,\xi}(\mathcal{G}, \xi)$ is an isomorphism.

For the statement where $\mathcal{G}$ appears as the second factor, we use the canonical isomorphism $\Theta(\xi) \cong \xi \star \Theta^{\circ}_{\mathcal{L}}$ given in Lemma 6.11. The rest of the argument is the same as above. $\square$

LEMMA 7.7. *Let $s \in W$ be a simple reflection and $s \in W^{\circ}_{\mathcal{L}}$. Let $\xi \in {}_{\mathcal{L}'}\mathfrak{P}^{\beta}_{\mathcal{L}}$ for some block $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $\mathcal{F} \in {}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}}$. Then the map $c_{\xi,\delta_{\mathcal{L}}}(\mathcal{F}, \mathrm{IC}(s)_{\mathcal{L}})$ is an isomorphism.*

Proof. Lemma 7.4 already gives us an isomorphism

$$\mu_{\mathcal{F},s} : \mathbb{M}_{\xi}(\mathcal{F}) \otimes_R \mathbb{M}^{\circ}(\mathrm{IC}(s)_{\mathcal{L}}) \cong \mathbb{M}_{\xi}(\mathcal{F}) \otimes_{R^s} R\langle 1\rangle \cong \mathbb{M}_{\xi}(\mathcal{F} \star \mathrm{IC}(s)_{\mathcal{L}}).$$

It remains to show that $\mu_{\mathcal{F},s}$ is the same as $c_{\xi,\delta_{\mathcal{L}}}(\mathcal{F}, \mathrm{IC}(s)_{\mathcal{L}})$. To prove this, after a diagram chase, it is enough to show that the following composition (we are using notation from the proof of Lemma 7.4)

$$\begin{aligned} \pi_s^*\overline{\Theta} = \Theta(\xi) &\xrightarrow{\varphi(\xi,\delta_{\mathcal{L}})} \Theta(\xi) \star \Theta^{\circ}_{\mathcal{L}} \xrightarrow{\mathrm{id}\star\psi_s} \Theta(\xi) \star \mathrm{IC}(s)_{\mathcal{L}}\langle -1\rangle \\ &\cong \pi_s^*\pi_{s*}\Theta(\xi) = \pi_s^*\pi_{s*}\pi_s^*\overline{\Theta} \end{aligned} \tag{7.7}$$

is the natural map given by the adjunction $\overline{\Theta} \to \pi_{s*}\pi_s^*\overline{\Theta}$. By Proposition 6.2(2)(3), $\pi_s^*\pi_{s*}\Theta(\xi) \cong \Theta(\xi) \star \mathrm{IC}(s)_{\mathcal{L}}\langle -1\rangle$ lies in perverse degree $\geqslant 0$, with $\omega {}^p\mathrm{H}^0\pi_s^*\pi_{s*}\Theta(\xi) \cong \underline{\Theta}(\xi)$. We see that $\mathrm{Hom}(\Theta(\xi), \pi_s^*\pi_{s*}\Theta(\xi))$ is one-dimensional. Therefore it suffices to show that the composition of (7.7) with the adjunction $\pi_s^*\pi_{s*}\Theta(\xi) \to \Theta(\xi)$ is the identity map of $\Theta(\xi)$. This boils down to the commutativity of the following diagram

$$\begin{tikzcd}
\Theta(\xi) \arrow[r, "\varphi(\xi,\delta_{\mathcal{L}})"] \arrow[d, equal, "\mathrm{id}"] & \Theta(\xi) \star \Theta^{\circ}_{\mathcal{L}} \arrow[r, "\mathrm{id}\star\psi_s"] \arrow[d, "\mathrm{id}\star\epsilon_{\mathcal{L}}"] & \Theta(\xi) \star \mathrm{IC}(s)_{\mathcal{L}}\langle -1\rangle \arrow[dl, "\mathrm{id}\star\epsilon_s"] \\
\Theta(\xi) \arrow[r, "\sim"] & \Theta(\xi) \star \delta_{\mathcal{L}} &
\end{tikzcd}$$

Here $\epsilon_s : \mathrm{IC}(s)_{\mathcal{L}}\langle -1\rangle \to \delta_{\mathcal{L}}$ is the map that induces the identity at $\dot{e} \in G$. The left square is commutative by Proposition 6.12(2); the right triangle is commutative by the characterization of ψ_s. This finishes the proof. $\square$

COROLLARY 7.8. *The map $c_{\eta,\xi}(\mathcal{G}, \mathcal{F})$ in (7.6) is an isomorphism if either $\mathcal{F} \in {}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}}$ is a semisimple complex or $\mathcal{G} \in {}_{\mathcal{L}''}\mathcal{D}^{\gamma}_{\mathcal{L}'}$ is a semisimple complex.*

Proof. By symmetry, we only need to treat the case $\mathcal{F}$ semisimple, and it suffices to work with nonmixed complexes. Since any simple perverse sheaf

$\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ is a direct summand of a successive convolution $\underline{\mathrm{IC}}(s_{i_1}, \ldots, s_{i_n})_{\mathcal{L}} := \underline{\mathrm{IC}}(s_{i_1})_{\mathcal{L}_1} \star \cdots \star \underline{\mathrm{IC}}(s_{i_n})_{\mathcal{L}}$, it suffices to prove the statement for $\mathcal{F} = \underline{\mathrm{IC}}(s_{i_1}, \ldots, s_{i_n})_{\mathcal{L}}$ for any sequence of simple reflections $(s_{i_1}, \ldots, s_{i_n})$ in W. But the latter case follows by successive application of either Lemma 7.6 or Lemma 7.7. ☐

The next result is the main result of this section. It is a monodromic version of Soergel's theorem [27, Erweiterungssatz 17]. For the nonmonodromic Hecke categories, the Erweiterungssatz (Extension Theorem) of Soergel is a special case of a more general result of Ginzburg for varieties with $\mathbb{G}_m$-actions [9]. Our argument below is specific to the Hecke categories.

THEOREM 7.9. *Let $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, $\xi \in {}_{\mathcal{L}'}\mathfrak{P}^{\beta}_{\mathcal{L}}$ and let $\mathcal{F}, \mathcal{G} \in {}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}}$ be semisimple complexes. Then the natural map*

$$m(\mathcal{F}, \mathcal{G}) : \mathrm{Hom}^{\bullet}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}^{\bullet}_{R\otimes R\text{-gmod}}(\mathbb{M}_{\xi}(\mathcal{F}), \mathbb{M}_{\xi}(\mathcal{G}))$$

is an isomorphism in $(R \otimes R, \mathrm{Fr})$-gmod.

Proof. Since $m(\mathcal{F}, \mathcal{G})$ is Fr-equivariant, if suffices to prove that $m(\mathcal{F}, \mathcal{G})$ is an isomorphism in $R \otimes R$-gmod. Therefore we may assume $\mathcal{F}, \mathcal{G} \in {}_{\mathcal{L}'}\underline{\mathcal{D}}^{\beta}_{\mathcal{L}}$. In the rest of the argument, we only consider the nonmixed Soergel functors, and we do not specify the nonmixed minimal IC sheaves defining them (the nonmixed minimal IC sheaves are unique up to isomorphism in each block); we simply write $\underline{\mathbb{M}}$ for the nonmixed Soergel functor.

Since every semisimple complex is a direct sum of shifts of $\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ (for $w \in \beta$), it suffices to prove the above isomorphism for $\mathcal{F} = \underline{\mathrm{IC}}(w)_{\mathcal{L}}$. For a sequence $\underline{w} = (s_{i_1}, \ldots, s_{i_n})$ of simple reflections, write $\underline{\mathrm{IC}}(\underline{w})_{\mathcal{L}} = \underline{\mathrm{IC}}(s_{i_1})_{s_{i_2}\cdots s_{i_n}\mathcal{L}} \star \cdots \star \underline{\mathrm{IC}}(s_{i_n})_{\mathcal{L}}$. By the decomposition theorem [1], every $\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ is a direct summand of $\underline{\mathrm{IC}}(\underline{w})_{\mathcal{L}}$ for some sequence $\underline{w}$. It suffices to treat the case $\mathcal{F} = \underline{\mathrm{IC}}(\underline{w})_{\mathcal{L}}$ for a sequence $\underline{w} = (s_{i_1}, \ldots, s_{i_n})$ of simple reflections, that is, showing the following is an isomorphism

$$\mathrm{Hom}^{\bullet}(\underline{\mathrm{IC}}(\underline{w})_{\mathcal{L}}, \mathcal{G}) \to \mathrm{Hom}^{\bullet}_{R\otimes R\text{-gmod}}(\underline{\mathbb{M}}(\underline{\mathrm{IC}}(\underline{w})_{\mathcal{L}}), \underline{\mathbb{M}}(\mathcal{G})). \tag{7.8}$$

We prove this by induction on the length of $\underline{w}$ (and varying block β accordingly). If $\underline{w} = \varnothing$, this means $\underline{\mathrm{IC}}(\underline{w})_{\mathcal{L}} \cong \underline{\delta}_{\mathcal{L}}$. This case will be treated in Lemma 7.10.

Now suppose (7.8) is an isomorphism for all $\underline{w}$ of length $< n$. Consider a sequence $\underline{w} = (s_{i_1}, \ldots, s_{i_n})$ of length n and arbitrary semisimple complex $\mathcal{G}$ in the same block as $\underline{\mathrm{IC}}(\underline{w})_{\mathcal{L}}$. Let $\underline{w}' = (s_{i_1}, \ldots, s_{i_{n-1}})$, $s = s_{i_n}$; then $\underline{\mathrm{IC}}(\underline{w})_{\mathcal{L}} \cong \underline{\mathrm{IC}}(\underline{w}')_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}}$. Consider the following diagram, where each solid arrow is well defined up to a nonzero scalar:

$$\begin{tikzcd}
\mathrm{Hom}^\bullet(\underline{\mathrm{IC}}(\underline{w}')_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}}, \mathcal{G}) \arrow[r, "a"] \arrow[d, "m"] & \mathrm{Hom}^\bullet(\underline{\mathrm{IC}}(\underline{w}')_{s\mathcal{L}}, \mathcal{G} \star \underline{\mathrm{IC}}(s)_{s\mathcal{L}}) \arrow[d, "m'"] \\
\mathrm{Hom}^\bullet(\underline{\mathbb{M}}(\underline{\mathrm{IC}}(\underline{w}')_{s\mathcal{L}} \star \underline{\mathrm{IC}}(s)_{\mathcal{L}}), \underline{\mathbb{M}}(\mathcal{G})) \arrow[d, "u"] & \mathrm{Hom}^\bullet(\underline{\mathbb{M}}(\underline{\mathrm{IC}}(\underline{w}')_{s\mathcal{L}}), \underline{\mathbb{M}}(\mathcal{G} \star \underline{\mathrm{IC}}(s)_{s\mathcal{L}})) \arrow[d, "u'"] \\
\mathrm{Hom}^\bullet(\underline{\mathbb{M}}(\underline{\mathrm{IC}}(\underline{w}')_{s\mathcal{L}}) \otimes_R \underline{\mathbb{M}}(\underline{\mathrm{IC}}(s)_{\mathcal{L}}), \underline{\mathbb{M}}(\mathcal{G})) \arrow[r, dashed, "b"] & \mathrm{Hom}^\bullet(\underline{\mathbb{M}}(\underline{\mathrm{IC}}(\underline{w}')_{s\mathcal{L}}), \underline{\mathbb{M}}(\mathcal{G}) \otimes_R \underline{\mathbb{M}}(\underline{\mathrm{IC}}(s)_{s\mathcal{L}}))
\end{tikzcd} \tag{7.9}$$

Here the map a is the adjunction isomorphism either from Lemma 3.6 if $s \notin W^\circ_{\mathcal{L}}$ or as in Corollary 3.9 if $s \in W^\circ_{\mathcal{L}}$. The maps m and m' are given by the functor $\underline{\mathbb{M}}$, and m' is an isomorphism by inductive hypothesis for $\underline{w}'$. The isomorphisms u and u' are induced by the monoidal structure of $\underline{\mathbb{M}}$ proved in Corollary 7.8. Therefore, to show that m is an isomorphism, it suffices to construct the dotted arrow b, which is an isomorphism and makes the diagram commutative up to a nonzero scalar.

If $s \notin W^\circ_{\mathcal{L}}$, using Lemma 7.6, we have

$$\begin{aligned}
&\mathrm{Hom}^\bullet(\underline{\mathbb{M}}(\underline{\mathrm{IC}}(\underline{w}')_{s\mathcal{L}}) \otimes_R \underline{\mathbb{M}}(\underline{\mathrm{IC}}(s)_{\mathcal{L}}), \underline{\mathbb{M}}(\mathcal{G})) \\
&\quad \cong \mathrm{Hom}^\bullet(\underline{\mathbb{M}}(\underline{\mathrm{IC}}(\underline{w}')_{s\mathcal{L}}) \otimes_R R(s), \underline{\mathbb{M}}(\mathcal{G})) \\
&\quad \cong \mathrm{Hom}^\bullet(\underline{\mathbb{M}}(\underline{\mathrm{IC}}(\underline{w}')_{s\mathcal{L}}), \underline{\mathbb{M}}(\mathcal{G}) \otimes_R R(s)) \\
&\quad \cong \mathrm{Hom}^\bullet(\underline{\mathbb{M}}(\underline{\mathrm{IC}}(\underline{w}')_{s\mathcal{L}}), \underline{\mathbb{M}}(\mathcal{G}) \otimes_R \underline{\mathbb{M}}(\underline{\mathrm{IC}}(s)_{s\mathcal{L}})).
\end{aligned}$$

Let b be the composition of the above isomorphisms. It is easy to check that b makes (7.9) commutative; hence m is an isomorphism.

If $s \in W^\circ_{\mathcal{L}}$, by Lemma 7.7, it suffices to construct an isomorphism

$$\begin{aligned}
b' : &\mathrm{Hom}^\bullet(\underline{\mathbb{M}}(\underline{\mathrm{IC}}(\underline{w}')_{\mathcal{L}} \otimes_{R^s} R\langle 1\rangle, \underline{\mathbb{M}}(\mathcal{G})) \\
&\xrightarrow{\sim} \mathrm{Hom}^\bullet(\underline{\mathbb{M}}(\underline{\mathrm{IC}}(\underline{w}')_{\mathcal{L}}), \underline{\mathbb{M}}(\mathcal{G}) \otimes_{R^s} R\langle 1\rangle).
\end{aligned}$$

By [28, Proposition 5.10(2)], for $M_1, M_2 \in R \otimes R$-gmod, there is a bifunctorial isomorphism of R-bimodules:

$$\mathrm{Hom}^\bullet_{R\otimes R\text{-gmod}}(M_1, M_2 \otimes_{R^s} R\langle 1\rangle) \cong \mathrm{Hom}^\bullet_{R\otimes R\text{-gmod}}(M_1 \otimes_{R^s} R\langle 1\rangle, M_2). \tag{7.10}$$

Indeed, since $R = R^s \oplus \alpha_s R^s$ (note $\alpha_s^2 \in R^s$), we may identify R with $R^s\langle 1\rangle \oplus R^s\langle -1\rangle$ as graded R^s-modules. For an R-bimodule map $f : M_1 \to M_2 \otimes_{R^s} R\langle 1\rangle = M_2\langle 1\rangle \oplus M_2\langle -1\rangle$, we write $f(x) = (f_{-1}(x), f_1(x))$, where $f_{\pm 1} : M_1 \to M_2\langle \pm 1\rangle$ is $R \otimes R^s$-linear. Then $f \mapsto f_1$ gives an isomorphism $\mathrm{Hom}^\bullet_{R\otimes R\text{-gmod}}(M_1, M_2 \otimes_{R^s} R\langle 1\rangle) \cong \mathrm{Hom}^\bullet_{R\text{-Mod-}R^s}(M_1, M_2\langle -1\rangle)$, with inverse

$f_1 \mapsto (f : x \mapsto (f_1(x\alpha_s), f_1(x)))$. On the other hand, by the adjunction between tensor and forgetful functors, we also have

$$\begin{aligned} \mathrm{Hom}^{\bullet}_{R\otimes R\text{-gmod}}(M_1 \otimes_{R^s} R\langle 1\rangle, M_2) &\cong \mathrm{Hom}^{\bullet}_{R\text{-Mod-}R^s}(M_1\langle 1\rangle, M_2) \\ &\cong \mathrm{Hom}^{\bullet}_{R\text{-Mod-}R^s}(M_1, M_2\langle -1\rangle). \end{aligned}$$

Combining these isomorphisms, we get the desired isomorphism (7.10). Moreover, (7.10) is compatible with the adjunction in Corollary 3.9 under the isomorphisms in Lemma 7.4. Isomorphism (7.10) gives the desired isomorphism b' and hence b, which makes diagram (7.9) commutative up to a nonzero scalar. Therefore m is again an isomorphism in this case. This finishes the proof. □

LEMMA 7.10. *For any semisimple complex $\mathcal{G} \in {}_{\mathcal{L}}\underline{\mathcal{D}}^{\circ}_{\mathcal{L}}$, the natural map*

$$m(\underline{\delta}_{\mathcal{L}}, \mathcal{G}) : \mathrm{Hom}^{\bullet}(\underline{\delta}_{\mathcal{L}}, \mathcal{G}) \to \mathrm{Hom}^{\bullet}_{R\otimes R\text{-gmod}}(R(e), \underline{\mathbb{M}}^{\circ}(\mathcal{G})) \tag{7.11}$$

is an isomorphism of graded R-bimodules.

Proof. Recall the adjunction $i_{e*} : {}_{\mathcal{L}}\underline{\mathcal{D}}(e)_{\mathcal{L}} \leftrightarrow {}_{\mathcal{L}}\underline{\mathcal{D}}^{\circ}_{\mathcal{L}} : i^{!}_{e}$. The adjunction map $i_{e*}i^{!}_{e}\mathcal{G} \to \mathcal{G}$ gives a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^{\bullet}(\underline{\delta}_{\mathcal{L}}, i_{e*}i^{!}_{e}\mathcal{G}) \arrow[r, "m(\underline{\delta}_{\mathcal{L}}, i_{e*}i^{!}_{e}\mathcal{G})"] \arrow[d, "a"] & \mathrm{Hom}^{\bullet}_{R\otimes R\text{-gmod}}(R, \underline{\mathbb{M}}^{\circ}(i_{e*}i^{!}_{e}\mathcal{G})) \arrow[d, "b"] \\
\mathrm{Hom}^{\bullet}(\underline{\delta}_{\mathcal{L}}, \mathcal{G}) \arrow[r, "m(\underline{\delta}_{\mathcal{L}}, \mathcal{G})"] & \mathrm{Hom}^{\bullet}_{R\otimes R\text{-gmod}}(R, \underline{\mathbb{M}}^{\circ}(\mathcal{G}))
\end{tikzcd}$$

We will show that $m(\underline{\delta}_{\mathcal{L}}, \mathcal{G})$ is an isomorphism by showing that the other three arrows in the above diagram are isomorphisms. Here the arrows a and b are induced by the adjunction map $i_{e*}i^{!}_{e}\mathcal{G} \to \mathcal{G}$ and a is an isomorphism by adjunction.

We show that $m(\underline{\delta}_{\mathcal{L}}, i_{e*}i^{!}_{e}\mathcal{G})$ is also an isomorphism. Indeed, by Proposition 3.11(2), $i^{!}_{e}\mathcal{G}$ is a direct sum of shifts of $\underline{C}(e)_{\mathcal{L}}$. It suffices to treat the case where $i^{!}_{e}\mathcal{G}$ is replaced by $\underline{C}(e)_{\mathcal{L}}$, or equivalently replacing $i_{e*}i^{!}_{e}\mathcal{G}$ with $\underline{\delta}_{\mathcal{L}}$, in which case both sides are identified with the regular R-bimodule $R(e) = R$.

Finally, we show that b is an isomorphism. Using the filtration $F_{\leqslant w}\underline{\mathbb{M}}^{\circ}(\mathcal{G})$ in Corollary 3.12, we have $F_{\leqslant e}\underline{\mathbb{M}}^{\circ}(\mathcal{G}) = \mathrm{Hom}^{\bullet}(\underline{\Theta}^{\circ}_{\mathcal{L}}, i_{e*}i^{!}_{e}\mathcal{G}) = \underline{\mathbb{M}}^{\circ}(i_{e*}i^{!}_{e}\mathcal{G})$, which implies that b is injective.

To see that b is surjective, we argue that any $R \otimes R$-linear map $\varphi : R(e) \to \underline{\mathbb{M}}^{\circ}(\mathcal{G})$ must land in $F_{\leqslant e}\underline{\mathbb{M}}^{\circ}(\mathcal{G})$. Extend the partial order on $W^{\circ}_{\mathcal{L}}$ to a total order, and suppose $w \in W^\circ_{\mathcal{L}}$ is the smallest element under this total order such that $\varphi(R(e)) \subset F_{\leqslant w}\underline{\mathbb{M}}^\circ(\mathcal{G})$. Then the projection $R \xrightarrow{\varphi} F_{\leqslant w}\underline{\mathbb{M}}^\circ(\mathcal{G}) \to \mathrm{Gr}^F_w\underline{\mathbb{M}}^\circ(\mathcal{G})$ must be nonzero for otherwise $\varphi(R(e))$ would land in the previous step of the filtration. However, by (3.2) and Proposition 3.11(2), $\mathrm{Gr}^F_w\underline{\mathbb{M}}^\circ(\mathcal{G})$ is a free $R(w)$-module. For there to exist a nonzero $R \otimes R$-linear map $R(e) \to R(w)$, we must have $w = e$, which implies that $\varphi(R(e)) \subset F_{\leqslant e}\underline{\mathbb{M}}^\circ(\mathcal{G})$, as desired. This finishes the proof of the lemma. □

8. Soergel bimodules

After reviewing basics about Soergel bimodules, the main result of this section is Proposition 8.7, which connects simple perverse sheaves in the monodromic Hecke category with indecomposable Soergel bimodules via the Soergel functor introduced in the previous section.

8.1. Soergel bimodules. Consider a finite Weyl group (W_0, S_0) with reflection representation V over $\overline{\mathbb{Q}}_\ell$. Let $R = \mathrm{Sym}(V^*)$ (graded with V^* in degree 1). We recall the notion of Soergel R-bimodules.

For any sequence $(s_{i_n}, \dots, s_{i_1})$ of simple reflections, we have the *Bott–Samelson bimodule* $\mathbb{S}(s_{i_n}, \dots, s_{i_1}) := R \otimes_{R^{s_{i_n}}} R \otimes_{R^{s_{i_{n-1}}}} \otimes \cdots \otimes_{R^{s_{i_1}}} R$.

The *indecomposable Soergel bimodules* are, up to degree shifts, indecomposable direct summands of $\mathbb{S}(s_{i_n}, \dots, s_{i_1})$ for some sequence $(s_{i_n}, \dots, s_{i_1})$ of simple reflections in W_0. A Soergel bimodule is a direct sum of indecomposable Soergel bimodules. Let $\mathrm{SB}(W_0) \subset R \otimes R$-gmod be the full subcategory consisting of the Soergel bimodules. Then $\mathrm{SB}(W_0)$ carries a monoidal structure given by the tensor product $(-) \otimes_R (-)$.

Soergel [28] shows that, for each $w \in W_0$, there is an indecomposable Soergel bimodule $\mathbb{S}(w)$ characterized (up to isomorphism) among graded $R \otimes R$-modules, by the following two properties.

(1) $\mathrm{Supp}(\mathbb{S}(w)) \subset V \times V$ contains $\Gamma(w) = \{(wx, x) | x \in V\}$, the graph of the w action on V.

(2) For some (equivalently any) reduced expression $w = s_{i_n} \cdots s_{i_1}$ in W_0, $\mathbb{S}(w)$ is a direct summand of the Bott–Samelson bimodule $\mathbb{S}(s_{i_n}, \dots, s_{i_1})$.

To emphasize the dependence on the Coxeter group W_0, we denote $\mathbb{S}(w)$ also by $\mathbb{S}(w)_{W_0}$.

8.2. Rigidified Soergel bimodules. It is easy to see that the degree zero part of $\mathbb{S}(w)$ is one-dimensional. Moreover, the endomorphism ring of $\mathbb{S}(w)$ inside

$R \otimes R$-gmod consists of scalars. Indeed in our case W_0 is a Weyl group, so we may interpret $\mathbb{S}(w)$ as the equivariant intersection cohomology of an IC sheaf of a Schubert variety in a flag variety, and deduce the statement about the endomorphism ring from it (see [27, Lemma 19 and Erweiterungssatz 17]).

For fixed $w \in W_0$, consider a pair $(M, \mathbf{1}_M)$, where $M \in \mathrm{SB}(W_0)$ is isomorphic to $\mathbb{S}(w)$ and $\mathbf{1}_M \in M^0$ is any nonzero element. Then the automorphism group of such a pair is trivial, and any two such pairs are isomorphic by a unique isomorphism. Therefore we may identify all such pairs with one pair, and denote it by $(\mathbb{S}(w), \mathbf{1})$.

8.3. Extended Soergel bimodules. For $\mathcal{L} \in \mathfrak{o}$ and $w \in W$, we define a graded $R \otimes R$-module $\mathbb{S}(w)_{\mathcal{L}}$ as follows. Let $\beta \in {}_{w\mathcal{L}}\underline{W}_{\mathcal{L}}$ be the block containing w. Write $w = xw^{\beta}$ for $x \in W^{\circ}_{w\mathcal{L}}$. Then we define

$$\mathbb{S}(w)_{\mathcal{L}} := \mathbb{S}(x)_{W^{\circ}_{w\mathcal{L}}} \otimes_R R(w^{\beta}).$$

Again we can rigidify $\mathbb{S}(w)_{\mathcal{L}}$ by equipping it with the degree zero element $\mathbf{1} \otimes 1$.

We also define a generalization of Bott–Samelson modules. For a sequence $(s_{i_n}, \ldots, s_{i_1})$ of simple reflections in W, and $\mathcal{L} \in \mathfrak{o}$, let $\mathcal{L}_j = s_{i_j} \cdots s_{i_1}\mathcal{L}$. Define

$$\mathbb{S}(s_{i_n}, \ldots, s_{i_1}) := \mathbb{S}(s_{i_n})_{\mathcal{L}_{n-1}} \otimes_R \mathbb{S}(s_{i_{n-1}})_{\mathcal{L}_{n-2}} \otimes_R \otimes \cdots \otimes_R \mathbb{S}(s_{i_1})_{\mathcal{L}}.$$

Note that $\mathbb{S}(s_{i_j})_{\mathcal{L}_{j-1}} \cong R(s_{i_j})$ if $s_{i_j} \notin W^{\circ}_{\mathcal{L}_j}$ and is otherwise isomorphic to $R \otimes_{R^{s_{i_j}}} R$.

LEMMA 8.4. *Let* $(s_{i_n}, \ldots, s_{i_1})$ *be a reduced word of simple reflections in* W *and* $\mathcal{L} \in \mathfrak{o}$, $\mathcal{L}_j = s_{i_j} \cdots s_{i_1}\mathcal{L}$ *for* $1 \leqslant j \leqslant n$. *Let* $\beta \in {}_{\mathcal{L}_n}\underline{W}_{\mathcal{L}}$ *be the block containing* $w = s_{i_n} \cdots s_{i_1}$. *Then there is a reduced word* $(t_m, \ldots, t_1)$ *of simple reflections in the Coxeter group* $W^{\circ}_{\mathcal{L}_n}$ *such that* $w = t_m t_{m-1} \cdots t_1 w^{\beta}$ *and*

$$\mathbb{S}(s_{i_n}, \ldots, s_{i_1})_{\mathcal{L}} \cong \mathbb{S}(t_m, \ldots, t_1)_{W^{\circ}_{\mathcal{L}_n}} \otimes_R R(w^{\beta})$$

as graded $R \otimes R$*-modules.*

Proof. We prove the lemma by induction on n. For $n = 1$ and $s = s_{i_1} \notin W^{\circ}_{\mathcal{L}}$, then $s = w^{\beta}$ and $\mathbb{S}(s)_{\mathcal{L}} \cong R(w^{\beta})$ (corresponding to $m = 0$). For $n = 1$ and $s = s_{i_1} \in W^{\circ}_{\mathcal{L}}$, we have $w^{\beta} = 1$ and s is a simple reflection in $W^{\circ}_{\mathcal{L}}$, and $\mathbb{S}(s)_{\mathcal{L}} \cong \mathbb{S}(s)_{W^{\circ}_{\mathcal{L}}}$.

Now suppose the statement is proved for reduced words of length less than n ($n \geqslant 2$). Let $\beta' \in {}_{\mathcal{L}_n}\underline{W}_{\mathcal{L}_1}$ be the block containing $w' = s_{i_n} \cdots s_{i_2}$. By inductive hypothesis, there is a reduced word $(t_m, \ldots, t_1)$ in $W^{\circ}_{\mathcal{L}_n}$ such that $t_m \cdots t_1 w^{\beta'} = w'$

$$\mathbb{S}(s_{i_n}, \ldots, s_{i_2})_{\mathcal{L}_1} \cong \mathbb{S}(t_m, \ldots, t_1)_{W^{\circ}_{\mathcal{L}_n}} \otimes_R R(w^{\beta'}).$$

Write $s = s_{i_1}$. If $s \notin W^\circ_{\mathcal{L}}$, then $w^\beta = w^{\beta'}s$, and we have

$$\begin{aligned}\mathbb{S}(s_{i_n}, \dots, s_{i_1})_{\mathcal{L}} &\cong \mathbb{S}(s_{i_n}, \dots, s_{i_2})_{\mathcal{L}_1} \otimes_R \mathbb{S}(s)_{\mathcal{L}} \\ &\cong \mathbb{S}(t_m, \dots, t_1)_{W^\circ_{\mathcal{L}_n}} \otimes_R R(w^{\beta'}) \otimes_R R(s) \\ &\cong \mathbb{S}(t_m, \dots, t_1)_{W^\circ_{\mathcal{L}_n}} \otimes_R R(w^{\beta'}s) \\ &= \mathbb{S}(t_m, \dots, t_1)_{W^\circ_{\mathcal{L}_n}} \otimes_R R(w^{\beta}).\end{aligned}$$

We have $t_m \cdots t_1 w^\beta = t_m \cdots t_1 w^{\beta'} s = w's = w$.

If $s \in W^\circ_{\mathcal{L}}$, then $\mathcal{L}_1 = \mathcal{L}$, $\beta' = \beta$. Moreover, $t = w^\beta s w^{\beta,-1}$ is a simple reflection in $W^\circ_{\mathcal{L}_n}$ by Corollary 4.4. Hence

$$\begin{aligned}\mathbb{S}(s_{i_n}, \dots, s_{i_1})_{\mathcal{L}} &\cong \mathbb{S}(s_{i_n}, \dots, s_{i_2})_{\mathcal{L}_1} \otimes_{R^s} R \\ &\cong \mathbb{S}(t_m, \dots, t_1)_{W^\circ_{\mathcal{L}_n}} \otimes_R R(w^{\beta}) \otimes_{R^s} R \\ &\cong \mathbb{S}(t_m, \dots, t_1)_{W^\circ_{\mathcal{L}_n}} \otimes_{R^t} R(w^{\beta}) \\ &= \mathbb{S}(t_m, \dots, t_1, t)_{W^\circ_{\mathcal{L}_n}}.\end{aligned}$$

Here we have used $\mathbb{S}(s)_{\mathcal{L}} = R \otimes_{R^s} R$ and $R(w^\beta) \otimes_{R^s} R \cong R \otimes_{R^t} R(w^\beta)$. We have $t_m \cdots t_1 t w^\beta = t_m \cdots t_1 w^\beta s = w's = w$. Since $\ell_\beta(w) = \ell_{\beta'}(w') + 1$ by Lemma 4.6(4), $(t_m, \dots, t_1, t)$ is a reduced word for $ww^{\beta,-1}$. This completes the inductive step. □

We have the following characterization for $\mathbb{S}(w)_{\mathcal{L}}$.

LEMMA 8.5. *Let $\mathcal{L} \in \mathfrak{o}$ and $w \in W$. Let M be an indecomposable graded $R \otimes R$-module such that*

(1) Supp$(M) \supset \Gamma(w)$ *as a subset of* Spec $(R \otimes R) = V \times V$;

(2) *for some reduced expression $w = s_{i_n} s_{i_{n-1}} \cdots s_{i_1}$ in W, M is a direct summand of $\mathbb{S}(s_{i_n}, \dots, s_{i_1})_{\mathcal{L}}$.*

Then $M \cong \mathbb{S}(w)_{\mathcal{L}}$.

Proof. Let $\beta \in {}_{w\mathcal{L}}\underline{W}_{\mathcal{L}}$ be the block containing w. Write $w = xw^\beta$ for $x \in W^\circ_{w\mathcal{L}}$. Let $M' = M \otimes_R R(w^{\beta,-1})$. Then M' is an indecomposable $R \otimes R$-module whose support contains $\Gamma(ww^{\beta,-1}) = \Gamma(x)$. By Lemma 8.4, $\mathbb{S}(s_{i_n}, \dots, s_{i_1})_{\mathcal{L}} \cong \mathbb{S}(t_m, \dots, t_1)_{W^\circ_{\mathcal{L}_n}} \otimes_R R(w^\beta)$ for a reduced expression $t_m \cdots t_1$ of $x = ww^{\beta,-1}$ in $W^\circ_{\mathcal{L}_n}$. Therefore, M' is a direct summand of $\mathbb{S}(t_m, \dots, t_1)_{W^\circ_{\mathcal{L}_n}}$. By Soergel's criterion in Section 8.1, $M' \cong \mathbb{S}(x)_{W^\circ_{\mathcal{L}_n}}$. Hence $M = M' \otimes_R R(w^\beta) \cong \mathbb{S}(x)_{W^\circ_{\mathcal{L}_n}} \otimes_R R(w^\beta) = \mathbb{S}(w)_{\mathcal{L}}$. □

8.6. Soergel bimodules with Frobenius action. Let $(R\otimes R,\mathrm{Fr})\text{-gmod}_{\mathrm{pure}}$ be the full subcategory of $(R\otimes R,\mathrm{Fr})$-gmod consisting of those $M=\oplus_n M^n$ such that M^n is pure of weight n as an Fr-module (Section 1.10.1). Forgetting the Frobenius action gives a functor

$$\omega:(R\otimes R,\mathrm{Fr})\text{-gmod}_{\mathrm{pure}}\to R\otimes R\text{-gmod}.$$

This functor admits a one-side inverse

$$(-)^{\natural}:R\otimes R\text{-gmod}\to(R\otimes R,\mathrm{Fr})\text{-gmod}_{\mathrm{pure}}$$

that sends a graded $R\otimes R$-module $M=\oplus_n M^n$ to the same graded $R\otimes R$-module M with Fr acting on M^n by $q^{n/2}$.

Let $\mathrm{SB}_m(W^\circ_{\mathcal{L}})\subset(R\otimes R,\mathrm{Fr})\text{-gmod}_{\mathrm{pure}}$ be the preimage of $\mathrm{SB}(W^\circ_{\mathcal{L}})$ under ω, that is, it is the full subcategory consisting of $M\in(R\otimes R,\mathrm{Fr})\text{-gmod}_{\mathrm{pure}}$ such that $\omega M\in\mathrm{SB}(W^\circ_{\mathcal{L}})$. Then $\mathrm{SB}_m(W^\circ_{\mathcal{L}})$ also carries a monoidal structure given by $(-)\otimes_R(-)$.

PROPOSITION 8.7. *Let $\mathcal{L}\in\mathfrak{o}$ and $w\in W^\circ_{\mathcal{L}}$. Then there is a unique isomorphism in $(R\otimes R,\mathrm{Fr})$-gmod*

$$\mathbb{M}^\circ(\mathrm{IC}(w)^\dagger_{\mathcal{L}}\langle-\ell_{\mathcal{L}}(w)\rangle)\cong\mathbb{S}(w)^{\natural}_{W^\circ_{\mathcal{L}}}\tag{8.1}$$

under which $\theta^\dagger_w$ corresponds to $\mathbf{1}\in\mathbb{S}(w)^{\natural}_{W^\circ_{\mathcal{L}}}$.

Proof. We first prove more generally for any $w\in W$, we have an isomorphism in $R\otimes R$-gmod:

$$\underline{\mathbb{M}}(\underline{\mathrm{IC}}(w)_{\mathcal{L}}[-\ell_\beta(w)])\cong\mathbb{S}(w)_{\mathcal{L}}.\tag{8.2}$$

Here $\beta\in{}_{w\mathcal{L}}\underline{W}_{\mathcal{L}}$ is the block containing w, and we are suppressing the choice of a minimal IC sheaf $\xi\in{}_{w\mathcal{L}}\mathfrak{P}^{\beta}_{\mathcal{L}}$ from $\underline{\mathbb{M}}_\xi$ because the isomorphism class of the functor $\underline{\mathbb{M}}_\xi$ is independent of ξ. To show (8.2), we apply the criterion in Lemma 8.5 to $M=\underline{\mathbb{M}}(\underline{\mathrm{IC}}(w)_{\mathcal{L}}[-\ell_\beta(w)])$. By Theorem 7.9, $\mathrm{End}(M)=\mathrm{End}(\underline{\mathrm{IC}}(w)_{\mathcal{L}})=\overline{\mathbb{Q}}_\ell$; hence M is indecomposable. By Corollary 3.12, M admits a filtration indexed by $\{v\in W;\ v\leqslant w\}$ with the last associated graded

$$\mathrm{Gr}^F_w M\cong\mathrm{Hom}^\bullet(i^*_w\underline{\mathrm{IC}}(w_\beta)[-N_{\mathcal{L}}],i^!_w\underline{\mathrm{IC}}(w)_{\mathcal{L}}[-\ell_\beta(w)]),$$

which by Proposition 6.4 is $\mathrm{Hom}^\bullet(\underline{C}(w)[-\ell_\beta(w)],\underline{C}(w)[-\ell_\beta(w)])\cong R(w)$. Therefore $\mathrm{Supp}(M)\supset\mathrm{Supp}(R(w))=\Gamma(w)$. Finally, for any reduced expression $w=s_{i_n}\cdots s_{i_1}$, $\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ is a direct summand of $\underline{\mathrm{IC}}(s_{i_n},\dots,s_{i_1})_{\mathcal{L}}=\underline{\mathrm{IC}}(s_{i_n})_{\mathcal{L}_{n-1}}\star\cdots\star\underline{\mathrm{IC}}(s_{i_1})_{\mathcal{L}}$ by the decomposition theorem. Therefore M

is a direct summand of $\underline{\mathbb{M}}(\underline{\mathrm{IC}}(s_{i_n},\dots,s_{i_1})_{\mathcal{L}}[-\ell_\beta(w)])$. By repeated applications of Lemmas 7.3 and 7.4, one sees that

$$\underline{\mathbb{M}}(\underline{\mathrm{IC}}(s_{i_n},\dots,s_{i_1})_{\mathcal{L}}[-\ell_\beta(w)]) \cong \mathbb{S}(s_{i_n},\dots,s_{i_1})_{\mathcal{L}}.$$

The shift by $\ell_\beta(w)$ matches the number of $1 \leqslant j \leqslant n$ such that $s_{i_j} \in W^\circ_{\mathcal{L}_{j-1}}$ by Lemma 4.6(4), which enters the above calculation because of the shift [1] that appears in Lemma 7.4(1). The above checks the conditions in Lemma 8.5 and hence (8.2) is proved.

Now consider the case $w \in W^\circ_{\mathcal{L}}$ and let $M^\dagger = \mathbb{M}^\circ(\mathrm{IC}(w)^\dagger_{\mathcal{L}}\langle -\ell_{\mathcal{L}}(w)\rangle)$. We have already proved that $\omega M^\dagger \cong \mathbb{S}(w)_{W^\circ_{\mathcal{L}}}$ as a graded $R \otimes R$-module. Now the Frobenius action on $\mathbb{S}(w)_{W^\circ_{\mathcal{L}}}$ compatible with the grading and the $R \otimes R$-action is unique up to a scalar (proof: if F and F' are two such Frobenius actions on $\mathbb{S}(w)_{W^\circ_{\mathcal{L}}}$, then $F' \circ F^{-1} \in \mathrm{Aut}_{R\otimes R\text{-gmod}}(\mathbb{S}(w)_{W^\circ_{\mathcal{L}}}) \cong \mathrm{Aut}(\underline{\mathrm{IC}}(w)_{\mathcal{L}}) = \overline{\mathbb{Q}}_\ell^\times$). Therefore, $M^\dagger \cong \mathbb{S}(w)^\natural_{W^\circ_{\mathcal{L}}} \otimes V$ for some one-dimensional Fr-module V. In particular, we have an identification of Fr-modules $(M^\dagger)^0 \cong V$. Now $0 \neq \theta^\dagger_w \in (M^\dagger)^0$ is Fr-invariant; hence V is a trivial Fr-module. Therefore $M^\dagger \cong \mathbb{S}(w)^\natural_{W^\circ_{\mathcal{L}}}$; such an isomorphism is unique up to a scalar, and it becomes unique if we require $\theta^\dagger_w$ to go to $\mathbf{1}$. □

8.8. More on Soergel bimodules. The rest of the section is only used in the proof of Proposition 9.5. Let $\mathcal{L} \in \mathfrak{o}$ and consider Soergel bimodules for $W^\circ_{\mathcal{L}}$. In the rest of the section, we shall denote $\mathbb{S}(w)_{W^\circ_{\mathcal{L}}}$ simply by $\mathbb{S}(w)$. To each indecomposable Soergel bimodule $S \cong \mathbb{S}(w)[n]$, we assign the integer $d(S) := -n + \ell_{\mathcal{L}}(w)$. This is the analogue of the perverse degree for Soergel bimodules.

LEMMA 8.9. *Let S, S' be indecomposable Soergel bimodules for $W^\circ_{\mathcal{L}}$.*

(1) *If $d(S) < d(S')$, then* $\mathrm{Hom}_{R\otimes R\text{-gmod}}(S, S') = 0$.

(2) *If $d(S) = d(S')$ and S and S' are not isomorphic, then* $\mathrm{Hom}_{R\otimes R\text{-gmod}}(S, S') = 0$.

Proof. If $S = \mathbb{S}(w)[n]$ and $S' = \mathbb{S}(w')[n']$, then by Proposition 8.7, $\underline{\mathbb{M}}^\circ(\underline{\mathrm{IC}}(w)_{\mathcal{L}}[n-\ell_{\mathcal{L}}(w)]) \cong S$, $\underline{\mathbb{M}}^\circ(\underline{\mathrm{IC}}(w')_{\mathcal{L}}[n'-\ell_{\mathcal{L}}(w')]) \cong S'$. By Theorem 7.9,

$$\begin{aligned}\mathrm{Hom}_{R\otimes R\text{-gmod}}(S, S') &= \mathrm{Hom}(\underline{\mathrm{IC}}(w)[n-\ell_{\mathcal{L}}(w)], \underline{\mathrm{IC}}(w')[n'-\ell_{\mathcal{L}}(w')]) \\ &= \mathrm{Hom}(\underline{\mathrm{IC}}(w)[-d(S)], \underline{\mathrm{IC}}(w')[-d(S')]).\end{aligned}$$

If $d(S) < d(S')$, then $\mathrm{Hom}(\underline{\mathrm{IC}}(w)[-d(S)], \underline{\mathrm{IC}}(w')[-d(S')]) = 0$ by perverse degree reasons; therefore $\mathrm{Hom}_{R\otimes R\text{-gmod}}(S, S') = 0$.

If $d(S) = d(S')$, then $\mathrm{Hom}(\underline{\mathrm{IC}}(w)[-d(S)], \underline{\mathrm{IC}}(w')[-d(S')])$ is the same as $\mathrm{Hom}(\underline{\mathrm{IC}}(w), \underline{\mathrm{IC}}(w'))$, which vanishes if $w \neq w'$. In this case for $w \neq w'$, we get $\mathrm{Hom}_{R\otimes R\text{-gmod}}(S, S') = 0$. □

PROPOSITION 8.10. *Let $M \in \mathrm{SB}_m(W^\circ_{\mathcal{L}})$. There exists a finite filtration $0 = F_0 M \subset F_1 M \subset \cdots \subset F_n M = M$ by subobjects in $\mathrm{SB}_m(W^\circ_{\mathcal{L}})$ with the following properties:*

(1) *For $1 \leqslant i \leqslant n$, $\mathrm{Gr}^F_i M \cong \mathbb{S}(w_i)^\natural\langle n_i\rangle \otimes V_i$ for some $w_i \in W^\circ_{\mathcal{L}}$, $n_i \in \mathbb{Z}$ and finite-dimensional* Fr*-module V_i pure of weight zero.*

(2) *The filtration $\omega F_\bullet M$ of ωM splits in $R \otimes R$*-gmod.

Proof. Let $M = \oplus_n M^n \in \mathrm{SB}_m(W^\circ_{\mathcal{L}})$. Since M is finitely generated as a graded $R \otimes R$-module, each M^n is finite-dimensional, and may be decomposed into generalized eigenspaces of Fr. We group the generalized Frobenius eigenvalues according to the cosets $\overline{\mathbb{Q}}^\times_\ell / q^{\mathbb{Z}}$:

$$M = \oplus_{\lambda \in \overline{\mathbb{Q}}^\times_\ell / q^{\mathbb{Z}}} M_\lambda.$$

Since Fr acts on $R \otimes R$ by integer powers of q, each M_λ is itself an object in $(R \otimes R, \mathrm{Fr})$-gmod; since ωM_λ is a direct summand of a Soergel bimodule, it is also a Soergel bimodule. Hence $M_\lambda \in \mathrm{SB}_m(W^\circ_{\mathcal{L}})$. We only need to produce a filtration for each M_λ. Without loss of generality, we consider the case $\lambda = 1$ and assume $M = M_1$, that is, Fr-eigenvalues on M are in $q^{\mathbb{Z}}$. In particular, M is evenly graded.

Consider any decomposition $\omega M = \oplus_{\alpha\in I} S_\alpha$, where each S_α is an indecomposable Soergel bimodule. Let $F_i M = \oplus_{\alpha\in I, d(S_\alpha)\leqslant i} S_\alpha \subset M$. By Lemma 8.9(1), $F_i M$ is independent of the decomposition of ωM into indecomposables. Therefore each $F_i M$ is stable under Fr, and hence an object in $\mathrm{SB}_m(W^\circ_{\mathcal{L}})$. Moreover, by Lemma 8.9(2), we can canonically write $\omega \mathrm{Gr}^F_i M = \oplus_{w\in W^\circ_{\mathcal{L}}} \mathbb{S}(w)[\ell_{\mathcal{L}}(w) - i] \otimes {}_iV_w$, where ${}_iV_w = \mathrm{Hom}_{R\otimes R\text{-gmod}}(\mathbb{S}(w)[\ell_{\mathcal{L}}(w) - i], \omega \mathrm{Gr}^F_i M)$. Equip ${}_iV_w$ with the Frobenius action by viewing it as $\mathrm{Hom}_{R\otimes R\text{-gmod}}(\mathbb{S}(w)^\natural\langle \ell_{\mathcal{L}}(w) - i\rangle, \mathrm{Gr}^F_i M)$; then $\mathrm{Gr}^F_i M \cong \oplus_{w\in W^\circ_{\mathcal{L}}} \mathbb{S}(w)^\natural\langle \ell_{\mathcal{L}}(w) - i\rangle \otimes {}_iV_w$ as objects in $(R \otimes R, \mathrm{Fr})$-gmod. After refining the filtration $F_\bullet M$ and renumbering, it becomes a filtration satisfying the required conditions. □

9. Equivalence for the neutral block

In this section, we prove Theorem 1.3.

9.1. The endoscopic group. Let $\mathcal{L} \in \mathfrak{o}$ and consider the neutral block ${}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}$. Let H be the reductive group over $\mathbb{F}_q$ with a maximal torus identified with T and the root system $\Phi(H, T) = \Phi_{\mathcal{L}} \subset \mathbb{X}^*(T)$. In particular, the Weyl group of H with respect to T is identified with $W^{\circ}_{\mathcal{L}}$. We call H the *endoscopic group of G corresponding to $\mathcal{L}$*. Let $B_H \subset H$ be the Borel subgroup containing T corresponding to the positive roots $\Phi^+_{\mathcal{L}}$. As defined, H is unique up to nonunique isomorphisms. We will give a rigidification of H later in Section 10.2.

To justify the terminology, we recall the usual definition of endoscopic groups of G. Let $\widehat{G}$ be the Langlands dual group of G defined over $\overline{\mathbb{Q}}_\ell$, with a maximal torus $\widehat{T}$ and roots $\Phi(\widehat{G}, \widehat{T}) \subset \mathbb{X}^*(\widehat{T}) = \mathbb{X}_*(T)$ identified with the coroots $\Phi^\vee(G, T)$ of (G, T). Let κ be a semisimple element in $\widehat{G}$, and $\widehat{H}$ be the neutral component of the centralizer $\widehat{G}_\kappa$. An endoscopic group of G associated with κ is a reductive group over $\mathbb{F}_q$ whose Langlands dual is isomorphic to $\widehat{H}$.

Now let $\widehat{H} \subset \widehat{G}$ be the connected reductive subgroup containing $\widehat{T}$ with roots $\Phi(\widehat{H}, \widehat{T}) = \Phi^\vee_{\mathcal{L}} \subset \Phi^\vee(G, T) = \Phi(\widehat{G}, \widehat{T})$. Then $\widehat{H}$ is dual to H. We claim that $\widehat{H}$ is the neutral component of the centralizer in $\widehat{G}$ of a semisimple element $\kappa \in \widehat{T}$, which would imply that H is an endoscopic group of G in the usual sense. In fact, bijection (2.1) allows us to identify $\mathrm{Ch}(T)$ with $\mathbb{X}^*(T) \otimes_{\mathbb{Z}} \mathrm{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}}_\ell^\times)$. Choosing a generator $\zeta \in \mathbb{F}_q^\times$, we get an isomorphism $\mathrm{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}}_\ell^\times) \cong \mu_{q-1}(\overline{\mathbb{Q}}_\ell)$ by evaluating at ζ; hence $\mathrm{Ch}(T) \xrightarrow{\sim} \mathbb{X}^*(T) \otimes_{\mathbb{Z}} \mu_{q-1}(\overline{\mathbb{Q}}_\ell) = \widehat{T}[q-1](\overline{\mathbb{Q}}_\ell)$. This allows us to turn $\mathcal{L} \in \mathrm{Ch}(T)$ into an element $\kappa \in \widehat{T}(\overline{\mathbb{Q}}_\ell)$ such that $\kappa^{q-1} = 1$. Then we have $\widehat{H} = \widehat{G}^{\circ}_\kappa$ as subgroups of $\widehat{G}$.

With the correspondence $\mathcal{L} \leftrightarrow \kappa \in \widehat{T}[q-1](\overline{\mathbb{Q}}_\ell)$ above, $W_{\mathcal{L}}$ is identified with the Weyl group of the possibly disconnected group $\widehat{G}_\kappa$ with respect to $\widehat{T}$, that is, $W_{\mathcal{L}} \cong N_{\widehat{G}_\kappa}(\widehat{T})/\widehat{T}$. If G has a connected center so that $\widehat{G}$ has a simply connected derived group, $\widehat{G}_\kappa$ is connected by Steinberg [**29**, Theorem 8.1]; hence $W_{\mathcal{L}} = W^{\circ}_{\mathcal{L}}$ in this case (see also [**7**, Theorem 5.13]).

Consider the usual Hecke category for H:

$$\mathcal{D}_H := D^b_m(B_H \backslash H / B_H).$$

We denote by $\mathrm{IC}(w)_H, \Delta(w)_H$ and $\nabla(w)_H$ the objects in $\mathcal{D}_H$ that are the intersection complex, standard perverse sheaf and costandard perverse sheaf supported on the closure of the Schubert cell $B_H w B_H / B_H \subset H/B_H$ defined similarly as in (2.4) for H in place of G and the trivial character sheaf on T in place of $\mathcal{L}$.

THEOREM 9.2 (Monodromic–endoscopic equivalence for the neutral block). *Let $\mathcal{L} \in \mathrm{Ch}(T)$ and H be the endoscopic group of G attached to $\mathcal{L}$ as in Section 9.1. Then there is a canonical monoidal equivalence of triangulated* *categories*

$$\Psi^{\circ}_{\mathcal{L}} : \mathcal{D}_H \xrightarrow{\sim} {}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}$$

satisfying the following:

(1) *For all* $w \in W^{\circ}_{\mathcal{L}}$,

$$\Psi^{\circ}_{\mathcal{L}}(\mathrm{IC}(w)_H) \cong \mathrm{IC}(w)^{\dagger}_{\mathcal{L}}, \tag{9.1}$$

$$\Psi^{\circ}_{\mathcal{L}}(\Delta(w)_H) \cong \Delta(w)^{\dagger}_{\mathcal{L}}, \tag{9.2}$$

$$\Psi^{\circ}_{\mathcal{L}}(\nabla(w)_H) \cong \nabla(w)^{\dagger}_{\mathcal{L}}. \tag{9.3}$$

In particular, $\Psi^{\circ}_{\mathcal{L}}$ *is* t*-exact for the perverse* t*-structures.*

(2) *There is a functorial isomorphism of graded* $(R \otimes R, \mathrm{Fr})$*-modules for all* $\mathcal{F}, \mathcal{F}' \in \mathcal{D}_H$,

$$\mathrm{Hom}^{\bullet}(\mathcal{F}, \mathcal{F}') \xrightarrow{\sim} \mathrm{Hom}^{\bullet}(\Psi^{\circ}_{\mathcal{L}}(\mathcal{F}), \Psi^{\circ}_{\mathcal{L}}(\mathcal{F}')). \tag{9.4}$$

Part (2) of the theorem does not automatically follow from the equivalence $\Psi^{\circ}_{\mathcal{L}}$ because, as in (1.1), $\mathrm{Hom}(-,-)$ denotes the Hom space after base change to $k = \overline{\mathbb{F}}_q$.

The proof will occupy Sections 9.3–9.8.

9.3. DG model for ${}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}$. We apply the construction of [6, Section B.1–B.2] to the category ${}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}$. Let ${}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}} \subset {}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}$ be the full subcategory consisting of objects that are pure of weight zero. By Proposition 3.11, any object $\mathcal{F} \in {}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}}$ is also *very pure* in the sense that $i^*_w\mathcal{F}$ and $i^!_w\mathcal{F}$ are pure of weight zero for all $w \in W^{\circ}_{\mathcal{L}}$. Then ${}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}}$ is an additive Karoubian category stable under the operation $(-) \otimes V$, where V is any bounded complex of finite-dimensional Fr-modules such that $\mathrm{H}^i V$ has weight i. In particular, ${}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}}$ is stable under $\langle n \rangle$, for all $n \in \mathbb{Z}$. By Lemma 3.3(2), ${}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}}$ is a monoidal category under convolution.

Let ${}_{\mathcal{L}}\underline{\mathcal{C}}^{\circ}_{\mathcal{L}}$ be the essential image of ${}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}}$ under $\omega : {}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}} \to {}_{\mathcal{L}}\underline{\mathcal{D}}^{\circ}_{\mathcal{L}}$. Then ${}_{\mathcal{L}}\underline{\mathcal{C}}^{\circ}_{\mathcal{L}}$ is the category of semisimple complexes in ${}_{\mathcal{L}}\underline{\mathcal{D}}^{\circ}_{\mathcal{L}}$. Let $K^b({}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}})$ be the homotopy category of bounded complexes in ${}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}}$. Let $K^b({}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}})_0 \subset K^b({}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}})$ be the thick subcategory consisting of complexes that are null-homotopic when mapped to $K^b({}_{\mathcal{L}}\underline{\mathcal{C}}^{\circ}_{\mathcal{L}})$.

As in [6, Section B.1], with the help of a filtered version of ${}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}$, there is a triangulated functor (the realization functor) $\widetilde{\rho} : K^b({}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}}) \to {}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}$.

LEMMA 9.4. *The functor* $\widetilde{\rho}$ *descends to an equivalence*

$$\rho : K^b({}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}})/K^b({}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}})_0 \to {}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}.$$

Proof. By [6, Proposition B.1.7], $\widetilde{\rho}$ induces an equivalence of triangulated categories

$$\rho : K^b({}_{\mathcal{L}}\mathcal{C}^\circ_{\mathcal{L}})/\ker(\widetilde{\rho}) \xrightarrow{\sim} {}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}}. \tag{9.5}$$

We claim that $\ker(\widetilde{\rho}) = K^b({}_{\mathcal{L}}\mathcal{C}^\circ_{\mathcal{L}})_0$. The inclusion $\ker(\widetilde{\rho}) \subset K^b({}_{\mathcal{L}}\mathcal{C}^\circ_{\mathcal{L}})_0$ is proved in [6, Lemma B.1.6]. We now show the inclusion in the other direction. Suppose $\mathcal{K}^\bullet \in K^b({}_{\mathcal{L}}\mathcal{C}^\circ_{\mathcal{L}})_0$, and $h : \omega\mathcal{K}^\bullet \to \omega\mathcal{K}^\bullet[-1]$ is a homotopy between $\mathrm{id}_{\mathcal{K}^\bullet}$ and 0. Then for any $\mathcal{F} \in {}_{\mathcal{L}}\underline{\mathcal{C}}^\circ_{\mathcal{L}}$, $\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{K}^\bullet)$ is calculated by a spectral sequence whose E_1-page consists of $E_1^{i,j} = \mathrm{Ext}^j(\mathcal{F}, \mathcal{K}^i)$ with differentials $E_1^{i,j} \to E_1^{i+1,j}$ induced by the differentials of $\mathcal{K}^\bullet$. The chain homotopy h implies that $E_1^{\bullet,j}$ is null-homotopic; hence $E_2 = 0$ and $\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{K}^\bullet) = 0$ for all $\mathcal{F} \in {}_{\mathcal{L}}\underline{\mathcal{C}}^\circ_{\mathcal{L}}$. This implies that $\omega\widetilde{\rho}(\mathcal{K}^\bullet) = 0$ in ${}_{\mathcal{L}}\underline{\mathcal{D}}^\circ_{\mathcal{L}}$. Now $\omega : {}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}} \to {}_{\mathcal{L}}\underline{\mathcal{D}}^\circ_{\mathcal{L}}$ is conservative; hence $\widetilde{\rho}(\mathcal{K}^\bullet) = 0$, and $\mathcal{K}^\bullet \in \ker(\widetilde{\rho})$. ☐

PROPOSITION 9.5. *The restriction of* $\mathbb{M}^\circ$ *gives a monoidal equivalence*

$$\varphi_0 : {}_{\mathcal{L}}\mathcal{C}^\circ_{\mathcal{L}} \xrightarrow{\sim} \mathrm{SB}_m(W^\circ_{\mathcal{L}})$$

such that for $\mathcal{F}, \mathcal{F}' \in {}_{\mathcal{L}}\mathcal{C}^\circ_{\mathcal{L}}$, *there is a canonical isomorphism in* $(R \otimes R, \mathrm{Fr})$-gmod*:*

$$\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{F}') \cong \mathrm{Hom}^\bullet_{R\otimes R\text{-gmod}}(\varphi_0(\mathcal{F}), \varphi_0(\mathcal{F}')). \tag{9.6}$$

Proof. The monoidal structure of $\mathbb{M}^\circ$ restricted to semisimple complexes is proved in Corollary 7.8. Let $\widetilde{\varphi}_0 := \mathbb{M}^\circ|_{{}_{\mathcal{L}}\mathcal{C}^\circ_{\mathcal{L}}} : {}_{\mathcal{L}}\mathcal{C}^\circ_{\mathcal{L}} \to (R \otimes R, \mathrm{Fr})$-gmod. Isomorphism (9.6) with φ_0 replaced by $\widetilde{\varphi}_0$ follows from Theorem 7.9.

Now for $\mathcal{F}, \mathcal{F}' \in {}_{\mathcal{L}}\mathcal{C}^\circ_{\mathcal{L}}$, $\mathrm{Ext}^i(\mathcal{F}, \mathcal{F}')$ is pure of weight i by the $*$-purity of $\mathcal{F}$ and !-purity of $\mathcal{F}'$ (cf. [6, Lemma 3.1.5]). This implies $\mathrm{hom}_{{}_{\mathcal{L}}\mathcal{C}^\circ_{\mathcal{L}}}(\mathcal{F}, \mathcal{F}') = \mathrm{Hom}(\mathcal{F}, \mathcal{F}')^{\mathrm{Fr}}$ since $\mathrm{Ext}^{-1}(\mathcal{F}, \mathcal{F}')$ is pure of weight -1. It also implies that $(\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{F}'))^{\mathrm{Fr}} = \mathrm{Hom}(\mathcal{F}, \mathcal{F}')^{\mathrm{Fr}} = \mathrm{hom}_{{}_{\mathcal{L}}\mathcal{C}^\circ_{\mathcal{L}}}(\mathcal{F}, \mathcal{F}')$. On the other hand, $\mathrm{hom}_{(R\otimes R,\mathrm{Fr})\text{-gmod}}(M, M') = \mathrm{Hom}_{R\otimes R\text{-gmod}}(M, M')^{\mathrm{Fr}}$ for $M, M' \in \mathrm{SB}_m(W^\circ_{\mathcal{L}})$. Taking Frobenius invariants of both sides of (9.6), we conclude that φ_0 is fully faithful.

We show that the image of $\mathbb{M}^\circ|_{{}_{\mathcal{L}}\mathcal{C}^\circ_{\mathcal{L}}}$ lies in $\mathrm{SB}_m(W^\circ_{\mathcal{L}})$. Indeed for $\mathcal{F} \in {}_{\mathcal{L}}\mathcal{C}^\circ_{\mathcal{L}}$, $\omega\mathcal{F}$ is a semisimple complex; hence $\omega\mathbb{M}^\circ(\mathcal{F}) = \underline{\mathbb{M}}^\circ(\omega\mathcal{F})$ is a direct sum of shifts of $\underline{\mathbb{M}}^\circ(\underline{\mathrm{IC}}(w)_{\mathcal{L}})$, which is isomorphic to a shift of $\mathbb{S}(w)_{W^\circ_{\mathcal{L}}}$ by Proposition 8.7. On the other hand, the very purity of $\Theta^\circ_{\mathcal{L}}$ and $\mathcal{F}$ implies that $\mathrm{Ext}^i(\Theta^\circ_{\mathcal{L}}, \mathcal{F})$ is pure of weight i; hence $\mathbb{M}^\circ(\mathcal{F}) \in (R \otimes R, \mathrm{Fr})\text{-gmod}_{\mathrm{pure}}$. We conclude that $\mathbb{M}^\circ(\mathcal{F}) \in \mathrm{SB}_m(W^\circ_{\mathcal{L}})$.

Finally, we show that any $M \in \mathrm{SB}_m(W^\circ_{\mathcal{L}})$ is in the essential image of φ_0. Let $0 = F_0M \subset F_1M \subset \cdots \subset F_nM = M$ be a filtration satisfying the conditions in Proposition 8.10. In particular, $\mathrm{Gr}^F_i M \cong \mathbb{S}(w_i)^\natural_{W^\circ_{\mathcal{L}}} \otimes V_i$ for some $w_i \in W^\circ_{\mathcal{L}}$

and $V_i \in D^b_m(\mathrm{pt})$ pure of weight zero. We prove by induction on i that $F_i M$ is in the essential image of φ_0. For $i = 0$, there is nothing to prove. Suppose $i \geqslant 1$ and we have found $\mathcal{F}_{i-1} \in {}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}}$ such that $\varphi_0(\mathcal{F}_{i-1}) \cong F_{i-1}M$. Let $\mathcal{K}_i = \mathrm{IC}(w_i)^{\dagger}_{\mathcal{L}}\langle -\ell_{\mathcal{L}}(w_i)\rangle \otimes V_i \in {}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}}$. Let $\epsilon \in \mathrm{Ext}^1_{(R\otimes R,\mathrm{Fr})}(\mathrm{Gr}^F_i M, F_{i-1}M)$ be the extension class of

$$0 \to F_{i-1}M \to F_i M \to \mathrm{Gr}^F_i M \to 0 \tag{9.7}$$

in $(R \otimes R, \mathrm{Fr})$-mod (nongraded modules). We have a short exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{R\otimes R}(\mathrm{Gr}^F_i M, F_{i-1}M)_{\mathrm{Fr}} &\to \mathrm{Ext}^1_{(R\otimes R,\mathrm{Fr})}(\mathrm{Gr}^F_i M, F_{i-1}M) \\ &\to \mathrm{Ext}^1_{R\otimes R}(\mathrm{Gr}^F_i M, F_{i-1}M)^{\mathrm{Fr}} \to 0. \end{aligned}$$

Since (9.7) splits in $R \otimes R$-mod, the image of ϵ in $\mathrm{Ext}^1_{R\otimes R}(\mathrm{Gr}^F_i M, F_{i-1}M)^{\mathrm{Fr}}$ is zero; therefore ϵ comes from a class $\widetilde{\epsilon} \in \mathrm{Hom}_{R\otimes R}(\mathrm{Gr}^F_i M, F_{i-1}M)_{\mathrm{Fr}}$. By Theorem 7.9, $\mathbb{M}^{\circ}$ induces an isomorphism of Fr-modules

$$\mathrm{Hom}^{\bullet}(\mathcal{K}_i, \mathcal{F}_{i-1}) \xrightarrow{\sim} \mathrm{Hom}^{\bullet}_{R\otimes R\text{-gmod}}(\mathrm{Gr}^F_i M, F_{i-1}M) = \mathrm{Hom}_{R\otimes R}(\mathrm{Gr}^F_i M, F_{i-1}M).$$

Therefore $\widetilde{\epsilon}$ can be viewed as a class $\widetilde{\epsilon}' \in \mathrm{Hom}^{\bullet}(\mathcal{K}_i, \mathcal{F}_{i-1})_{\mathrm{Fr}} = \mathrm{Hom}(\mathcal{K}_i, \mathcal{F}_{i-1})_{\mathrm{Fr}}$ (because $\mathrm{Ext}^j(\mathcal{K}_i, \mathcal{F}_{i-1})$ has weight j). Let ϵ' be the image of $\widetilde{\epsilon}'$ under the map $\mathrm{Hom}(\mathcal{K}_i, \mathcal{F}_{i-1})_{\mathrm{Fr}} \to \mathrm{hom}(\mathcal{K}_i, \mathcal{F}_{i-1}[1])$ (the latter is calculated in ${}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}$). Let $\mathcal{F}_i = \mathrm{Cone}(\epsilon')[-1] \in {}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}$. Then $\mathcal{F}_i$ fits into a distinguished triangle $\mathcal{F}_{i-1} \to \mathcal{F}_i \to \mathcal{K}_i \to \mathcal{F}_{i-1}[1]$. Therefore $\mathcal{F}_i \in {}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}}$ and $\varphi_0(\mathcal{F}_i) \cong F_i M$ by construction. □

To state the next theorem, we need some notation. For $\mathcal{F}, \mathcal{F}' \in {}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}$, we let $\mathrm{Ext}^n(\mathcal{F}, \mathcal{F}')_m$ be the weight m summand of the Fr-module $\mathrm{Ext}^n(\mathcal{F}, \mathcal{F}')$. For M, $M' \in K^b(R \otimes R\text{-mod})$, their morphism space in $K^b(R \otimes R\text{-mod})$ is denoted by $\mathrm{HOM}_{K^b(R\otimes R\text{-mod})}(M, M')$, and it is the homotopy classes of $R \otimes R$-linear chain maps $M \to M'$. We denote the degree shift of complexes in $K^b(R \otimes R\text{-mod})$ by $\{1\}$. We denote

$$\mathrm{HOM}^{\bullet}_{K^b(R\otimes R\text{-mod})}(M, M') = \bigoplus_{n\in\mathbb{Z}} \mathrm{HOM}_{K^b(R\otimes R\text{-mod})}(M, M'\{n\}).$$

When $M, M' \in K^b(R \otimes R\text{-gmod})$, $\mathrm{HOM}_{K^b(R\otimes R\text{-mod})}(M, M')$ also carries an internal grading from the gradings of each component M^i and M'^i (which are in $R \otimes R$-gmod), and we denote the graded pieces by $\mathrm{HOM}_{K^b(R\otimes R\text{-mod})}(M, M')_m$. If $M, M' \in K^b((R \otimes R, \mathrm{Fr})\text{-gmod})$, $\mathrm{HOM}_{K^b(R\otimes R\text{-mod})}(M, M')$ also inherits an Fr-module structure.

THEOREM 9.6. *The equivalence $K^b(\varphi_0)$ on the homotopy categories of ${}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}}$ and* $\mathrm{SB}_m(W^{\circ}_{\mathcal{L}})$ *induces a monoidal equivalence of triangulated categories*

$$\varphi_{\mathcal{L}} : {}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}} \xrightarrow{\rho^{-1}} K^b({}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}})/K^b({}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}})_0 \xrightarrow{K^b(\varphi_0)} K^b(\mathrm{SB}_m(W^{\circ}_{\mathcal{L}}))/K^b(\mathrm{SB}_m(W^{\circ}_{\mathcal{L}}))_0.$$

Here $K^b(\mathrm{SB}_m(W^{\circ}_{\mathcal{L}}))_0$ consists of complexes of objects in $\mathrm{SB}_m(W^{\circ}_{\mathcal{L}})$ that become null-homotopic in $\mathrm{SB}(W^{\circ}_{\mathcal{L}})$.

Moreover, for $\mathcal{F}, \mathcal{F}' \in {}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}$, we have a functorial isomorphism of $(R \otimes R, \mathrm{Fr})$-modules

$$\mathrm{Hom}^{\bullet}(\mathcal{F}, \mathcal{F}') \xrightarrow{\sim} \mathrm{HOM}^{\bullet}_{K^b(R\otimes R\text{-mod})}(\omega\varphi_{\mathcal{L}}(\mathcal{F}), \omega\varphi_{\mathcal{L}}(\mathcal{F}')) \tag{9.8}$$

under which

$$\mathrm{Ext}^n(\mathcal{F}, \mathcal{F}')_m \xrightarrow{\sim} \mathrm{HOM}_{K^b(R\otimes R\text{-mod})}(\omega\varphi_{\mathcal{L}}(\mathcal{F}), \omega\varphi_{\mathcal{L}}(\mathcal{F}')\{n-m\})_m, \quad \forall n, m \in \mathbb{Z}. \tag{9.9}$$

Proof. In view of the equivalence φ_0, to prove $\varphi_{\mathcal{L}}$ is an equivalence, it suffices to show that the image of $K^b({}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}})_0$ under $K^b(\varphi_0)$ is $K^b(\mathrm{SB}_m(W^{\circ}_{\mathcal{L}}))_0$. If $\mathcal{K}^{\bullet} \in K^b({}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}})_0$, then there exists a chain homotopy $h_i : \omega\mathcal{K}^i \to \omega\mathcal{K}^{i-1}$ between $\mathrm{id}_{\omega\mathcal{K}^{\bullet}}$ and 0. Applying φ_0 to h_i, we get $\varphi_0(h_i) : \omega\varphi_0(\mathcal{K}^i) \to \omega\varphi_0(\mathcal{K}^{i-1})$, giving a chain homotopy between $\mathrm{id}_{\omega\varphi_0(\mathcal{K}^{\bullet})}$ and 0. The same argument shows that φ_0^{-1} sends $K^b(\mathrm{SB}_m(W^{\circ}_{\mathcal{L}}))_0$ to $K^b({}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}})_0$. This shows that $\varphi_{\mathcal{L}}$ is an equivalence. The monoidal structure of $\varphi_{\mathcal{L}}$ comes from that of φ_0 since $K^b({}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}})_0$ and $K^b(\mathrm{SB}_m(W^{\circ}_{\mathcal{L}}))_0$ are monoidal ideals.

We prove isomorphism (9.8). For $\mathcal{F}^{\bullet}, \mathcal{F}'^{\bullet}$, two bounded complexes in ${}_{\mathcal{L}}\mathcal{C}^{\circ}_{\mathcal{L}}$, let $\mathcal{F} = \rho(\mathcal{F}^{\bullet}), \mathcal{F}' = \rho(\mathcal{F}'^{\bullet}) \in {}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}}$. Then there is a spectral sequence with $E_1^{a,b} = \oplus_{j-i=a}\mathrm{Ext}^b(\mathcal{F}^i, \mathcal{F}'^j)$ that converges to $\mathrm{Ext}^{a+b}(\mathcal{F}, \mathcal{F}')$. The differential $d_1 : E_1^{a,b} \to E_1^{a+1,b}$ is given by an alternating sum of maps induced by the differentials in $\mathcal{F}^{\bullet}$ and $\mathcal{F}'^{\bullet}$. Since $E_1^{a,b}$ is pure of weight b, the spectral sequence degenerates at E_2. This implies that

$$E_2^{a,b} = \mathrm{Ext}^{a+b}(\mathcal{F}, \mathcal{F}')_b. \tag{9.10}$$

Let $M = \omega\varphi_{\mathcal{L}}(\mathcal{F})$, that is, M is a complex with terms $M^i = \omega\mathbb{M}^{\circ}(\mathcal{F}^i) \in \mathrm{SB}(W^{\circ}_{\mathcal{L}})$; similarly, let $M' = \omega\varphi_{\mathcal{L}}(\mathcal{F}')$. By Theorem 7.9, $\mathbb{M}^{\circ}$ induces an isomorphism of Fr-modules $E_1^{a,b} \cong \oplus_{j-i=a}\mathrm{Hom}_{R\otimes R\text{-gmod}}(M^i, M'^j[b])$, and the differential d_1 is given by an alternating sum of differentials in $M^{\bullet}$ and $M'^{\bullet}$. Therefore $E_2^{a,b}$ is isomorphic to $\mathrm{HOM}_{K^b(R\otimes R\text{-gmod})}(M, M'[b]\{a\})$, which is the same as the degree b part of $\mathrm{HOM}_{K^b(R\otimes R\text{-mod})}(M, M'\{a\})$ for the internal grading

$$E_2^{a,b} \cong \mathrm{HOM}_{K^b(R\otimes R\text{-mod})}(M, M'\{a\})_b. \tag{9.11}$$

Comparing (9.10) and (9.11), we get (9.9). Taking the direct sum over all n and m in (9.9), we get (9.8). □

REMARK 9.7. The functor $K^b(\omega) : K^b(\mathrm{SB}_m(W^\circ_{\mathcal{L}})) \to K^b(\mathrm{SB}(W^\circ_{\mathcal{L}}))$ clearly factors through $K^b(\mathrm{SB}_m(W^\circ_{\mathcal{L}}))_0$. It induces a functor

$$\overline{\varphi}_{\mathcal{L}} : {}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}} \to K^b(\mathrm{SB}(W^\circ_{\mathcal{L}})).$$

The homotopy category $K^b(\mathrm{SB}(W^\circ_{\mathcal{L}}))$ can be viewed as an Fr-semisimplified version of the monodromic Hecke category ${}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}}$.

9.8. Finishing the proof of Theorem 9.2. Apply Theorem 9.6 to the endoscopic group H and the trivial character sheaf $\mathcal{L} = \overline{\mathbb{Q}}_\ell \in \mathrm{Ch}(T)$. We get a monoidal equivalence

$$\varphi_H : \mathcal{D}_H \xrightarrow{\sim} K^b(\mathrm{SB}_m(W_H))/K^b(\mathrm{SB}_m(W_H))_0$$

such that for $\mathcal{K}, \mathcal{K}' \in \mathcal{D}_H$, there is a natural $R \otimes R$-linear isomorphism

$$\mathrm{Hom}^\bullet(\mathcal{K}, \mathcal{K}') \xrightarrow{\sim} \mathrm{HOM}^\bullet_{K^b(R\otimes R\text{-mod})}(\omega\varphi_H(\mathcal{K}), \omega\varphi_H(\mathcal{K}')) \tag{9.12}$$

with an analogue of (9.9). Since $W_H = W^\circ_{\mathcal{L}}$, we may identify the target categories of $\varphi_{\mathcal{L}}$ and φ_H.

Let $\Psi^\circ_{\mathcal{L}} = \varphi_{\mathcal{L}}^{-1} \circ \varphi_H$. Then $\Psi^\circ_{\mathcal{L}}$ is a monoidal equivalence of triangulated categories. Combining (9.8) and (9.12), we get (9.4).

It remains to show (9.1)–(9.3). By Proposition 8.7 and its analogue for $\mathcal{D}_H$, we know $\varphi_{\mathcal{L}}(\mathrm{IC}(w)^\dagger_{\mathcal{L}}) \cong \mathbb{S}(w)^\natural_{W^\circ_{\mathcal{L}}}\langle \ell_{\mathcal{L}}(w)\rangle \cong \varphi_H(\mathrm{IC}(w)_H)$ for all $w \in W^\circ_{\mathcal{L}}$. Therefore $\Psi^\circ_{\mathcal{L}}(\mathrm{IC}(w)_H) \cong \mathrm{IC}(w)^\dagger_{\mathcal{L}}$ for all $w \in W^\circ_{\mathcal{L}}$. This proves (9.1).

Now we show (9.2). Let $\mathcal{F} = (\Psi^\circ_{\mathcal{L}})^{-1}(\Delta(w)^\dagger_{\mathcal{L}})$. From the properties of $\Delta(w)^\dagger_{\mathcal{L}}$ and the fact that $(\Psi^\circ)^{-1}$ preserves IC sheaves and Hom spaces, we have the following:

(1) $\mathcal{F}$ lies in the full triangulated subcategory generated by $\mathrm{IC}(w')_H \otimes V$ for $w' \leqslant w$ (Bruhat order of W) and Fr-modules V.

(2) $\mathrm{Hom}^\bullet(\mathcal{F}, \mathrm{IC}(w')_H) = 0$ for all $w' < w$.

(3) There is a canonical isomorphism $\mathrm{Hom}^\bullet(\mathcal{F}, \mathrm{IC}(w)_H) \cong R(w)$ as graded $(R \otimes R, \mathrm{Fr})$-modules.

We show that these properties imply a canonical isomorphism $\mathcal{F} \cong \Delta(w)_H$. Let $Y \subset H$ (respectively, $Z \subset H$) be the union of $B_H w' B_H$ for $w' \leqslant w$ (respectively, $w' < w$) in the Bruhat order of W. By Lemma 4.8(3), Y and Z are closed, and $Y - Z = B_H w B_H = H(w)$ is open in Y. Note that Y is not necessarily the closure of $H(w)$; see Remark 4.9. Property (1) above implies that $\mathcal{F}$ is supported on $B_H \backslash Y / B_H$; property (2) implies that $\mathcal{F}|_Z = 0$. Therefore $\mathcal{F} = j_! \mathcal{G}$ for some $\mathcal{G} \in D^b_m(B_H \backslash H(w)/B_H) =: \mathcal{D}_H(w)$ (where $j : H(w) \hookrightarrow Y$ is the open embedding). Now $\mathrm{Hom}^\bullet(\mathcal{F}, \mathrm{IC}(w)_H) \cong \mathrm{Hom}^\bullet_{\mathcal{D}_H(w)}(\mathcal{G}, \overline{\mathbb{Q}}_\ell)$, and property (3) above gives a map $\mathcal{F} \to \mathrm{IC}(w)_H$ in $\mathcal{D}_H$ (corresponding to $1 \in \overline{\mathbb{Q}}_\ell$ under the isomorphism $\mathrm{Hom}(\mathcal{F}, \mathrm{IC}(w)_H) \cong \overline{\mathbb{Q}}_\ell$, which is Fr-invariant and hence lifts uniquely to $\mathrm{hom}(\mathcal{F}, \mathrm{IC}(w)_H)$ for $\mathrm{Hom}^{-1}(\mathcal{F}, \mathrm{IC}(w)_H) = 0$) and hence a nonzero map $c : \mathcal{G} \to \overline{\mathbb{Q}}_\ell$ in $\mathcal{D}_H(w) \cong D^b_{\Gamma(w),m}(\mathrm{pt})$. Moreover, property (3) implies that $\mathrm{Hom}^\bullet(\mathcal{G}, \overline{\mathbb{Q}}_\ell)$ is a free left R-module of rank one and is generated in degree zero, which implies that $\mathcal{G}$ has rank one and is concentrated in degree zero. Therefore the canonical map c is an isomorphism, and it induces an isomorphism $\mathcal{F} = j_! \mathcal{G} \xrightarrow{\sim} j_! \overline{\mathbb{Q}}_\ell = \Delta(w)_H$. Therefore we get a canonical isomorphism $\Psi^\circ_{\mathcal{L}}(\Delta(w)_H) \cong \Psi^\circ_{\mathcal{L}}(\mathcal{F}) = \Delta(w)^\dagger_{\mathcal{L}}$.

A similar argument proves (9.3). ☐

Remark 9.9 (Parabolic version). It is possible to extend Theorem 9.2 to a parabolic version. Namely, consider two standard parabolic subgroups P and Q of G with unipotent radicals U_P and U_Q and Levi subgroups L and M containing T. Suppose that $\mathcal{L} \in \mathrm{Ch}(T)$ extends to rank-one local systems $\mathcal{K} \in \mathrm{Ch}(L)$ and $\mathcal{K}' \in \mathrm{Ch}(M)$. Then we may consider the category ${}_{\mathcal{K}'}\mathcal{D}_{\mathcal{K}} = D^b_{(M \times L, \mathcal{K}' \boxtimes \mathcal{K}^{-1}),m}(U_Q \backslash G / U_P)$. We still have a block decomposition of ${}_{\mathcal{K}'}\mathcal{D}_{\mathcal{K}}$ indexed by $\Omega_{\mathcal{L}} = W_{\mathcal{L}} / W^\circ_{\mathcal{L}}$.

By Lemma 2.3, we have $\Phi(L, T), \Phi(M, T) \subset \Phi_{\mathcal{L}}$. Hence L and M determine standard parabolic subgroups P_H and Q_H of H whose Levi factors have roots $\Phi(L, T)$ and $\Phi(M, T)$. Then there is an equivalence between the neutral block ${}_{\mathcal{K}'}\mathcal{D}^\circ_{\mathcal{K}}$ and $D^b_m(Q_H \backslash H / P_H)$, which can be proved using similar techniques used in this paper.

We have the following strengthening of the purity result in Proposition 3.11 to include Frobenius semisimplicity for the stalks of IC sheaves. To state it, recall that $P^{W^\circ_{\mathcal{L}}}_{x,y}(t)$ is the Kazhdan–Lusztig polynomial [11] for the Coxeter group $W^\circ_{\mathcal{L}}$, of degree less than $\frac{1}{2}(\ell_{\mathcal{L}}(y) - \ell_{\mathcal{L}}(x))$ if $x < y$. The numerical part of the following result was first proved in [13, Lemma 1.11].

Proposition 9.10. *Let* $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$.

(1) *For $v \leqslant w \in W^{\circ}_{\mathcal{L}}$, write*

$$P^{W^{\circ}_{\mathcal{L}}}_{v,w}(t) = \sum_{n\geqslant 0} a_{v,w}(n)t^{n}.$$

Then we have

$$i^{*}_{v}\mathrm{IC}(w)^{\dagger}_{\mathcal{L}} \cong C(v)^{\dagger}_{\mathcal{L}}\langle \ell_{\mathcal{L}}(w) - \ell_{\mathcal{L}}(v)\rangle \otimes \left(\bigoplus_{n\geqslant 0} \overline{\mathbb{Q}}_{\ell}\langle -2n\rangle^{\oplus a_{v,w}(n)}\right), \quad (9.13)$$

$$i^{!}_{v}\mathrm{IC}(w)^{\dagger}_{\mathcal{L}} \cong C(v)^{\dagger}_{\mathcal{L}}\langle -\ell_{\mathcal{L}}(w) + \ell_{\mathcal{L}}(v)\rangle \otimes \left(\bigoplus_{n\geqslant 0} \overline{\mathbb{Q}}_{\ell}\langle 2n\rangle^{\oplus a_{v,w}(n)}\right). \quad (9.14)$$

(2) *Let w, v be in the same block $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$. Write $v = w^{\beta}x$ and $w = w^{\beta}y$ for $x, y \in W^{\circ}_{\mathcal{L}}$. Then there is a one-dimensional* Fr*-module V_1 (depending on $\dot{w}$ and $\dot{v}$) of weight zero such that*

$$i^{*}_{v}\mathrm{IC}(\dot{w})_{\mathcal{L}} \cong C(\dot{v})_{\mathcal{L}}\langle \ell_{\mathcal{L}}(y) - \ell_{\mathcal{L}}(x)\rangle \otimes \left(\bigoplus_{n\geqslant 0} \overline{\mathbb{Q}}_{\ell}\langle -2n\rangle^{\oplus a_{x,y}(n)}\right) \otimes V_1,$$

$$i^{!}_{v}\mathrm{IC}(\dot{w})_{\mathcal{L}} \cong C(\dot{v})_{\mathcal{L}}\langle -\ell_{\mathcal{L}}(y) + \ell_{\mathcal{L}}(x)\rangle \otimes \left(\bigoplus_{n\geqslant 0} \overline{\mathbb{Q}}_{\ell}\langle 2n\rangle^{\oplus a_{x,y}(n)}\right) \otimes V_1.$$

Proof. (1) We first treat the costalk $i^{!}_{v}\mathrm{IC}(w)^{\dagger}_{\mathcal{L}}$. In Proposition 3.11, we already proved that $\mathcal{K}_v = i^{!}_{v}\mathrm{IC}(w)^{\dagger}_{\mathcal{L}} \in {}_{\mathcal{L}}\mathcal{D}(v)_{\mathcal{L}}$ is a successive extension of $C(v)^{\dagger}\langle n\rangle \otimes V_n$ for finite-dimensional Fr-modules V_n pure of weight zero, and $n \equiv \ell(w) - \ell(v)$ mod 2. We shall first show

$$\mathcal{K}_v \text{ is a direct sum of } C(v)^{\dagger}_{\mathcal{L}}\langle n\rangle \text{ for } n \equiv \ell(w) - \ell(v) \mod 2. \quad (9.15)$$

Let ${}_{\mathcal{L}}\mathcal{C}(v)_{\mathcal{L}} \subset {}_{\mathcal{L}}\mathcal{D}(v)_{\mathcal{L}}$ be the subcategory of complexes that are pure of weight zero. Applying Proposition 9.5 to the case $G = T$ (now $C(v)^{\dagger}_{\mathcal{L}}$ plays the role of $\Theta^{\circ}_{\mathcal{L}}$), we see that $\mathrm{Hom}^{\bullet}(C(v)^{\dagger}_{\mathcal{L}}, -)$ induces a full embedding

$$h : {}_{\mathcal{L}}\mathcal{C}(v)_{\mathcal{L}} \hookrightarrow (R(v), \mathrm{Fr})\text{-mod}. \quad (9.16)$$

Here $R(v) = \mathrm{H}^{*}_{\Gamma(v)_k}(\mathrm{pt}_k)$ is introduced in Section 7.1. Under this embedding, to show (9.15), it suffices to show that $h(\mathcal{K}_v)$ is a direct sum of $R(v)\langle n\rangle$ for $n \equiv \ell(w) - \ell(v) \mod 2$. By Proposition 3.11, $h(\mathcal{K}_v)$ is a successive extension of $R(v)\langle n\rangle \otimes V_n$ for $n \equiv \ell(w) - \ell(v) \mod 2$ and for Fr-modules V_n pure of weight zero. In particular, $h(\mathcal{K}_v)$ is free as an $R(v)$-module. Therefore it suffices to show that $h(\mathcal{K}_v)$, as an Fr-module, is a direct sum of $\overline{\mathbb{Q}}_\ell\langle n\rangle$ for $n\in\mathbb{Z}$ (necessarily of the same parity as $\ell(w)-\ell(v)$): for then $h(\mathcal{K}_v)\otimes_{R(v)}\overline{\mathbb{Q}}_\ell$ is a direct sum of $\overline{\mathbb{Q}}_\ell\langle n\rangle$, and we can lift a basis of $h(\mathcal{K}_v)\otimes_{R(v)}\overline{\mathbb{Q}}_\ell$ consisting of Frobenius eigenvectors to Frobenius eigenvectors in $h(\mathcal{K}_v)$, giving an $R(v)$-basis of $h(\mathcal{K}_v)$.

To summarize, to show (9.15), we only need to show that $h(\mathcal{K}_v)$ is a direct sum of $\overline{\mathbb{Q}}_\ell\langle n\rangle$ as an Fr-module. By Corollary 3.12, $\mathrm{Hom}^\bullet(i_v^*\Theta^\circ_{\mathcal{L}}, \mathcal{K}_v)=\mathrm{Gr}^F_v\mathbb{M}^\circ(\mathrm{IC}(w)^\dagger_{\mathcal{L}})$, the latter being a subquotient of $\mathbb{S}(w)^\natural_{W^\circ_{\mathcal{L}}}\langle\ell_{\mathcal{L}}(w)\rangle$ (by Proposition 8.7), and is hence a direct sum of $\overline{\mathbb{Q}}_\ell\langle n\rangle$. Since $i_v^*\Theta^\circ_{\mathcal{L}}=C(v)^\dagger_{\mathcal{L}}\langle-\ell_{\mathcal{L}}(v)\rangle$, $h(\mathcal{K}_v)=\mathrm{Gr}^F_v\mathbb{M}^\circ(\mathrm{IC}(w)^\dagger_{\mathcal{L}})\langle-\ell_{\mathcal{L}}(v)\rangle$ is a direct sum of $\overline{\mathbb{Q}}_\ell\langle n\rangle$. This proves (9.15).

By Theorem 9.2, we have

$$h(\mathcal{K}_v)=\mathrm{Hom}^\bullet(\Delta(v)^\dagger_{\mathcal{L}},\mathrm{IC}(w)^\dagger_{\mathcal{L}})\cong\mathrm{Hom}^\bullet(\Delta(v)_H,\mathrm{IC}(w)_H),$$

which by adjunction is $\mathrm{Hom}^\bullet(C(v)_H, i_v^!\mathrm{IC}(w)_H)$. Therefore the multiplicity of $C(v)^\dagger_{\mathcal{L}}\langle n\rangle$ in $\mathcal{K}_v$ is the same as the multiplicity of $C(v)_H\langle n\rangle$ in $i_v^!\mathrm{IC}(w)_H$, which is well known to be expressed in terms of the coefficients of $P^{W^\circ_{\mathcal{L}}}_{v,w}$, as in (9.14).

The statement for $i_v^*\mathrm{IC}(w)^\dagger_{\mathcal{L}}$ can be proved in the same way by analyzing $\mathrm{Hom}^\bullet(\mathrm{IC}(w)^\dagger_{\mathcal{L}},\nabla(v)^\dagger_{\mathcal{L}})$ and comparing it to $\mathrm{Hom}^\bullet(\mathrm{IC}(w)_H,\nabla(v)_H)$. We omit details.

(2) By Proposition 5.2, there is a minimal IC sheaf $\xi\in{}_{\mathcal{L}'}\mathfrak{P}^\beta_{\mathcal{L}}$ such that $\mathrm{IC}(\dot{w})_{\mathcal{L}}\cong\xi\star\mathrm{IC}(y)^\dagger_{\mathcal{L}}$. Then $\xi\star\Delta(x)^\dagger_{\mathcal{L}}\cong\Delta(\dot{v})_{\mathcal{L}}\otimes V_1$ for some one-dimensional Fr-module V_1. We have $\mathrm{Hom}^\bullet(C(\dot{v})_{\mathcal{L}}\otimes V_1, i_v^!\mathrm{IC}(\dot{w})_{\mathcal{L}})=\mathrm{Hom}^\bullet(\Delta(\dot{v})_{\mathcal{L}}\otimes V_1, \mathrm{IC}(\dot{w})_{\mathcal{L}})\cong\mathrm{Hom}^\bullet(\xi\star\Delta(x)^\dagger_{\mathcal{L}},\xi\star\mathrm{IC}(y)^\dagger_{\mathcal{L}})=\mathrm{Hom}^\bullet(\Delta(x)^\dagger_{\mathcal{L}},\mathrm{IC}(y)^\dagger_{\mathcal{L}})$, which is a direct sum of $\overline{\mathbb{Q}}_\ell\langle n\rangle$ as an Fr-module by (1). By the same argument as in (1) using embedding (9.16), this implies that $i_v^!\mathrm{IC}(\dot{w})_{\mathcal{L}}$ is a direct sum of $C(\dot{v})_{\mathcal{L}}\langle n\rangle\otimes V_1$, with multiplicities given by the coefficients of $P^{W^\circ_{\mathcal{L}}}_{x,y}$. The argument for $i_v^*\mathrm{IC}(\dot{w})_{\mathcal{L}}$ is similar, using $\mathrm{Hom}^\bullet(\mathrm{IC}(\dot{w})_{\mathcal{L}},\nabla(\dot{v})_{\mathcal{L}}\otimes V_1)\cong\mathrm{Hom}^\bullet(\mathrm{IC}(y)^\dagger_{\mathcal{L}},\nabla(x)^\dagger_{\mathcal{L}})$. □

Similarly, we have the Frobenius semisimplicity of convolution.

PROPOSITION 9.11. (1) *For $w,w'\in W^\circ_{\mathcal{L}}$, the convolution* $\mathrm{IC}(w')^\dagger_{\mathcal{L}}\star\mathrm{IC}(w)^\dagger_{\mathcal{L}}$ *is a direct sum of* $\mathrm{IC}(v)^\dagger_{\mathcal{L}}\langle n\rangle$ *for $v\in W^\circ_{\mathcal{L}}$ and* $n\equiv\ell_{\mathcal{L}}(w)+\ell_{\mathcal{L}}(w')-\ell_{\mathcal{L}}(v)\mod 2$.

(2) *Let $\mathcal{L},\mathcal{L}',\mathcal{L}''\in\mathfrak{o}$, $w\in{}_{\mathcal{L}'}W_{\mathcal{L}}$ and $w'\in{}_{\mathcal{L}''}W_{\mathcal{L}'}$. Let $\beta\in{}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $\beta'\in{}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$ be the blocks containing w and w'. Then the convolution* $\mathrm{IC}(\dot{w}')_{\mathcal{L}'}\star\mathrm{IC}(\dot{w})_{\mathcal{L}}$ *is a direct sum of* $\mathrm{IC}(\dot{v})_{\mathcal{L}}\langle n\rangle\otimes V^{\dot{v}}_{\dot{w}',\dot{w}}$ *for $v\in\beta'\beta\subset{}_{\mathcal{L}''}W_{\mathcal{L}}$,* $n\equiv\ell_\beta(w)+\ell_{\beta'}(w')-\ell_{\beta'\beta}(v)\mod 2$ *and a one-dimensional* Fr-*module* $V^{\dot{v}}_{\dot{w}',\dot{w}}$ *depending only on $\dot{w}$, $\dot{w}'$ and $\dot{v}$.*

Proof. (1) The same statement for $\mathcal{D}_H$ holds by [6, Proposition 3.2.5]; hence (1) follows from the equivalence $\Psi^{\circ}_{\mathcal{L}}$.

(2) Write $w = xw^{\beta}$ for $x \in W^{\circ}_{\mathcal{L}'}$; $w' = w^{\beta'}y$ for $y \in W^{\circ}_{\mathcal{L}'}$. Let $\xi \in {}_{\mathcal{L}'}\mathfrak{P}^{\beta}_{\mathcal{L}}$ and $\eta \in {}_{\mathcal{L}''}\mathfrak{P}^{\beta'}_{\mathcal{L}'}$ be such that $\mathrm{IC}(\dot{w})_{\mathcal{L}} \cong \mathrm{IC}(x)^{\dagger}_{\mathcal{L}'} \star \xi$ and $\mathrm{IC}(\dot{w}')_{\mathcal{L}'} \cong \eta \star \mathrm{IC}(y)^{\dagger}_{\mathcal{L}'}$. For $v \in \beta'\beta$, we have $v = w^{\beta'}zw^{\beta}$ for $z \in W^{\circ}_{\mathcal{L}'}$, and let $V^{\dot{v}}_{\dot{w}',\dot{w}}$ be the one-dimensional Fr-module such that

$$\mathrm{IC}(\dot{v})_{\mathcal{L}} \otimes V^{\dot{v}}_{\dot{w}',\dot{w}} \cong \eta \star \mathrm{IC}(z)^{\dagger}_{\mathcal{L}'} \star \xi.$$

By (1), $\mathrm{IC}(y)^{\dagger}_{\mathcal{L}'} \star \mathrm{IC}(x)^{\dagger}_{\mathcal{L}'}$ is a direct sum of $\mathrm{IC}(z)^{\dagger}_{\mathcal{L}'}\langle n\rangle$ for $z \in W^{\circ}_{\mathcal{L}'}$ and $n \equiv \ell_{\mathcal{L}'}(x) + \ell_{\mathcal{L}'}(y) - \ell_{\mathcal{L}'}(z) \mod 2$. Therefore $\mathrm{IC}(\dot{w}')_{\mathcal{L}'} \star \mathrm{IC}(\dot{w})_{\mathcal{L}} \cong \eta \star \mathrm{IC}(y)^{\dagger}_{\mathcal{L}'} \star \mathrm{IC}(x)^{\dagger}_{\mathcal{L}'} \star \xi$ is a direct sum of $\eta \star \mathrm{IC}(z)^{\dagger}_{\mathcal{L}'}\langle n\rangle \star \xi$ for $z \in W^{\circ}_{\mathcal{L}'}$ and $n \equiv \ell_{\mathcal{L}'}(x) + \ell_{\mathcal{L}'}(y) - \ell_{\mathcal{L}'}(z) \mod 2$, or equivalently a direct sum of $\mathrm{IC}(\dot{v})_{\mathcal{L}}\langle n\rangle \otimes V^{\dot{v}}_{\dot{w}',\dot{w}}$ for $v \in \beta'\beta$ and $n \equiv \ell_{\mathcal{L}'}(x)+\ell_{\mathcal{L}'}(y)-\ell_{\mathcal{L}'}(z) \mod 2$, where $v = w^{\beta'}zw^{\beta}$. It remains to note that $\ell_{\mathcal{L}'}(x) = \ell_{\beta}(w)$, $\ell_{\mathcal{L}'}(y) = \ell_{\beta'}(w')$ and $\ell_{\mathcal{L}'}(z) = \ell_{\beta'\beta}(v)$. □

10. Equivalence for all blocks

In this section, we extend the monoidal equivalence for the neutral blocks in Theorem 9.2 to an equivalence for all blocks (Theorem 10.12). To do this, we will need to extend the endoscopic group to a groupoid, and it will be convenient to organize the various blocks into a 2-category.

10.1. The groupoid $\widetilde{\Xi}$. We define a groupoid $\widetilde{\Xi}$ in $\mathbb{F}_q$-schemes as follows. Its object set is $\mathfrak{o}$, and the morphism ${}_{\mathcal{L}'}\widetilde{\Xi}_{\mathcal{L}}$ between $\mathcal{L}$ and $\mathcal{L}' \in \mathfrak{o}$ is the union of connected components of $N_G(T)$ whose image in W is in ${}_{\mathcal{L}'}\Xi_{\mathcal{L}}$. In other words, ${}_{\mathcal{L}'}\widetilde{\Xi}_{\mathcal{L}}$ parametrizes liftings of w^{β} for blocks $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$. The composition map is defined by the multiplication in $N_G(T)$. We have an obvious map of groupoids $\widetilde{\Xi} \to \Xi$, which is a T-torsor.

For $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, let ${}_{\mathcal{L}'}\widetilde{\Xi}^{\beta}_{\mathcal{L}} \subset {}_{\mathcal{L}'}\widetilde{\Xi}_{\mathcal{L}}$ be the component corresponding to w^{β}. Then ${}_{\mathcal{L}'}\widetilde{\Xi}_{\mathcal{L}} = \coprod_{\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}} {}_{\mathcal{L}'}\widetilde{\Xi}^{\beta}_{\mathcal{L}}$.

10.2. Relative pinning. We give a rigidification of the endoscopic group $H = H_{\mathcal{L}}$ attached to $\mathcal{L} \in \mathfrak{o}$ as follows. Recall that H contains T as a maximal torus, and has $\Phi^{+}_{\mathcal{L}}$ as its positive roots with respect to the Borel B_H. Let $\Delta_{\mathcal{L}} \subset \Phi^{+}_{\mathcal{L}}$ be the set of simple roots. A *relative pinning* for the endoscopic group H is a collection of isomorphisms $\iota_{\alpha} : H_{\alpha} \cong G_{\alpha}$ for each $\alpha \in \Delta_{\mathcal{L}}$. Here H_{α} (respectively, G_{α}) is the root subgroup for α (isomorphic to the additive group) of H (respectively, G). The automorphism group of the data $(H, T, B_H, \{\iota_{\alpha}\}_{\alpha\in\Delta_{\mathcal{L}}})$

is trivial. Therefore a relatively pinned endoscopic group attached to $\mathcal{L}$ is unique up to a unique isomorphism.

For each $\mathcal{L} \in \mathfrak{o}$, we use the notation $H^\circ_{\mathcal{L}}$ to denote the relatively pinned endoscopic group attached to $\mathcal{L}$. Its canonical Borel subgroup is denoted by $B^H_{\mathcal{L}}$.

Let $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$ and $\ddot{w} \in {}_{\mathcal{L}'}\widetilde{\Xi}_{\mathcal{L}}$ with image $w \in {}_{\mathcal{L}'}W_{\mathcal{L}}$. There is a unique isomorphism

$$\sigma(\ddot{w}) : H^\circ_{\mathcal{L}} \to H^\circ_{\mathcal{L}'}$$

characterized as follows. It is w when restricted to T. Since w is minimal in its block, it induces an isomorphism between the based root systems $(\Phi_{\mathcal{L}}, \Delta_{\mathcal{L}})$ and $(\Phi_{\mathcal{L}'}, \Delta_{\mathcal{L}'})$. For each simple root $\alpha \in \Delta_{\mathcal{L}}$, $\sigma(\ddot{w})$ is required to restrict to an isomorphism of root subgroups $H^\circ_{\mathcal{L},\alpha} \xrightarrow{\sim} H^\circ_{\mathcal{L}',w\alpha}$, and we require that the following diagram be commutative

$$\begin{tikzcd}
H^\circ_{\mathcal{L},\alpha} \arrow[r, "\iota_\alpha"] \arrow[d, "\sigma(\ddot{w})"] & G_\alpha \arrow[d, "\mathrm{Ad}(\ddot{w})"] \\
H^\circ_{\mathcal{L}',w\alpha} \arrow[r, "\iota_{w\alpha}"] & G_{w\alpha}
\end{tikzcd}$$

When $\mathcal{L}' = \mathcal{L}$, the above construction gives an action of ${}_{\mathcal{L}}\widetilde{\Xi}_{\mathcal{L}}$ on $H^\circ_{\mathcal{L}}$. When restricted to $T \subset {}_{\mathcal{L}}\widetilde{\Xi}_{\mathcal{L}}$, it is the conjugation action of T on $H^\circ_{\mathcal{L}}$.

10.3. The groupoid $\mathfrak{H}$. We construct a groupoid $\mathfrak{H}$ in $\mathbb{F}_q$-schemes together with a map of groupoids $\omega_{\mathfrak{H}} : \mathfrak{H} \to \Xi$ as follows. Set $\mathrm{Ob}(\mathfrak{H}) = \mathfrak{o}$ and $\omega_{\mathfrak{H}}$ is the identity on objects. For $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$, define the morphism $\mathbb{F}_q$-scheme in $\mathfrak{H}$ as

$${}_{\mathcal{L}'}\mathfrak{H}_{\mathcal{L}} = {}_{\mathcal{L}'}\widetilde{\Xi}_{\mathcal{L}} \overset{T}{\times} H^\circ_{\mathcal{L}},$$

where the action of T on $H^\circ_{\mathcal{L}}$ is by left translation, and its action on ${}_{\mathcal{L}'}\widetilde{\Xi}_{\mathcal{L}} \subset N_G(T)$ is by right translation.

For $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, we get a component

$${}_{\mathcal{L}'}\mathfrak{H}^\beta_{\mathcal{L}} := {}_{\mathcal{L}'}\widetilde{\Xi}^\beta_{\mathcal{L}} \overset{T}{\times} H^\circ_{\mathcal{L}}.$$

The map $\omega_{\mathfrak{H}} : \mathfrak{H} \to \Xi$ then sends ${}_{\mathcal{L}'}\mathfrak{H}^\beta_{\mathcal{L}}$ to $w^\beta \in {}_{\mathcal{L}'}\Xi_{\mathcal{L}}$. There is a canonical isomorphism

$${}_{\mathcal{L}'}\mathfrak{H}^\beta_{\mathcal{L}} = {}_{\mathcal{L}'}\widetilde{\Xi}^\beta_{\mathcal{L}} \overset{T}{\times} H^\circ_{\mathcal{L}} \cong H^\circ_{\mathcal{L}'} \overset{T}{\times} {}_{\mathcal{L}'}\widetilde{\Xi}^\beta_{\mathcal{L}}$$

sending $(\ddot{w}, h) \mapsto (\sigma(\ddot{w})(h), \ddot{w})$. Under this isomorphism, ${}_{\mathcal{L}'}\mathfrak{H}^\beta_{\mathcal{L}}$ is an $(H^\circ_{\mathcal{L}'}, H^\circ_{\mathcal{L}})$-bitorsor.

For $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $\gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$, the composition map

$${}_{\mathcal{L}''}\mathfrak{H}^{\gamma}_{\mathcal{L}'} \times {}_{\mathcal{L}'}\mathfrak{H}^{\beta}_{\mathcal{L}} \to {}_{\mathcal{L}''}\mathfrak{H}^{\gamma\beta}_{\mathcal{L}}$$

is defined as

$$\begin{aligned}({}_{\mathcal{L}''}\widetilde{\Xi}^{\gamma}_{\mathcal{L}'} \overset{T}{\times} H^{\circ}_{\mathcal{L}'}) \times ({}_{\mathcal{L}'}\widetilde{\Xi}^{\beta}_{\mathcal{L}} \overset{T}{\times} H^{\circ}_{\mathcal{L}}) &\to {}_{\mathcal{L}''}\widetilde{\Xi}^{\gamma\beta}_{\mathcal{L}} \overset{T}{\times} H^{\circ}_{\mathcal{L}} \\ (\ddot{w}', h', \ddot{w}, h) &\mapsto (\ddot{w}'\ddot{w}, \sigma(\ddot{w}^{-1})h'h).\end{aligned}$$

It is easy to check that the composition map is associative. Under the composition map, ${}_{\mathcal{L}}\mathfrak{H}_{\mathcal{L}}$ becomes a group scheme over $\mathbb{F}_q$ with a neutral component $H^{\circ}_{\mathcal{L}}$ and a component group $W_{\mathcal{L}}/W^{\circ}_{\mathcal{L}}$. Each ${}_{\mathcal{L}'}\mathfrak{H}_{\mathcal{L}}$ is a $({}_{\mathcal{L}'}\mathfrak{H}_{\mathcal{L}'}, {}_{\mathcal{L}}\mathfrak{H}_{\mathcal{L}})$-bitorsor.

The double cosets $B^{H}_{\mathcal{L}'}\backslash {}_{\mathcal{L}'}\mathfrak{H}_{\mathcal{L}}/B^{H}_{\mathcal{L}}$ are in natural bijection with ${}_{\mathcal{L}'}W_{\mathcal{L}}$: for $w \in {}_{\mathcal{L}'}W_{\mathcal{L}}$, we can write it uniquely as $w^{\beta}v$ for the block $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ containing w and $v \in W^{\circ}_{\mathcal{L}} = W(H^{\circ}_{\mathcal{L}}, T)$. Then w corresponds to the $(B^{H}_{\mathcal{L}'}, B^{H}_{\mathcal{L}})$-double coset containing $(\dot{w}^{\beta}, \dot{v}) \in {}_{\mathcal{L}'}\widetilde{\Xi}_{\mathcal{L}} \overset{T}{\times} H^{\circ}_{\mathcal{L}} = {}_{\mathcal{L}'}\mathfrak{H}_{\mathcal{L}}$, which we denote by $\mathfrak{H}(w)_{\mathcal{L}}$.

10.4. 2-categories over a groupoid. What we call a 2-category $\mathfrak{C}$ is called a ‘bicategory’ in [25, Ch.XII.6]. It has an object set $\mathrm{Ob}(\mathfrak{C})$, and for $x, y \in \mathrm{Ob}(\mathfrak{C})$, the morphisms from x to y form an ordinary category, which we denote by ${}_{y}\mathfrak{C}_{x}$. The category ${}_{x}\mathfrak{C}_{x}$ carries an identity $\mathbf{1}_{x}$. For $x, y, z \in \mathrm{Ob}(\mathfrak{C})$, there is a bifunctor called composition: ${}_{z}\mathfrak{C}_{y} \times {}_{y}\mathfrak{C}_{x} \to {}_{z}\mathfrak{C}_{x}$. For a quadruple of objects, there is a natural isomorphism of functors giving the associativity of composition. These data are required to satisfy the pentagon axiom for associativity and another axiom involving the identities $\{\mathbf{1}_{x}\}$.

From a 2-category $\mathfrak{C}$, we get an ordinary category $\pi_{\leqslant 1}\mathfrak{C}$ with the same object set and morphism sets $\mathrm{Hom}_{\pi_{\leqslant 1}\mathfrak{C}}(x, y) := |{}_{y}\mathfrak{C}_{x}|$, the set of isomorphism classes of objects of ${}_{y}\mathfrak{C}_{x}$.

Let Γ be a small groupoid, viewed as a category where all morphisms are isomorphisms. A 2-category $\mathfrak{C}$ over Γ is a 2-category with a functor $\omega : \pi_{\leqslant 1}\mathfrak{C} \to \Gamma$. In other words, for each object $x \in \mathrm{Ob}(\mathfrak{C})$, we assign an object $\omega(x) \in \mathrm{Ob}(\Gamma)$, and for a pair of objects $x, y \in \mathrm{Ob}(\mathfrak{C})$, a map ${}_{y}h_{x} : |{}_{y}\mathfrak{C}_{x}| \to {}_{\omega(y)}\Gamma_{\omega(x)}$ compatible with compositions and sending identities to identities.

If $(\mathfrak{C}, \omega : \pi_{\leqslant 1}\mathfrak{C} \to \Gamma)$ is a 2-category over Γ, and $x, y \in \mathrm{Ob}(\mathfrak{C})$, $\xi \in {}_{\omega(y)}\Gamma_{\omega(x)}$, we denote by ${}_{y}\mathfrak{C}^{\xi}_{x} \subset {}_{y}\mathfrak{C}_{x}$ the full subcategory of objects whose isomorphism class maps to ξ via ${}_{y}h_{x}$. Then ${}_{y}\mathfrak{C}_{x} = \coprod_{\xi \in {}_{\omega(y)}\Gamma_{\omega(x)}} {}_{y}\mathfrak{C}^{\xi}_{x}$. The composition functor restricts to a bifunctor

$$\circ : {}_{z}\mathfrak{C}^{\eta}_{y} \times {}_{y}\mathfrak{C}^{\xi}_{x} \to {}_{z}\mathfrak{C}^{\eta\xi}_{x}, \qquad \forall \xi \in {}_{\omega(y)}\Gamma_{\omega(x)}, \eta \in {}_{\omega(z)}\Gamma_{\omega(y)}.$$

EXAMPLE 10.5. The categories $\{{}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}}\}_{\mathcal{L},\mathcal{L}'\in\mathfrak{o}}$ can be organized into a 2-category $\mathfrak{D}$ over $\varXi$ in an obvious way. The object set is $\mathfrak{o}$ and ω is the identity map on the object sets. For $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$, the morphism category is ${}_{\mathcal{L}'}\mathfrak{D}_{\mathcal{L}} = \coprod_{\beta\in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}} {}_{\mathcal{L}'}\mathcal{D}^{\beta}_{\mathcal{L}}$ with composition given by convolution (using Proposition 4.13).

EXAMPLE 10.6. For $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$ and $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, define

$${}_{\mathcal{L}'}\mathcal{E}^{\beta}_{\mathcal{L}} := D^b_m(B^H_{\mathcal{L}'}\backslash {}_{\mathcal{L}'}\mathfrak{H}^{\beta}_{\mathcal{L}}/B^H_{\mathcal{L}}).$$

Then the groupoid structure on $\mathfrak{H}$ gives a convolution functor for $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $\gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$,

$$\star : {}_{\mathcal{L}''}\mathcal{E}^{\gamma}_{\mathcal{L}'} \times {}_{\mathcal{L}'}\mathcal{E}^{\beta}_{\mathcal{L}} \to {}_{\mathcal{L}''}\mathcal{E}^{\gamma\beta}_{\mathcal{L}},$$

carrying an associativity natural transformation satisfying the pentagon axiom. This defines a 2-category $\mathfrak{E}$ over $\varXi$ with object set $\mathfrak{o}$ and morphism categories ${}_{\mathcal{L}'}\mathfrak{E}_{\mathcal{L}} = \coprod_{\beta\in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}} {}_{\mathcal{L}'}\mathcal{E}^{\beta}_{\mathcal{L}}$.

If $\beta = W^{\circ}_{\mathcal{L}} \subset {}_{\mathcal{L}}\underline{W}_{\mathcal{L}}$ is the neutral block, we denote ${}_{\mathcal{L}}\mathcal{E}^{\beta}_{\mathcal{L}}$ by ${}_{\mathcal{L}}\mathcal{E}^{\circ}_{\mathcal{L}}$. This is the usual Hecke category $\mathcal{D}_{H^{\circ}_{\mathcal{L}}}$ for the reductive group $H^{\circ}_{\mathcal{L}}$.

10.7. Twisting data. Let E be a field. An *E-linear twisting data* for a groupoid $\varGamma$ is a normalized 2-cocycle of $\varGamma$ with values in $\mathrm{Pic}(E)$, the Picard groupoid of one-dimensional E-vector spaces. More precisely, it is the following data (λ, μ):

(1) for arrows $x \xrightarrow{\xi} y \xrightarrow{\eta} z$ in $\varGamma$, an E-line $\lambda(\eta, \xi)$;

(2) for any arrow $x \xrightarrow{\xi} y$ in $\varGamma$, trivializations of the lines $\lambda(\xi, \mathrm{id}_x)$ and $\lambda(\mathrm{id}_y, \xi)$;

(3) for arrows $x \xrightarrow{\xi} y \xrightarrow{\eta} z \xrightarrow{\zeta} t$ in $\varGamma$, an isomorphism of E-lines

$$\mu_{\zeta,\eta,\xi} : \lambda(\zeta, \eta\xi) \otimes_E \lambda(\eta, \xi) \xrightarrow{\sim} \lambda(\zeta\eta, \xi) \otimes_E \lambda(\zeta, \eta).$$

The data (λ, μ) should satisfy the following conditions:

- For arrows $x \xrightarrow{\xi} y \xrightarrow{\eta} z$ in $\varGamma$, $\mu_{\eta,\mathrm{id}_y,\xi}$ is the identity map of $\lambda(\eta, \xi)$ using the trivializations of $\lambda(\mathrm{id}_y, \xi)$ and $\lambda(\eta, \mathrm{id}_y)$.
- For four composable morphisms ξ, η, ζ, τ in $\varGamma$, the following diagram is

commutative:

$$\begin{tikzcd}
& \lambda(\tau, \zeta\eta\xi) \otimes \lambda(\zeta, \eta\xi) \otimes \lambda(\eta, \xi) \arrow[dl, "\mathrm{id}\otimes\mu_{\zeta,\eta,\xi}"'] \arrow[dr, "\mu_{\tau,\zeta,\eta\xi}\otimes\mathrm{id}"] & \\
\lambda(\tau, \zeta\eta\xi) \otimes \lambda(\zeta\eta, \xi) \otimes \lambda(\zeta, \eta) \arrow[d, "\mu_{\tau,\zeta\eta,\xi}\otimes\mathrm{id}"] & & \lambda(\tau\zeta, \eta\xi) \otimes \lambda(\tau, \zeta) \otimes \lambda(\eta, \xi) \arrow[d, "\mu_{\tau\zeta,\eta,\xi}\otimes\mathrm{id}_{\lambda(\tau,\zeta)}"] \\
\lambda(\tau\zeta\eta, \xi) \otimes \lambda(\tau, \zeta\eta) \otimes \lambda(\zeta, \eta) \arrow[rr, "\mathrm{id}\otimes\mu_{\tau,\zeta,\eta}"] & & \lambda(\tau\zeta\eta, \xi) \otimes \lambda(\tau\zeta, \eta) \otimes (\tau, \zeta)
\end{tikzcd}$$

Let $Z^2_{\mathrm{norm}}(\varGamma, \mathrm{Pic}(E))$ be the category of E-twisting data.

Suppose we have chosen a basis for each $\lambda(\eta, \xi)$ compatible with the trivializations of $\lambda(\mathrm{id}_y, \xi)$ and $\lambda(\xi, \mathrm{id}_x)$. Using these bases, $\mu_{\zeta,\eta,\xi}$ then gives an element in $E^\times$. The collection $\{\mu_{\zeta,\eta,\xi}\}$ defines a normalized 3-cocycle of $\varGamma$ with values in $E^\times$ (normalized means $\mu_{\zeta,\eta,\xi} = 1$ whenever one of ζ, η, ξ is the identity arrow). A different choice of bases of $\lambda(\eta, \xi)$ gives another 3-cocycle, which differs from the previous one by a coboundary of a normalized 2-cochain. This gives an equivalence of groupoids

$$Z^3_{\mathrm{norm}}(\varGamma, E^\times)/C^2_{\mathrm{norm}}(\varGamma, E^\times) \xrightarrow{\sim} Z^2_{\mathrm{norm}}(\varGamma, \mathrm{Pic}(E)). \tag{10.1}$$

In particular, the isomorphism classes of $Z^2_{\mathrm{norm}}(\varGamma, \mathrm{Pic}(E))$ are parametrized by $\mathrm{H}^3(\varGamma, E^\times)$, and the automorphism groups are $Z^2_{\mathrm{norm}}(\varGamma, E^\times)$.

There is an action of $Z^3_{\mathrm{norm}}(\varGamma, E^\times)$ on $Z^2_{\mathrm{norm}}(\varGamma, \mathrm{Pic}(E))$ as follows. For $z \in Z^3_{\mathrm{norm}}(\varGamma, E^\times)$ and $(\lambda, \mu) \in Z^2_{\mathrm{norm}}(\varGamma, \mathrm{Pic}(E))$, $z \cdot (\lambda, \mu) = (\lambda, z\mu)$, where $(z\mu)_{\zeta,\eta,\xi} = z(\zeta, \eta, \xi)\mu_{\zeta,\eta,\xi}$.

10.8. Twisting a 2-category by twisting data. Let $(\mathfrak{C}, \omega : \pi_{\leqslant 1}\mathfrak{C} \to \varGamma)$ be a 2-category over a groupoid $\varGamma$, such that ${}_y\mathfrak{C}^\xi_x$ is a module category for $\mathrm{Pic}(E)$, for every $x, y \in \mathrm{Ob}(\mathfrak{C})$ and $\xi \in {}_{\omega(y)}\varGamma_{\omega(x)}$. Let $(\lambda, \mu) \in Z^2_{\mathrm{norm}}(\varGamma, \mathrm{Pic}(E))$ be a twisting data for $\varGamma$. We define a new 2-category $\mathfrak{C}^{(\lambda,\mu)}$ over $\varGamma$ as follows:

(1) $\mathfrak{C}^{(\lambda,\mu)}$ has the same objects and the same morphism categories as $\mathfrak{C}$.

(2) For $x, y, z \in \mathrm{Ob}(\mathfrak{C})$ and $\omega(x) \xrightarrow{\xi} \omega(y) \xrightarrow{\eta} \omega(z)$ in $\varGamma$, the composition functor $\circ_\lambda$ for $\mathfrak{C}^{(\lambda,\mu)}$ is defined as

$$\begin{aligned} \circ_\lambda : {}_z\mathfrak{C}^\eta_y \times {}_y\mathfrak{C}^\xi_x &\to {}_z\mathfrak{C}^{\eta\xi}_x \\ (\mathcal{G}, \mathcal{F}) &\mapsto (\mathcal{G} \circ \mathcal{F}) \otimes_E \lambda(\eta, \xi). \end{aligned}$$

Here $\mathcal{G} \circ \mathcal{F}$ is the composition functor in $\mathfrak{C}$.

(3) The identity morphism in ${}_x\mathfrak{C}_x$ remains the same, and the natural isomorphisms $f \circ_\lambda 1_x \cong f \cong 1_y \circ_\lambda f$ for $f \in {}_y\mathfrak{C}_x^\xi$ are defined using similar isomorphisms for $\circ$ and the trivializations of $\lambda(\xi, \mathrm{id}_x)$ and $\lambda(\mathrm{id}_y, \xi)$.

(4) The associativity isomorphisms for $\mathfrak{C}^{(\lambda,\mu)}$ between two three-term composition functors $(h \circ_\lambda g) \circ_\lambda f \cong h \circ_\lambda (g \circ_\lambda f)$, where $\omega(f) = \xi$, $\omega(g) = \eta$ and $\omega(h) = \zeta$ are three composable arrows in Γ, are obtained using the associativity isomorphisms for $\circ_\lambda$ and the isomorphism $\mu_{\zeta,\eta,\xi}$.

The pentagon identities for $\mathfrak{C}$ and for $\{\mu_{\zeta,\eta,\xi}\}$ imply the pentagon identities for $\mathfrak{C}^{(\lambda,\mu)}$.

CONSTRUCTION 10.9. We define a $\overline{\mathbb{Q}}_\ell$-linear twisting data (λ, μ) for Ξ, which depends on the choice of a lifting $\dot{w}^\beta$ for the minimal elements w^β in each block $\beta \in {}_{\mathcal{L}'}W_{\mathcal{L}}$. From the liftings $\{\dot{w}^\beta\}$ (normalized such that $\dot{e}$ is the identity of G), we get a normalized $T(\mathbb{F}_q)$-valued 2-cocycle c for the groupoid Ξ: for $\beta \in {}_{\mathcal{L}'}\Xi_{\mathcal{L}}$ and $\gamma \in {}_{\mathcal{L}''}\Xi_{\mathcal{L}'}$, let

$$c(\gamma, \beta) = (\dot{w}^{\gamma\beta})^{-1}\dot{w}^\gamma\dot{w}^\beta.$$

Now define

$$\lambda(\gamma, \beta) := \mathcal{L}_{c(\gamma,\beta)} \quad \text{(the stalk of } \mathcal{L} \text{ at } c(\gamma, \beta)).$$

Since $c(\gamma, \beta) = 1$ if one of γ, β is the neutral block, $\lambda(\gamma, \beta) = \mathcal{L}_{\dot{e}}$ carries a trivialization in this case. The construction of λ gives canonical isomorphisms $\mu^\natural_{\delta,\gamma,\beta} : \lambda(\delta, \gamma\beta) \otimes \lambda(\gamma, \beta) \xrightarrow{\sim} \lambda(\delta\gamma, \beta) \otimes \lambda(\delta, \gamma)$ coming from the fact that c is a cocycle and $\mathcal{L}$ is a character sheaf. The pair $(\lambda, \mu^\natural)$ is a $\overline{\mathbb{Q}}_\ell$-linear twisting data, but it is not what we will use.

Instead, by combining the canonical isomorphism $\mathrm{can}_{\dot{w}^\gamma,\dot{w}^\beta}$ in (5.7) and isomorphism (2.6), we have a canonical isomorphism

$$c(\gamma, \beta) : \mathrm{IC}(\dot{w}^\gamma)_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^\beta)_{\mathcal{L}} \cong \mathrm{IC}(\dot{w}^{\gamma\beta})_{\mathcal{L}} \otimes \lambda(\gamma, \beta). \tag{10.2}$$

Our $\mu_{\delta,\gamma,\beta}$ will come from the above isomorphism and the associativity for the convolution. More precisely, let $\sigma(w^\delta, w^\gamma, w^\beta) \in \overline{\mathbb{Q}}_\ell^\times$ be the normalized 3-cocycle on Ξ introduced in Section 5.8 as the ratio of the two maps in (5.8). Let $\mu = \sigma\mu^\natural$, that is, $\mu_{\delta,\gamma,\beta} = \sigma(w^\delta, w^\gamma, w^\beta)\mu^\natural_{\delta,\gamma,\beta}$. Then $\{\lambda(\gamma, \beta)\}$ together with $\{\mu_{\delta,\gamma,\beta}\}$ define a twisting data $(\lambda, \mu) \in Z^2_{\mathrm{norm}}(\Xi, \mathrm{Pic}(\overline{\mathbb{Q}}_\ell))$.

From the construction of μ, we have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{IC}(\dot{w}^{\delta})_{\mathcal{L}''} \star \mathrm{IC}(\dot{w}^{\gamma})_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^{\beta})_{\mathcal{L}} \arrow[dl, "c(\delta,\gamma)\star\mathrm{id}"'] \arrow[dr, "\mathrm{id}\star c(\gamma,\beta)"] & \\
(\mathrm{IC}(\dot{w}^{\delta\gamma})_{\mathcal{L}'} \otimes \lambda(\delta,\gamma)) \star \mathrm{IC}(\dot{w}^{\beta})_{\mathcal{L}} \arrow[d, "c(\delta\gamma,\beta)"'] & & \mathrm{IC}(\dot{w}^{\delta})_{\mathcal{L}''} \star (\mathrm{IC}(\dot{w}^{\gamma\beta})_{\mathcal{L}} \otimes \lambda(\gamma,\beta)) \arrow[d, "c(\delta,\gamma\beta)"] \\
\mathrm{IC}(\dot{w}^{\delta\gamma\beta})_{\mathcal{L}} \otimes \lambda(\delta,\gamma) \otimes \lambda(\delta\gamma,\beta) & & \mathrm{IC}(\dot{w}^{\delta\gamma\beta})_{\mathcal{L}} \otimes \lambda(\delta,\gamma\beta) \otimes \lambda(\gamma,\beta) \arrow[ll, "\mathrm{id}\otimes\mu_{\delta,\gamma,\beta}"']
\end{tikzcd} \tag{10.3}$$

LEMMA 10.10. *The cohomology class of* $(\lambda, \mu^{\natural})$ *in* $\mathrm{H}^3(\Xi, \overline{\mathbb{Q}}_\ell^\times)$ *is trivial. In particular, the cohomology class of* (λ, μ) *in* $\mathrm{H}^3(\Xi, \overline{\mathbb{Q}}_\ell^\times)$ *is equal to the class of the* 3*-cocycle* σ *introduced in Section* 5.8.

Proof. By construction, $(\lambda, \mu^{\natural})$ is the image of a cocycle $c \in Z^2_{\mathrm{norm}}(W, T(\mathbb{F}_q))$ under the homomorphism $T(\mathbb{F}_q) \to \mathrm{Pic}(\overline{\mathbb{Q}}_\ell)$ given by the character sheaf $\mathcal{L}$. It suffices to show that $\mathcal{L}$ can be trivialized (as a character sheaf) when restricted to $T(\mathbb{F}_q)$, or more generally to any finite subgroup $A \subset T_k$. Let $T'_k = T_k/A$, another torus over k, and let $\pi : T_k \to T'_k$ be the projection. It suffices to show that the pullback $\pi^* : \mathrm{Ch}(T'_k) \to \mathrm{Ch}(T_k)$ is surjective, for then any $\mathcal{L} \in \mathrm{Ch}(T_k)$ is isomorphic to $\pi^*\mathcal{L}'$ for some $\mathcal{L}' \in \mathrm{Ch}(T'_k)$, and $\mathcal{L}|_A \cong \pi^*\mathcal{L}'|_A$ is visibly trivial. Now $\mathrm{Ch}(T_k) = \mathrm{Hom}_{\mathrm{cont}}(\pi_1^t(T_k), \overline{\mathbb{Q}}_\ell^\times)$ (where π_1^t stands for the tame fundamental group). Since $\overline{\mathbb{Q}}_\ell^\times$ is divisible, any homomorphism $\rho : \pi_1^t(T_k) \to \overline{\mathbb{Q}}_\ell^\times$ can be extended to $\pi_1^t(T'_k)$, and if ρ is continuous, any such extension is also continuous because $\pi_1^t(T_k) \subset \pi_1^t(T'_k)$ has a finite index. Therefore $\pi^* : \mathrm{Ch}(T'_k) \to \mathrm{Ch}(T_k)$ is surjective. □

REMARK 10.11. If G has a connected center, then $W_{\mathcal{L}} = W^\circ_{\mathcal{L}}$ and Ξ is a groupoid that is equivalent to a point. Since $\mathrm{H}^3(\Xi, \overline{\mathbb{Q}}_\ell^\times) = 1$ in this case, (λ, μ) can be trivialized by equivalence (10.1); however, the trivializations of (λ, μ) are not unique but form a torsor under $Z^2_{\mathrm{norm}}(\Xi, \overline{\mathbb{Q}}_\ell^\times)$.

On the other hand, when $\Omega_{\mathcal{L}}$ is nontrivial, the cohomology class of σ is calculated in [32], which by Lemma 10.10 also gives the cohomology class of the twisting data (λ, μ).

For example, let $G = \mathrm{SL}_2$, and $\mathcal{L} \in \mathrm{Ch}(T)$ be the unique element of order two. Then $\mathfrak{o} = \{\mathcal{L}\}$, and Ξ is the groupoid with one object $\mathcal{L}$ and automorphism group $W = \mathbb{Z}/2\mathbb{Z} = \{1, s\}$. The calculation in Example 5.7 shows that $\sigma(s, s, s) = -1$. Therefore the class of (λ, μ) in $\mathrm{H}^3(\mathbb{Z}/2\mathbb{Z}, \overline{\mathbb{Q}}_\ell^\times) \cong \{\pm 1\}$ is nontrivial.

We are ready to state the extension of Theorem 9.2 to all blocks. Recall from Section 10.3 that for $\mathcal{L} \in \mathfrak{o}$ and $w \in W$, we have a $(B^H_{w\mathcal{L}}, B^H_{\mathcal{L}})$-double coset $\mathfrak{H}(w)_{\mathcal{L}} \subset {}_{w\mathcal{L}}\mathfrak{H}_{\mathcal{L}}$. Let $C(w)^H_{\mathcal{L}} = \overline{\mathbb{Q}}_\ell\langle \ell_\beta(w)\rangle$ (where $\beta \in {}_{w\mathcal{L}}\underline{W}_{\mathcal{L}}$ is the block containing w) be the shifted and twisted constant sheaf on $\mathfrak{H}(w)_{\mathcal{L}}$. Let $\Delta(w)^H_{\mathcal{L}}$, $\nabla(w)^H_{\mathcal{L}}$ and $\mathrm{IC}(w)^H_{\mathcal{L}}$ be the !-, $*$-, and middle extensions of $C(w)^H_{\mathcal{L}}$ to the closure of $\mathfrak{H}(w)_{\mathcal{L}}$, viewed as objects in ${}_{w\mathcal{L}}\mathcal{E}^\beta_{\mathcal{L}} \subset {}_{w\mathcal{L}}\mathfrak{E}_{\mathcal{L}}$.

THEOREM 10.12 (Monodromic–endoscopic equivalence in general). *Fix a lifting $\dot{w}^\beta$ for the minimal element w^β in each block β, and use them to define the twisting data $(\lambda, \mu) \in Z^2_{\mathrm{norm}}(\Xi, \mathrm{Pic}(\overline{\mathbb{Q}}_\ell))$ as in Construction 10.9. Then there is a canonical equivalence of 2-categories over Ξ,*

$$\Psi : \mathfrak{E}^{(\lambda,\mu)} \cong \mathfrak{D},$$

such that we have the following:

(1) *For $\mathcal{L} \in \mathfrak{o}$ and β the unit coset in ${}_{\mathcal{L}}\underline{W}_{\mathcal{L}}$, the equivalence Ψ restricts to the equivalence $\Psi^\circ_{\mathcal{L}}$ in Theorem 9.2 as monoidal functors.*

(2) *For $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$ and $w \in {}_{\mathcal{L}'}W_{\mathcal{L}}$, write $w = xw^\beta$ for $x \in W^\circ_{\mathcal{L}'}$. Then the equivalence Ψ sends $\Delta(w)^H_{\mathcal{L}}, \nabla(w)^H_{\mathcal{L}}$ and $\mathrm{IC}(w)^H_{\mathcal{L}}$ in ${}_{\mathcal{L}'}\mathfrak{E}_{\mathcal{L}}$ to $\Delta(x)^\dagger_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^\beta)_{\mathcal{L}}, \nabla(x)^\dagger_{\mathcal{L}} \star \mathrm{IC}(\dot{w}^\beta)_{\mathcal{L}}$ and $\mathrm{IC}(x)^\dagger_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^\beta)_{\mathcal{L}}$ in ${}_{\mathcal{L}'}\mathcal{D}_{\mathcal{L}}$.*

REMARK 10.13. In the statement of the above theorem, Ψ being an equivalence of 2-categories implies that for $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$ and $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, it restricts to an equivalence of triangulated categories ${}_{\mathcal{L}'}\Psi^\beta_{\mathcal{L}} : {}_{\mathcal{L}'}\mathcal{E}^\beta_{\mathcal{L}} \xrightarrow{\sim} {}_{\mathcal{L}'}\mathcal{D}^\beta_{\mathcal{L}}$; moreover, the equivalences $\{{}_{\mathcal{L}'}\Psi^\beta_{\mathcal{L}}\}$ are compatible with convolution structures after modifying the convolution structure of the $\{{}_{\mathcal{L}'}\mathcal{E}^\beta_{\mathcal{L}}\}$ by the twisting data (λ, μ).

By the last paragraph of Section 10.9, when G has a connected center, one can choose a (noncanonical) trivialization of the twisting data (λ, μ) and conclude that $\mathfrak{E} \cong \mathfrak{D}$ in this case.

The rest of the section is devoted to the proof of Theorem 10.12.

10.14. Action of minimal IC sheaves on neutral blocks. For $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$, $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, we define the functor

$$\begin{aligned} {}^\beta(-) : {}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}} &\to {}_{\mathcal{L}'}\mathcal{D}^\circ_{\mathcal{L}'} \\ \mathcal{F} &\mapsto {}^\beta\mathcal{F} := \xi \star \mathcal{F} \star \xi^{-1}, \end{aligned}$$

where $\xi \in {}_{\mathcal{L}'}\mathfrak{P}^\beta_{\mathcal{L}}$, and $\xi^{-1} \in {}_{\mathcal{L}}\mathfrak{P}^{\beta^{-1}}_{\mathcal{L}'}$ is the inverse of ξ under convolution (that is, ξ^{-1} is equipped with canonical isomorphisms $\xi^{-1} \star \xi \cong \delta_{\mathcal{L}}$ and $\xi \star \xi^{-1} \cong \delta_{\mathcal{L}'}$

satisfying the usual axioms). We claim that the functor ${}^{\beta}(-)$ is independent of the choice of ξ up to a canonical isomorphism. Indeed, if $\xi' \in {}_{\mathcal{L}'}\mathfrak{P}^{\beta}_{\mathcal{L}}$ is another minimal IC sheaf, then we may canonically write $\xi' = \xi \otimes V$ for a one-dimensional Fr-module $V = \mathrm{Hom}(\xi, \xi')$. Then $\xi'^{-1} = \xi^{-1} \otimes V^{\vee}$, and $\xi' \star \mathcal{F} \star \xi'^{-1} \cong \xi \star \mathcal{F} \star \xi^{-1} \otimes (V \otimes V^{\vee}) \cong \xi \star \mathcal{F} \star \xi^{-1}$ canonically.

If $\mathcal{L}, \mathcal{L}', \mathcal{L}'' \in \mathfrak{o}$, $\gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$ and $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, then there is a canonical isomorphism making the following diagram commutative:

$$\begin{tikzcd}
{}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}} \arrow[r, "{}^{\beta}(-)"] \arrow[rr, bend right, "{}^{\gamma\beta}(-)"'] & {}_{\mathcal{L}'}\mathcal{D}^{\circ}_{\mathcal{L}'} \arrow[r, "{}^{\gamma}(-)"] & {}_{\mathcal{L}''}\mathcal{D}^{\circ}_{\mathcal{L}''}
\end{tikzcd}$$

Moreover, these isomorphisms are compatible with three-step compositions. All these statements can be checked easily using the independence of ξ in defining the functor ${}^{\beta}(-)$.

By Corollary 4.4, we have an isomorphism of Coxeter groups $W^{\circ}_{\mathcal{L}} \to W^{\circ}_{\mathcal{L}'}$ given by $\mathrm{Ad}(w^{\beta})$. It induces an equivalence

$${}^{\beta}(-) : \mathrm{SB}_m(W^{\circ}_{\mathcal{L}}) \xrightarrow{\sim} \mathrm{SB}_m(W^{\circ}_{\mathcal{L}'}).$$

LEMMA 10.15. *There is a canonical isomorphism making the following diagram commutative:*

$$\begin{tikzcd}
{}_{\mathcal{L}}\mathcal{D}^{\circ}_{\mathcal{L}} \arrow[r, "\varphi_{\mathcal{L}}"] \arrow[d, "{}^{\beta}(-)"] & K^b(\mathrm{SB}_m(W^{\circ}_{\mathcal{L}}))/K^b(\mathrm{SB}_m(W^{\circ}_{\mathcal{L}}))_0 \arrow[d, "K^b({}^{\beta}(-))"] \\
{}_{\mathcal{L}'}\mathcal{D}^{\circ}_{\mathcal{L}'} \arrow[r, "\varphi_{\mathcal{L}'}"] & K^b(\mathrm{SB}_m(W^{\circ}_{\mathcal{L}'}))/K^b(\mathrm{SB}_m(W^{\circ}_{\mathcal{L}'}))_0
\end{tikzcd}$$

Moreover, these isomorphisms are compatible for composable blocks β, γ.

Proof. Unwinding the definitions of the functors involved, it suffices to give a canonical isomorphism ${}^{\beta}\Theta^{\circ}_{\mathcal{L}} \cong \Theta^{\circ}_{\mathcal{L}'}$. Now $\Theta^{\circ}_{\mathcal{L}}\langle N_{\mathcal{L}}\rangle$ is a maximal IC sheaf equipped with a nonzero map $\epsilon_{\mathcal{L}} : \Theta^{\circ}_{\mathcal{L}} \to \delta_{\mathcal{L}}$. Therefore ${}^{\beta}\Theta^{\circ}_{\mathcal{L}}\langle N_{\mathcal{L}}\rangle = {}^{\beta}\Theta^{\circ}_{\mathcal{L}}\langle N_{\mathcal{L}'}\rangle$ is a maximal IC sheaf equipped with a nonzero map ${}^{\beta}\epsilon_{\mathcal{L}} : {}^{\beta}\Theta^{\circ}_{\mathcal{L}} \to {}^{\beta}\delta_{\mathcal{L}} = \delta_{\mathcal{L}'}$, that is, $({}^{\beta}\Theta^{\circ}_{\mathcal{L}}, {}^{\beta}\epsilon_{\mathcal{L}})$ is a rigidified maximal IC sheaf in ${}_{\mathcal{L}'}\mathcal{D}^{\circ}_{\mathcal{L}'}$. Therefore, by the discussion in Section 6.5, there is a unique isomorphism $({}^{\beta}\Theta^{\circ}_{\mathcal{L}}, {}^{\beta}\epsilon_{\mathcal{L}}) \cong (\Theta^{\circ}_{\mathcal{L}'}, \epsilon_{\mathcal{L}'})$. □

For $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}, \beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, the isomorphism $\sigma(\dot{w}^{\beta}) : H^{\circ}_{\mathcal{L}} \xrightarrow{\sim} H^{\circ}_{\mathcal{L}'}$ (see Section 10.2) induces an equivalence of neutral blocks

$${}^{\beta}(-) : {}_{\mathcal{L}}\mathcal{E}^{\circ}_{\mathcal{L}} \xrightarrow{\sim} {}_{\mathcal{L}'}\mathcal{E}^{\circ}_{\mathcal{L}'}.$$

From the definition of $\mathrm{IC}(w^\beta)^H_{\mathcal{L}}$, we get canonically

$${}^{\beta}\mathcal{F} \cong \mathrm{IC}(w^\beta)^H_{\mathcal{L}} \star \mathcal{F} \star \mathrm{IC}(w^{\beta,-1})^H_{\mathcal{L}'}, \quad \forall \mathcal{F} \in {}_{\mathcal{L}}\mathcal{E}^\circ_{\mathcal{L}}.$$

From this, we see that the functor ${}^{\beta}(-)$ is independent of the choice of the lifting $\dot{w}^\beta$ up to a canonical isomorphism.

Lemma 10.15 immediately implies the following.

COROLLARY 10.16. *Let* $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}, \beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$. *There is a canonical isomorphism making the following diagram commutative:*

$$\begin{tikzcd}
{}_{\mathcal{L}}\mathcal{E}^\circ_{\mathcal{L}} \arrow[r, "\Psi^\circ_{\mathcal{L}}"] \arrow[d, "{}^{\beta}(-)"] & {}_{\mathcal{L}}\mathcal{D}^\circ_{\mathcal{L}} \arrow[d, "{}^{\beta}(-)"] \\
{}_{\mathcal{L}'}\mathcal{E}^\circ_{\mathcal{L}'} \arrow[r, "\Psi^\circ_{\mathcal{L}'}"] & {}_{\mathcal{L}'}\mathcal{D}^\circ_{\mathcal{L}'}
\end{tikzcd}$$

Moreover, these isomorphisms are compatible with compositions of 1*-morphisms in* $\mathfrak{E}$ *and* $\mathfrak{D}$ *for composable blocks* β, γ.

10.17. Proof of Theorem 10.12. For $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$ and $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, define the functor

$$\begin{aligned}
{}_{\mathcal{L}'}\Psi^\beta_{\mathcal{L}} : \quad & {}_{\mathcal{L}'}\mathcal{E}^\beta_{\mathcal{L}} \to {}_{\mathcal{L}'}\mathcal{D}^\beta_{\mathcal{L}} \\
& \mathcal{F} \star \mathrm{IC}(w^\beta)^H_{\mathcal{L}} \mapsto \Psi^\circ_{\mathcal{L}'}(\mathcal{F}) \star \mathrm{IC}(\dot{w}^\beta)_{\mathcal{L}}, \quad \forall \mathcal{F} \in {}_{\mathcal{L}'}\mathcal{E}^\circ_{\mathcal{L}'} = \mathcal{D}_{H^\circ_{\mathcal{L}'}}.
\end{aligned}$$

Note that $(-) \star \mathrm{IC}(w^\beta)^H_{\mathcal{L}} : {}_{\mathcal{L}'}\mathcal{E}^\circ_{\mathcal{L}'} \to {}_{\mathcal{L}'}\mathcal{E}^\beta_{\mathcal{L}}$ is an equivalence. These equivalences satisfy the requirements (1)(2) in the statement of the theorem. It remains to extend these equivalences to an equivalence of 2-categories, that is, we need to give natural isomorphisms between composition functors and check compatibilities with associativity.

For $\mathcal{L}, \mathcal{L}', \mathcal{L}'' \in \mathfrak{o}$ and $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}, \gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$, consider $\widetilde{\mathcal{F}} := \mathcal{F} \star \mathrm{IC}(w^\beta)^H_{\mathcal{L}} \in {}_{\mathcal{L}'}\mathcal{E}^\circ_{\mathcal{L}}$ (for some $\mathcal{F} \in {}_{\mathcal{L}'}\mathcal{E}^\circ_{\mathcal{L}'}$) and $\widetilde{\mathcal{G}} := \mathcal{G} \star \mathrm{IC}(w^\gamma)^H_{\mathcal{L}'} \in {}_{\mathcal{L}''}\mathcal{E}^\circ_{\mathcal{L}'}$ (for some $\mathcal{G} \in {}_{\mathcal{L}''}\mathcal{E}^\circ_{\mathcal{L}''}$). The λ-twisted composition of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ is

$$\widetilde{\mathcal{F}} \circ_\lambda \widetilde{\mathcal{G}} = (\mathcal{G} \star \mathrm{IC}(w^\gamma)^H_{\mathcal{L}'} \star \mathcal{F} \star \mathrm{IC}(w^\beta)^H_{\mathcal{L}}) \otimes \lambda(\gamma, \beta).$$

Using that $\mathrm{IC}(w^\gamma)^H_{\mathcal{L}'} \star \mathcal{F} = {}^{\gamma}\mathcal{F} \star \mathrm{IC}(w^\gamma)^H_{\mathcal{L}'}$, we get a canonical isomorphism

$$\widetilde{\mathcal{F}} \circ_\lambda \widetilde{\mathcal{G}} \cong (\mathcal{G} \star {}^{\gamma}\mathcal{F}) \star \mathrm{IC}(w^{\gamma\beta})^H_{\mathcal{L}} \otimes \lambda(\gamma, \beta).$$

Hence

$${}_{\mathcal{L}''}\Psi^{\gamma\beta}_{\mathcal{L}}(\widetilde{\mathcal{F}} \circ_\lambda \widetilde{\mathcal{G}}) \cong \Psi^\circ_{\mathcal{L}''}(\mathcal{G} \star {}^{\gamma}\mathcal{F}) \star \mathrm{IC}(\dot{w}^{\gamma\beta})_{\mathcal{L}} \otimes \lambda(\gamma, \beta). \qquad (10.4)$$

On the other hand,

$$ {}_{\mathcal{L}''}\Psi^{\gamma}_{\mathcal{L}'}(\widetilde{\mathcal{F}}) \circ {}_{\mathcal{L}'}\Psi^{\beta}_{\mathcal{L}}(\widetilde{\mathcal{G}}) = (\Psi^{\circ}_{\mathcal{L}''}(\mathcal{G}) \star \mathrm{IC}(\dot{w}^{\gamma})_{\mathcal{L}'}) \star (\Psi^{\circ}_{\mathcal{L}'}(\mathcal{F}) \star \mathrm{IC}(\dot{w}^{\beta})_{\mathcal{L}}). $$

Using that $\mathrm{IC}(\dot{w}^{\gamma})_{\mathcal{L}'} \star \Psi^{\circ}_{\mathcal{L}'}(\mathcal{F}) \cong {}^{\gamma}(\Psi^{\circ}_{\mathcal{L}'}(\mathcal{F})) \star \mathrm{IC}(\dot{w}^{\gamma})_{\mathcal{L}'}$, we get

$$ {}_{\mathcal{L}''}\Psi^{\gamma}_{\mathcal{L}'}(\widetilde{\mathcal{F}}) \circ {}_{\mathcal{L}'}\Psi^{\beta}_{\mathcal{L}}(\widetilde{\mathcal{G}}) \cong \Psi^{\circ}_{\mathcal{L}''}(\mathcal{G}) \star {}^{\gamma}(\Psi^{\circ}_{\mathcal{L}'}(\mathcal{F})) \star \mathrm{IC}(\dot{w}^{\gamma})_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^{\beta})_{\mathcal{L}}. $$

Using the canonical isomorphism ${}^{\gamma}(\Psi^{\circ}_{\mathcal{L}'}(\mathcal{F})) \cong \Psi^{\circ}_{\mathcal{L}''}({}^{\gamma}\mathcal{F})$ in Corollary 10.16, we get

$$ {}_{\mathcal{L}''}\Psi^{\gamma}_{\mathcal{L}'}(\widetilde{\mathcal{F}}) \circ {}_{\mathcal{L}'}\Psi^{\beta}_{\mathcal{L}}(\widetilde{\mathcal{G}}) \cong \Psi^{\circ}_{\mathcal{L}''}(\mathcal{G} \star {}^{\gamma}\mathcal{F}) \star (\mathrm{IC}(\dot{w}^{\gamma})_{\mathcal{L}'} \star \mathrm{IC}(\dot{w}^{\beta})_{\mathcal{L}}). \tag{10.5} $$

Comparing (10.4) and (10.5), using (10.2), we get a canonical isomorphism

$$ {}_{\mathcal{L}''}\Psi^{\gamma}_{\mathcal{L}'}(\widetilde{\mathcal{F}}) \circ {}_{\mathcal{L}'}\Psi^{\beta}_{\mathcal{L}}(\widetilde{\mathcal{G}}) \cong {}_{\mathcal{L}''}\Psi^{\gamma\beta}_{\mathcal{L}}(\widetilde{\mathcal{F}} \circ_{\lambda} \widetilde{\mathcal{G}}). $$

The compatibility of these isomorphisms with the associativity in $\mathfrak{E}$ and $\mathfrak{D}$ follows from the pentagon diagram (10.3). This finishes the proof. □

11. Application to character sheaves

In this section, we apply Theorem 9.2 to get an equivalence between the asymptotic versions of character sheaves on G with semisimple parameter $\mathfrak{o}$ and unipotent character sheaves on its endoscopic group. To state the theorem, we review three versions of the statement 'character sheaves are categorical center of Hecke categories' (after passing to asymptotic versions).

In this section, *all schemes are defined over* $k = \overline{\mathbb{F}}_q$.

11.1. Truncated convolution for the usual Hecke category. Let H be a connected reductive group over k with maximal torus T and Borel subgroup B_H containing T (later H will be an endoscopic group of G). Let $\mathbf{c}$ be a two-sided cell in the Weyl group W_H. Let $\underline{\mathcal{S}}^{\mathbf{c}}_H$ be the full subcategory of $\underline{\mathcal{D}}_H$ consisting of perverse sheaves that are direct sums of $\mathrm{IC}(w)_H$ for $w \in \mathbf{c}$. Then $\underline{\mathcal{S}}^{\mathbf{c}}_H$ is a semisimple Abelian category equipped with a truncated convolution $(-) \circ (-)$ defined in [20, 3.2]. Note that the truncated convolution in [20] is first defined for the mixed version of $\underline{\mathcal{S}}^{\mathbf{c}}_H$ via a perverse degree truncation and a weight truncation; the weight truncation is in fact unnecessary because convolution preserves complexes pure of weight zero. Therefore one can directly define truncated convolution on $\underline{\mathcal{S}}^{\mathbf{c}}_H$.

11.2. Unipotent character sheaves. We recall the relationship between the usual Hecke category $\mathcal{D}_H$ for a connected reductive group H and unipotent character sheaves on H, following [20].

Character sheaves on H are certain simple perverse sheaves on H equivariant under the conjugation action by H. Each character sheaf has a semisimple parameter that is a W_H-orbit of $\mathrm{Ch}(T)$. When the semisimple parameter is the trivial local system on T, we call the character sheaf unipotent. Each unipotent character sheaf on H can be assigned a two-sided cell in W_H; see [20, 1.5]. Let $\underline{\mathcal{CS}}^{\mathbf{c}}_u(H)$ be the full subcategory of $D^b_H(H)$ (for the conjugation action) consisting of finite direct sums of unipotent character sheaves belonging to $\mathbf{c}$. Then $\underline{\mathcal{CS}}^{\mathbf{c}}_u(H)$ is a semisimple $\overline{\mathbb{Q}}_\ell$-linear Abelian category. By [20, 4.6, 9.1], truncated convolution is defined on $\underline{\mathcal{CS}}^{\mathbf{c}}_u(H)$ and makes it a braided monoidal category.

THEOREM 11.3 [20, Theorem 9.5]. *There is a canonical equivalence of braided monoidal categories*

$$\underline{\mathcal{CS}}^{\mathbf{c}}_u(H) \xrightarrow{\sim} \mathcal{Z}(\underline{\mathcal{S}}^{\mathbf{c}}_H),$$

where $\mathcal{Z}(-)$ denotes the categorical center introduced by Joyal and Street [10], Majid [26] and Drinfeld.

11.4. Truncated convolution for monodromic Hecke categories. Now consider the situation for G. Let $\mathfrak{o} \subset \mathrm{Ch}(T)$ be a W-orbit. In [22, 1.11, Case (v)], the notion of two-sided cells inside $W \times \mathfrak{o}$ is defined (see also [22, third paragraph in p. 620]). Such a two-sided cell $\mathfrak{c} \subset W \times \mathfrak{o}$ can be characterized as follows. For $\mathcal{L}, \mathcal{L}' \in \mathfrak{o}$ and any block $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, let $\mathfrak{c}(\beta) = \mathfrak{c} \cap (\beta \times \{\mathcal{L}\}) \subset {}_{\mathcal{L}'}W_{\mathcal{L}} \times \{\mathcal{L}\} \subset W \times \mathfrak{o}$. Then $\{\mathfrak{c}(\beta)\}$ satisfies the following:

(1) For any triple $\mathcal{L}, \mathcal{L}', \mathcal{L}'' \in \mathfrak{o}$ and $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$ and $\gamma \in {}_{\mathcal{L}''}\underline{W}_{\mathcal{L}'}$, we have $w^\gamma \mathfrak{c}(\beta) = \mathfrak{c}(\gamma\beta) = \mathfrak{c}(\gamma) w^\beta$.

(2) For $\mathcal{L} \in \mathfrak{o}$ and β the neutral block $\beta = W^\circ_{\mathcal{L}}$, $\mathfrak{c}(\beta)$ is the union of a $\Omega_{\mathcal{L}} = W_{\mathcal{L}}/W^\circ_{\mathcal{L}}$-orbit of the usual two-sided cells for $W^\circ_{\mathcal{L}}$.

In other words, starting from a fixed $\mathcal{L} \in \mathfrak{o}$ and a two-sided cell $\mathbf{c} \subset W^\circ_{\mathcal{L}}$, there is a unique two-sided cell $\mathfrak{c} \subset W \times \mathfrak{o}$, which we denote by $\mathfrak{c} = [\mathbf{c}]$, such that $\mathfrak{c} \cap (W^\circ_{\mathcal{L}} \times \{\mathcal{L}\}) = \cup_{\omega \in \Omega_{\mathcal{L}}} \omega(\mathbf{c})$.

Fix a two-sided cell $\mathfrak{c}$ for $W \times \mathfrak{o}$. For $\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}$, let ${}_{\mathcal{L}'}\underline{\mathcal{S}}^{\mathfrak{c}(\beta)}_{\mathcal{L}}$ be the full subcategory of ${}_{\mathcal{L}'}\underline{\mathcal{D}}^{\beta}_{\mathcal{L}}$ consisting of finite direct sums of simple perverse sheaves of the form $\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ for $w \in \mathfrak{c}(\beta)$. Let ${}_{\mathcal{L}'}\underline{\mathcal{S}}^{\mathfrak{c}}_{\mathcal{L}} = \oplus_{\beta \in {}_{\mathcal{L}'}\underline{W}_{\mathcal{L}}}({}_{\mathcal{L}'}\underline{\mathcal{S}}^{\mathfrak{c}(\beta)}_{\mathcal{L}})$ and $\underline{\mathcal{S}}^{\mathfrak{c}}_{\mathfrak{o}} = \oplus_{\mathcal{L}, \mathcal{L}' \in \mathfrak{o}}({}_{\mathcal{L}'}\underline{\mathcal{S}}^{\mathfrak{c}}_{\mathcal{L}})$. Truncated convolution [22, 2.24, 4.6] defines a monoidal structure $(-) \circ (-)$ on $\underline{\mathcal{S}}^{\mathfrak{c}}_{\mathfrak{o}}$. (In [22], the convolution on the monodromic Hecke category is defined in a different way from Section 3.1. Namely, in *loc. cit.*, the pushforward along $G \overset{U}{\times} G \to G$ was used instead of $G \overset{B}{\times} G \to G$. As a result, convolution as defined in *loc. cit.* does not preserve purity while the convolution in this paper does. Therefore, instead of using the definition of the truncated convolution in [22, 2.24, 4.6], we may work with the convolution defined in this paper and ignore weight truncation (doing only the cell truncation). In particular, truncated convolution can be defined directly for the nonmixed category $\underline{\mathcal{S}}^{\mathfrak{c}}_{\mathfrak{o}}$.)

11.5. Character sheaves with general monodromy. Let $\underline{\mathcal{CS}}_{\mathfrak{o}}(G)$ be the semisimple Abelian category of finite direct sums of character sheaves whose semisimple parameter is $\mathfrak{o}$ (see [22, middle of p.698]). To each character sheaf $\mathcal{A}$ with semisimple parameter $\mathfrak{o}$, one can attach a two-sided cell $\mathfrak{c}_{\mathcal{A}}$ for $W \times \mathfrak{o}$ following [22, first paragraph of p.699]. Let $\underline{\mathcal{CS}}^{\mathfrak{c}}_{\mathfrak{o}}(G)$ be the full subcategory of $\underline{\mathcal{CS}}_{\mathfrak{o}}(G)$ consisting of finite direct sums of character sheaves $\mathcal{A}$ such that $\mathfrak{c}_{\mathcal{A}} = \mathfrak{c}$. By [22, 5.20, 6.11], truncated convolution equips $\underline{\mathcal{CS}}^{\mathfrak{c}}_{\mathfrak{o}}(G)$ with the structure of a braided monoidal category.

THEOREM 11.6 [22, Theorem 6.13]. *There is a canonical equivalence of braided monoidal categories*

$$\underline{\mathcal{CS}}^{\mathfrak{c}}_{\mathfrak{o}}(G) \xrightarrow{\sim} \mathcal{Z}(\underline{\mathcal{S}}^{\mathfrak{c}}_{\mathfrak{o}}).$$

11.7. Unipotent character sheaves on a disconnected group as a twisted center. Let H be a reductive group with a finite-order automorphism σ. Then there is the notion of σ-twisted unipotent character sheaves: these are certain simple perverse sheaves on H equivariant under the σ-twisted conjugation action $h \cdot x = hx\sigma(h)^{-1}$, $h, x \in H$. Let $\mathbf{c}$ be a two-sided cell of W_H invariant under σ. Then one can define the category $\underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H; \sigma)$ consisting of finite direct sums of σ-twisted unipotent character sheaves on H whose two-sided cell is $\mathbf{c}$. If σ changes to the automorphism $\sigma\mathrm{Ad}(h)$ for some $h \in H(k)$, then right translation by h induces an equivalence between $\underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H; \sigma)$ and $\underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H; \sigma\mathrm{Ad}(h))$.

On the other hand, if σ stabilizes B_H, then it induces an autoequivalence σ_* of the monoidal category $\underline{\mathcal{D}}_H$. For a two-sided cell $\mathbf{c}$ for W_H fixed by σ, $\underline{\mathcal{S}}^{\mathbf{c}}_{H}$ is stable under the σ-action, and one can talk about the *σ-twisted center* of the monoidal category $\underline{\mathcal{S}}^{\mathbf{c}}_{H}$, denoted by $\mathcal{Z}(\underline{\mathcal{S}}^{\mathbf{c}}_{H}; \sigma)$. Objects $\mathcal{F}$ in $\mathcal{Z}(\underline{\mathcal{S}}^{\mathbf{c}}_{H}; \sigma)$ are $\mathcal{F} \in \underline{\mathcal{S}}^{\mathbf{c}}_{H}$ equipped with functorial isomorphisms $\mathcal{F} \circ \sigma_*\mathcal{G} \cong \mathcal{G} \circ \mathcal{F}$ for $\mathcal{G} \in \underline{\mathcal{S}}^{\mathbf{c}}_{H}$. If σ changes to $\sigma\mathrm{Ad}(b)$ for some $b \in B_H(k)$, then the actions of σ and $\sigma\mathrm{Ad}(b)$ on $\underline{\mathcal{D}}_H$

are canonically equivalent (using the $\mathrm{Ad}(B_H)$-equivariant structures of objects in $\mathcal{D}_H$) and hence a canonical equivalence $\mathcal{Z}(\underline{\mathcal{S}}^{\mathbf{c}}_H; \sigma) \cong \mathcal{Z}(\underline{\mathcal{S}}^{\mathbf{c}}_H; \sigma\mathrm{Ad}(b))$.

THEOREM 11.8 [23, Theorem 7.3]. *Under the above assumptions (in particular,* $\mathbf{c}$ *is fixed by* σ*), there is a canonical equivalence of categories*

$$\underline{\mathcal{CS}}^{\mathbf{c}}_u(H; \sigma) \xrightarrow{\sim} \mathcal{Z}(\underline{\mathcal{S}}^{\mathbf{c}}_H; \sigma).$$

11.9. More notations Now we set up notation for our application to character sheaves. Fix $\mathcal{L} \in \mathfrak{o}$, and let $\mathbf{c}$ be a two-sided cell of $W^\circ_{\mathcal{L}}$. Let $[\mathbf{c}]$ be the two-sided cell for $W \times \mathfrak{o}$ constructed from $\mathbf{c}$ by the procedure described in Section 11.5. Let $\Omega_{\mathbf{c}} \subset \Omega_{\mathcal{L}}$ be the stabilizer of $\mathbf{c}$ under $\Omega_{\mathcal{L}}$.

Let H be the endoscopic group of G attached to $\mathcal{L}$. In Section 10.3, we have introduced an algebraic group ${}_{\mathcal{L}}\mathfrak{H}_{\mathcal{L}}$ containing $H = H^\circ_{\mathcal{L}}$ as its neutral component. The component group of ${}_{\mathcal{L}}\mathfrak{H}_{\mathcal{L}}$ is $\Omega_{\mathcal{L}}$. For $\beta \in \Omega_{\mathcal{L}}$, any lifting $\dot{w}^\beta \in {}_{\mathcal{L}}\Xi^\beta_{\mathcal{L}} = w^\beta T$ induces an automorphism of H preserving B_H. The category of β-twisted character sheaves $\underline{\mathcal{CS}}^{\mathbf{c}}_u(H; \beta)$ is independent of the choice of $\dot{w}^\beta$ up to canonical equivalences as we discussed in Section 11.7. Therefore we may unambiguously identify all these categories and write it as $\underline{\mathcal{CS}}^{\mathbf{c}}_u(H; \beta)$. Note that $\underline{\mathcal{CS}}^{\mathbf{c}}_u(H; \beta)$ carries an action of $\Omega_{\mathbf{c}}$: for each $\beta' \in \Omega_{\mathbf{c}}$ with lifting $\dot{w}^{\beta'} \in {}_{\mathcal{L}}\widetilde{\Xi}^{\beta'}_{\mathcal{L}}$, the $\dot{w}^{\beta'}$-action on H induces an autoequivalence of $\underline{\mathcal{CS}}^{\mathbf{c}}_u(H; \beta)$, which depends only on β' up to canonical equivalences.

For $\beta \in \Omega_{\mathcal{L}}$, we have defined an autoequivalence ${}^\beta(-) : {}_{\mathcal{L}}\underline{\mathcal{D}}_{\mathcal{L}} \to {}_{\mathcal{L}}\underline{\mathcal{D}}_{\mathcal{L}}$ in Section 10.14. For any minimal IC sheaf $\xi_\gamma \in {}_{\mathcal{L}}\underline{\mathfrak{P}}^\gamma_{\mathcal{L}}$ for $\gamma \in \Omega_{\mathcal{L}}$ (so $\xi_\gamma \cong \underline{\mathrm{IC}}(w^\gamma)_{\mathcal{L}}$), define a $\overline{\mathbb{Q}}_\ell$-line

$$\Lambda_\beta(\gamma) := \mathrm{Hom}(\xi_\gamma, {}^\beta\xi_\gamma). \tag{11.1}$$

Note that $\Lambda_\beta(\gamma)$ is independent of the choice of ξ_γ up to canonical isomorphisms. We have canonical isomorphisms $\Lambda_\beta(\gamma_1) \otimes \Lambda_\beta(\gamma_2) \xrightarrow{\sim} \Lambda_\beta(\gamma_1\gamma_2)$ satisfying associativity, and $\Lambda_\beta(1)$ is canonically trivialized. Therefore the assignment $\gamma \mapsto \Lambda_\beta(\gamma)$ defines a normalized 1-cocycle on $\Omega_{\mathcal{L}}$ valued in $\mathrm{Pic}(\overline{\mathbb{Q}}_\ell)$. By restriction, we may view Λ_β as a normalized 1-cocycle on $\Omega_{\mathbf{c}}$ valued in $\mathrm{Pic}(\overline{\mathbb{Q}}_\ell)$.

Suppose $\mathcal{C}$ is an E-linear category (E is a field) on which a group A acts (so each $\gamma \in A$ gives an autoequivalence of $\mathcal{C}$, which we denote by ${}^\gamma(-)$, ${}^{1_A}(-) = \mathrm{id}_{\mathcal{C}}$, together with natural isomorphisms ${}^{\gamma_1\gamma_2}(-) \cong {}^{\gamma_1}({}^{\gamma_2}(-))$ satisfying associativity and unital conditions). Let $\Lambda \in \mathcal{Z}^1_{\mathrm{norm}}(A, \mathrm{Pic}(E))$ be a normalized 1-cocycle of A valued in $\mathrm{Pic}(E)$. Then an (A, Λ)-equivariant structure on an object $X \in \mathcal{C}$ is a collection of isomorphisms $\alpha_\gamma : {}^\gamma X \cong X \otimes \Lambda(\gamma)$ for $\gamma \in A$, which is the identity for $\gamma = 1_A$ ($\Lambda(1_A)$ is trivialized) such that for $\gamma_1, \gamma_2 \in A$, the following diagram is commutative

$$\begin{tikzcd}
{}^{\gamma_1\gamma_2}X \arrow[r, "{}^{\gamma_1}\alpha_{\gamma_2}"] \arrow[d, "\alpha_{\gamma_1\gamma_2}"] & {}^{\gamma_1}X \otimes \Lambda(\gamma_2) \arrow[d, "\alpha_{\gamma_1}\otimes \mathrm{id}"] \\
X \otimes \Lambda(\gamma_1\gamma_2) \arrow[r, "\sim"] & X \otimes \Lambda(\gamma_1) \otimes \Lambda(\gamma_2)
\end{tikzcd}$$

where the bottom map is the one from the cocycle structure of Λ. Let $\mathcal{C}^{(A,\Lambda)}$ be the category of objects in $\mathcal{C}$ equipped with (A,Λ)-equivariant structures, with the obvious notion of morphisms compatible with the equivariant structures.

THEOREM 11.10. (1) *Let $\mathfrak{o} \subset \mathrm{Ch}(T)$ be the W-orbit of $\mathcal{L}$, and $\mathbf{c}$ a two-sided cell in $W^{\circ}_{\mathcal{L}}$. There is an equivalence of semisimple Abelian categories depending on the liftings $\{\dot{w}^{\beta}\}_{\beta\in\Omega_{\mathbf{c}}}$:*

$$\underline{\mathcal{CS}}^{[\mathbf{c}]}_{\mathfrak{o}}(G) \xrightarrow{\sim} \bigoplus_{\beta\in\Omega_{\mathbf{c}}} \underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H;\beta)^{(\Omega_{\mathbf{c}},\Lambda_{\beta})}.$$

(2) *The class of Λ_{β} in $\mathrm{H}^2(\Omega_{\mathcal{L}},\overline{\mathbb{Q}}_{\ell}^{\times})$ is always trivial. In particular, we have a noncanonical equivalence of semisimple Abelian categories*

$$\underline{\mathcal{CS}}^{[\mathbf{c}]}_{\mathfrak{o}}(G) \xrightarrow{\sim} \bigoplus_{\beta\in\Omega_{\mathbf{c}}} \underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H;\beta)^{\Omega_{\mathbf{c}}}, \tag{11.2}$$

where $(-)^{\Omega_{\mathbf{c}}}$ means the category of objects with $\Omega_{\mathbf{c}}$-equivariant structures.

REMARK 11.11. Equivalence (11.2) induces a *canonical* bijection between simple objects on both sides (independent of how one trivializes Λ_{β}). Simple objects in $\underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H;\beta)$ are classified in [**19**, Section 46], from which one can get a classification of simple objects in $\underline{\mathcal{CS}}^{[\mathbf{c}]}_{\mathfrak{o}}(G)$ using (11.2). In the case where $\Omega_{\mathbf{c}}$ is trivial, simple objects in both $\underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H)$ and $\underline{\mathcal{CS}}^{[\mathbf{c}]}_{\mathfrak{o}}(G)$ are parametrized by the set $\mathcal{M}(\mathcal{G}_{\mathbf{c}})$ by [**15**, Theorem 23.1] (see [**13**, Sections 4.4–4.13] for $\mathcal{G}_{\mathbf{c}}$ and $\mathcal{M}(\mathcal{G}_{\mathbf{c}})$). This is consistent with (11.2).

The rest of the section is devoted to the proof of Theorem 11.10.

LEMMA 11.12. *The projection from $\underline{\mathcal{S}}^{[\mathbf{c}]}_{\mathfrak{o}}$ to ${}_{\mathcal{L}}\underline{\mathcal{S}}^{[\mathbf{c}]}_{\mathcal{L}}$ induces an equivalence on their categorical centers:*

$$r_{\mathcal{L}} : \mathcal{Z}(\underline{\mathcal{S}}^{[\mathbf{c}]}_{\mathfrak{o}}) \xrightarrow{\sim} \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}^{[\mathbf{c}]}_{\mathcal{L}}). \tag{11.3}$$

Proof. We construct an inverse to $r_{\mathcal{L}}$ as follows. Let ${}_{\mathcal{L}}\mathcal{F}_{\mathcal{L}} \in \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}^{[\mathbf{c}]}_{\mathcal{L}})$. Define $\mathcal{F} = \oplus_{\mathcal{L}'}\mathcal{F}_{\mathcal{L}} \in \underline{\mathcal{S}}^{[\mathbf{c}]}_{\mathfrak{o}} = \oplus_{\mathcal{L},\mathcal{L}'}({}_{\mathcal{L}'}\underline{\mathcal{S}}^{[\mathbf{c}]}_{\mathcal{L}})$ by ${}_{\mathcal{L}'}\mathcal{F}_{\mathcal{L}} = 0$ if $\mathcal{L}' \neq \mathcal{L}$, and ${}_{\mathcal{L}'}\mathcal{F}_{\mathcal{L}'} = \xi \star {}_{\mathcal{L}}\mathcal{F}_{\mathcal{L}} \star \xi^{-1}$ for some $\xi \in {}_{\mathcal{L}'}\mathfrak{P}_{\mathcal{L}}$. Using the central structure of ${}_{\mathcal{L}}\mathcal{F}_{\mathcal{L}}$, we see that ${}_{\mathcal{L}'}\mathcal{F}_{\mathcal{L}'}$ is independent of the choice of ξ up to canonical isomorphisms. Moreover, we show that $\mathcal{F}$ carries a central structure. For $\mathcal{G} \in {}_{\mathcal{L}''}\underline{\mathcal{S}}^{[\mathbf{c}]}_{\mathcal{L}'}$, upon choosing $\xi \in {}_{\mathcal{L}'}\mathfrak{P}_{\mathcal{L}}$ and $\eta \in {}_{\mathcal{L}''}\mathfrak{P}_{\mathcal{L}}$, we may write $\mathcal{G} = \eta \star \mathcal{H} \star \xi^{-1}$ for $\mathcal{H} \in {}_{\mathcal{L}}\underline{\mathcal{S}}^{[\mathbf{c}]}_{\mathcal{L}}$. Then we have an isomorphism $\mathcal{F} \circ \mathcal{G} = {}_{\mathcal{L}''}\mathcal{F}_{\mathcal{L}''} \circ \mathcal{G} = (\eta \star {}_{\mathcal{L}}\mathcal{F}_{\mathcal{L}} \star \eta^{-1}) \circ (\eta \star \mathcal{H} \star \xi^{-1}) \cong \eta \star ({}_{\mathcal{L}}\mathcal{F}_{\mathcal{L}} \circ \mathcal{H}) \star \xi^{-1} \cong \eta \star (\mathcal{H} \circ {}_{\mathcal{L}}\mathcal{F}_{\mathcal{L}}) \star \xi^{-1} = \mathcal{G} \circ {}_{\mathcal{L}'}\mathcal{F}_{\mathcal{L}'} = \mathcal{G} \circ \mathcal{F}$ coming from the central structure of ${}_{\mathcal{L}}\mathcal{F}_{\mathcal{L}}$. Again this isomorphism is independent of the choices of ξ and η. The construction ${}_{\mathcal{L}}\mathcal{F}_{\mathcal{L}} \mapsto \mathcal{F}$ gives an inverse to $r_{\mathcal{L}}$ and shows that $r_{\mathcal{L}}$ is an equivalence. □

Lemma 11.13. *Using liftings* $\{\dot{w}^{\beta}\}_{\beta \in \Omega_{\mathbf{c}}}$, *there is an equivalence*

$$\mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}^{[\mathbf{c}]}_{\mathcal{L}}) \xrightarrow{\sim} \bigoplus_{\beta \in \Omega_{\mathbf{c}}} \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}^{\circ,\mathbf{c}}_{\mathcal{L}};\beta)^{(\Omega_{\mathbf{c}},\Lambda_{\beta})}, \tag{11.4}$$

where ${}_{\mathcal{L}}\underline{\mathcal{S}}^{\circ,\mathbf{c}}_{\mathcal{L}}$ *consists of direct sums of* $\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ *for* $w \in \mathbf{c} \subset W^{\circ}_{\mathcal{L}}$.

Proof. For each $\beta \in \Omega_{\mathcal{L}}$, let $\xi_{\beta} = \omega\mathrm{IC}(\dot{w}^{\beta})_{\mathcal{L}} \in {}_{\mathcal{L}}\mathfrak{P}^{\beta}_{\mathcal{L}}$. For $\mathcal{F} \in \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}^{[\mathbf{c}]}_{\mathcal{L}})$, write $\mathcal{F} = \oplus_{\beta \in \Omega_{\mathcal{L}}}\mathcal{F}_{\beta} \star \xi_{\beta}$, where $\mathcal{F}_{\beta} \in {}_{\mathcal{L}}\underline{\mathcal{S}}^{\circ,[\mathbf{c}]}_{\mathcal{L}} := {}_{\mathcal{L}}\underline{\mathcal{S}}^{\circ}_{\mathcal{L}} \cap {}_{\mathcal{L}}\underline{\mathcal{S}}^{[\mathbf{c}]}_{\mathcal{L}}$. By the description of $[\mathbf{c}]$ in Section 11.5, $\mathcal{F}_{\beta}$ can be written uniquely as a direct sum $\oplus_{\mathbf{c}' \in \Omega_{\mathcal{L}} \cdot \mathbf{c}}\mathcal{F}^{\mathbf{c}'}_{\beta}$, where $\mathcal{F}^{\mathbf{c}'}_{\beta} \in {}_{\mathcal{L}}\underline{\mathcal{S}}^{\circ,\mathbf{c}'}_{\mathcal{L}}$. Let $(-) \circ (-)$ denote the truncated convolution in ${}_{\mathcal{L}}\underline{\mathcal{S}}^{[\mathbf{c}]}_{\mathcal{L}}$. The central structure of $\mathcal{F}$ gives the following isomorphisms:

$$(\mathcal{F}_{\beta} \star \xi_{\beta}) \circ (\mathcal{G} \star \xi_{\gamma}) \cong (\mathcal{G} \star \xi_{\gamma}) \circ (\mathcal{F}_{\beta} \star \xi_{\beta}), \quad \forall \beta, \gamma \in \Omega_{\mathcal{L}}, \mathcal{G} \in {}_{\mathcal{L}}\underline{\mathcal{S}}^{\circ,[\mathbf{c}]}_{\mathcal{L}}. \tag{11.5}$$

Using the action of $\Omega_{\mathcal{L}}$ on ${}_{\mathcal{L}}\underline{\mathcal{D}}^{\circ}_{\mathcal{L}}$ introduced in Section 10.14, we may rewrite the above isomorphism as

$$(\mathcal{F}_{\beta} \circ {}^{\beta}\mathcal{G}) \star \xi_{\beta} \star \xi_{\gamma} \cong (\mathcal{G} \circ {}^{\gamma}\mathcal{F}_{\beta}) \star \xi_{\gamma} \star \xi_{\beta}.$$

By (11.1), we have $\xi_{\beta} \star \xi_{\gamma} \star \xi_{\beta}^{-1} \star \xi_{\gamma}^{-1} = \delta_{\mathcal{L}} \otimes \Lambda_{\beta}(\gamma)$. We may rewrite the above isomorphism as

$$\mathcal{F}_{\beta} \circ {}^{\beta}\mathcal{G} \otimes \Lambda_{\beta}(\gamma) \cong \mathcal{G} \circ {}^{\gamma}\mathcal{F}_{\beta}, \quad \forall \beta, \gamma \in \Omega_{\mathcal{L}}, \mathcal{G} \in {}_{\mathcal{L}}\underline{\mathcal{S}}^{\circ,[\mathbf{c}]}_{\mathcal{L}}. \tag{11.6}$$

Taking $\gamma = 1$, we get isomorphisms

$$\eta_{\mathcal{G}} : \mathcal{F}_{\beta} \circ {}^{\beta}\mathcal{G} \cong \mathcal{G} \circ \mathcal{F}_{\beta}, \quad \forall \beta \in \Omega_{\mathcal{L}}, \mathcal{G} \in {}_{\mathcal{L}}\underline{\mathcal{S}}^{\circ,[\mathbf{c}]}_{\mathcal{L}}, \tag{11.7}$$

which equip $\mathcal{F}_{\beta}$ with a β-twisted central structure, that is, $\mathcal{F}_{\beta}$ has a natural lift to an object $\mathcal{F}^{\#}_{\beta} \in \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}^{\circ,[\mathbf{c}]}_{\mathcal{L}};\beta)$.

Taking $\mathcal{G} = \delta_{\mathcal{L}}$ in (11.6), we get isomorphisms

$$\zeta_{\gamma} : \mathcal{F}_{\beta} \otimes \Lambda_{\beta}(\gamma) \cong {}^{\gamma}\mathcal{F}_{\beta}, \quad \forall \gamma \in \Omega_{\mathcal{L}}, \tag{11.8}$$

which equip $\mathcal{F}_{\beta}$ with an $(\Omega_{\mathcal{L}}, \Lambda_{\beta})$-equivariant structure. The central structure implies that isomorphisms (11.6) satisfy compatibilities with convolution of the $\mathcal{G} \star \xi_{\gamma}$'s, which are equivalent to the commutative diagram

$$\begin{tikzcd}
\mathcal{F}_{\beta} \circ {}^{\beta\gamma}\mathcal{G} \otimes \Lambda_{\beta}(\gamma) \arrow[r, "\eta_{{}^{\gamma}\mathcal{G}} \otimes \mathrm{id}"] \arrow[d, equal] & {}^{\gamma}\mathcal{G} \circ \mathcal{F}_{\beta} \otimes \Lambda_{\beta}(\gamma) \arrow[r, "\mathrm{id} \circ \zeta_{\gamma}"] & {}^{\gamma}\mathcal{G} \circ {}^{\gamma}\mathcal{F}_{\beta} \arrow[d, equal] \\
\mathcal{F}_{\beta} \circ {}^{\gamma\beta}\mathcal{G} \otimes \Lambda_{\beta}(\gamma) \arrow[r, "\zeta_{\gamma} \circ \mathrm{id}"] & {}^{\gamma}\mathcal{F}_{\beta} \circ {}^{\gamma\beta}\mathcal{G} \arrow[r, "{}^{\gamma}\eta_{\mathcal{G}}"] & {}^{\gamma}(\mathcal{G} \circ \mathcal{F}_{\beta})
\end{tikzcd}$$

for all $\gamma \in \Omega_{\mathcal{L}}$ and $\mathcal{G} \in {}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ,[\mathbf{c}]}$. The commutativity of these diagrams means exactly that $\mathcal{F}_{\beta}^{\#}$ carries an $(\Omega_{\mathcal{L}}, \Lambda_{\beta})$-equivariant structure as an object in $\mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ,[\mathbf{c}]}; \beta)$, that is, $\mathcal{F}_{\beta}^{\#}$ further lifts to an object $\mathcal{F}_{\beta}^{\heartsuit} \in \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ,[\mathbf{c}]}; \beta)^{(\Omega_{\mathcal{L}}, \Lambda_{\beta})}$.

Take a cell $\mathbf{c}' \subset W_{\mathcal{L}}^{\circ}$ in the $\Omega_{\mathcal{L}}$-orbit of $\mathbf{c}$ and take $\mathcal{G} \in {}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ,\mathbf{c}'}$. Now ${}^{\beta}\mathcal{G} \in {}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ,\beta(\mathbf{c}')}$. For $w, w' \in W_{\mathcal{L}}^{\circ}$ in different cells, the truncated convolution of $\underline{\mathrm{IC}}(w)_{\mathcal{L}}$ and $\underline{\mathrm{IC}}(w')_{\mathcal{L}}$ vanishes. Therefore the left side of (11.7) lies in ${}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ,\beta(\mathbf{c}')}$ while the right side lies in ${}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ,\mathbf{c}'}$. If $\beta(\mathbf{c}') \neq \mathbf{c}'$, then both sides of (11.7) must vanish; hence $\underline{\mathrm{IC}}(w)_{\mathcal{L}} \circ \mathcal{F}_{\beta}^{\mathbf{c}'} = 0$ for all $w \in \mathbf{c}'$. This implies $\mathcal{F}_{\beta}^{\mathbf{c}'} = 0$ if $\beta(\mathbf{c}') \neq \mathbf{c}'$ since ${}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ,\mathbf{c}'}$ has a monoidal unit. Since $\Omega_{\mathcal{L}}$ is Abelian, $\beta \in \Omega_{\mathcal{L}}$ either fixes all $\mathbf{c}'$ in the orbit of $\mathbf{c}$ or none; therefore $\mathcal{F}_{\beta} = 0$ if $\beta \notin \Omega_{\mathbf{c}}$.

Now we consider $\beta \in \Omega_{\mathbf{c}}$. Isomorphisms (11.8) allow us to recover $\mathcal{F}_{\beta}^{\mathbf{c}'}$ for any cell $\mathbf{c}'$ in the $\Omega_{\mathcal{L}}$-orbit of $\mathbf{c}$ from $\mathcal{F}_{\beta}^{\mathbf{c}}$. The object $\mathcal{F}_{\beta}^{\mathbf{c}}$ lifts to $\mathcal{F}_{\beta}^{\mathbf{c},\heartsuit} \in \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ,\mathbf{c}}; \beta)^{(\Omega_{\mathbf{c}}, \Lambda_{\beta})}$. The functor $\mathcal{F}_{\beta}^{\heartsuit} \mapsto \mathcal{F}_{\beta}^{\mathbf{c},\heartsuit}$ is an equivalence

$$\mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ,[\mathbf{c}]}; \beta)^{(\Omega_{\mathcal{L}}, \Lambda_{\beta})} \xrightarrow{\sim} \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ,\mathbf{c}}; \beta)^{(\Omega_{\mathbf{c}}, \Lambda_{\beta})}.$$

Combining the above discussions, we arrive at equivalence (11.4) given by $\mathcal{F} \mapsto \bigoplus_{\beta \in \Omega_{\mathbf{c}}} \mathcal{F}_{\beta}^{\mathbf{c},\heartsuit}$. □

11.14. Proof of Theorem 11.10(1). Theorem 9.2 implies a monoidal equivalence between semisimple Abelian categories:

$$\underline{\mathcal{S}}_{H}^{\mathbf{c}} \xrightarrow{\sim} {}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ,\mathbf{c}}. \tag{11.9}$$

In [22], the value of the a-function for $[\mathbf{c}]$ used in the construction of the truncated convolution is the same as the value of the a-function on $\mathbf{c}$ as a cell for

W_H. By Corollary 10.16, (11.9) is equivariant under the actions of $\Omega_{\mathbf{c}}$. Therefore, we get a canonical braided monoidal equivalence for $\beta \in \Omega_{\mathbf{c}}$:

$$\mathcal{Z}(\underline{\mathcal{S}}_H^{\mathbf{c}}; \beta)^{(\Omega_{\mathbf{c}}, \Lambda_\beta)} \xrightarrow{\sim} \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ, \mathbf{c}}; \beta)^{(\Omega_{\mathbf{c}}, \Lambda_\beta)}. \tag{11.10}$$

Composing the known equivalences, we get

$$\begin{tikzcd}
\underline{\mathcal{CS}}_{\mathfrak{o}}^{[\mathbf{c}]}(G) \arrow[r, "\text{Th.11.6}", "\sim"'] & \mathcal{Z}(\underline{\mathcal{S}}_{\mathfrak{o}}^{[\mathbf{c}]}) \arrow[r, "(11.3)", "\sim"'] & \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{[\mathbf{c}]}) \arrow[r, "(11.4)", "\sim"'] & \oplus_{\beta \in \Omega_{\mathbf{c}}} \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ, \mathbf{c}}; \beta)^{(\Omega_{\mathbf{c}}, \Lambda_\beta)} \\
 & & \oplus_{\beta \in \Omega_{\mathbf{c}}} \mathcal{Z}(\underline{\mathcal{S}}_H^{\mathbf{c}}; \beta)^{(\Omega_{\mathbf{c}}, \Lambda_\beta)} \arrow[ur, "(11.10)", "\sim"'] & \oplus_{\beta \in \Omega_{\mathbf{c}}} \underline{\mathcal{CS}}_u^{\mathbf{c}}(H; \beta)^{(\Omega_{\mathbf{c}}, \Lambda_\beta)} \arrow[l, "\text{Th.11.8}"', "\sim"]
\end{tikzcd}$$

11.15. Proof of Theorem 11.10(2). If $\Omega_{\mathcal{L}}$ is cyclic, then $\mathrm{H}^2(\Omega_{\mathcal{L}}, \overline{\mathbb{Q}}_\ell^\times) = \{1\}$.

When G is almost simple, the only case where $\Omega_{\mathcal{L}}$ is not cyclic is when $G = \mathrm{Spin}_{4n}$ and $\Omega_{\mathcal{L}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for certain $\mathcal{L}$. In this case, if $\beta = 1$, then ${}^\beta(-)$ is naturally isomorphic to the identity functor; hence Λ_β carries a trivialization. If $\beta \neq 1$, then $\Lambda_\beta(\beta)$ carries a canonical trivialization such that $\Lambda_\beta : \Omega_{\mathcal{L}} \to \mathrm{Pic}(\overline{\mathbb{Q}}_\ell)$ factors through $\overline{\Lambda}_\beta : \Omega_{\mathcal{L}}/\langle\beta\rangle \cong \mathbb{Z}/2\mathbb{Z} \to \mathrm{Pic}(\overline{\mathbb{Q}}_\ell)$. Therefore the class of $\Omega_{\mathcal{L}}$ is the pullback from the class of $\overline{\Lambda}_\beta$ in $\mathrm{H}^2(\Omega_{\mathcal{L}}/\langle\beta\rangle, \overline{\mathbb{Q}}_\ell^\times) = \{1\}$, which has to be trivial.

In general, let $\widetilde{G} \to G$ be the simply connected cover of the derived subgroup of G. Then $\widetilde{G} = \prod_i G_i$, where each G_i is almost simple and simply connected. Let $\widetilde{T} \subset \widetilde{G}$ be the maximal torus whose image in G is contained in T, and let $\widetilde{\mathcal{L}} \in \mathrm{Ch}(\widetilde{T})$ be the pullback of $\mathcal{L}$. Then under the identification of the Weyl groups of $\widetilde{G}$ and G, there are an inclusion $W_{\mathcal{L}} \subset W_{\widetilde{\mathcal{L}}}$ and an equality $W_{\mathcal{L}}^\circ = W_{\widetilde{\mathcal{L}}}^\circ$. Therefore $\Omega_{\mathcal{L}} \subset \Omega_{\widetilde{\mathcal{L}}}$. Moreover, from the definitions we see that for $\beta \in \Omega_{\mathcal{L}}$, Λ_β is the restriction to $\Omega_{\mathcal{L}}$ of the similarly defined cocycle $\widetilde{\Lambda}_\beta$ for $\Omega_{\widetilde{\mathcal{L}}}$. By the almost simple case settled above, the class of $\widetilde{\Lambda}_\beta$ is always trivial in $\mathrm{H}^2(\Omega_{\widetilde{\mathcal{L}}}, \overline{\mathbb{Q}}_\ell^\times)$; hence the same is true for Λ_β by restriction. The proof of Theorem 11.10 is now complete. □

12. Application to representations

In [21] and [23], the first author has related the category of representations of $G(\mathbb{F}_q)$ to the twisted categorical center of asymptotic versions of the monodromic Hecke category of G, in a similar way that character sheaves on G are related to the categorical center of the monodromic Hecke category. In this section, we apply the monodromic–endoscopic equivalence to prove a relationship between representations of $G(\mathbb{F}_q)$ and its endoscopic groups.

In this section, *all schemes are over* $k = \overline{\mathbb{F}}_q$. We will work in the more general context of disconnected groups as in [23], and establish a relationship between character sheaves on disconnected groups and unipotent character sheaves on their endoscopic groups.

12.1. Disconnected groups and forms. Let G be a connected reductive group over k with maximal torus T and Borel subgroup B containing T (all defined over k). Let $\epsilon : G \to G$ be a morphism over k preserving (T, B) and satisfying one of the following two conditions:

(A) ϵ is the Frobenius map for some rational structure of G over $\mathbb{F}_q$;

(B) ϵ is a finite-order automorphism of G over k.

We will refer to the two conditions above as 'Situation (A)' and 'Situation (B)'.

We form the semidirect semigroup product $G \rtimes \epsilon^{\mathbb{Z}_{\geqslant 0}}$ (where $\epsilon^{\mathbb{Z}_{\geqslant 0}}$ is a copy of $\mathbb{Z}_{\geqslant}$ acting on G via ϵ). There is the notion of character sheaves on the coset $G \cdot \epsilon$; see [23, Section 6.1]. These are certain G-equivariant simple perverse sheaves on $G \cdot \epsilon$. In Situation (B), this notion is the same as the ϵ-twisted character sheaves on G considered in Section 11.7.

When ϵ is in Situation (A), character sheaves on $G \cdot \epsilon$ are exactly the irreducible $\overline{\mathbb{Q}}_\ell$-representations of the finite group G^ϵ, the group of $\mathbb{F}_q$-points of the form of G with Frobenius map ϵ.

The map ϵ induces an action on $\mathrm{Ch}(T)$: $\epsilon\mathcal{L} := \epsilon_*\mathcal{L}$. In the following, we fix a W-orbit $\mathfrak{o} \subset \mathrm{Ch}(T)$ that is stable under the action of ϵ. Fix $\mathcal{L} \in \mathrm{Ch}(T)$. We have the relatively pinned endoscopic group $H = H^\circ_{\mathcal{L}}$ as defined in Section 10.2 (now over k).

The map ϵ induces an automorphism of the based root system of G, which we still denote by ϵ. It restricts to a bijection of based root systems $\epsilon : \Phi_{\mathcal{L}} \xrightarrow{\sim} \Phi_{\epsilon\mathcal{L}}$ (it sends $\Phi^+_{\mathcal{L}} = \Phi_{\mathcal{L}} \cap \Phi^+$ to $\Phi^+_{\epsilon\mathcal{L}} = \Phi_{\epsilon\mathcal{L}} \cap \Phi^+$ because ϵ is a bijection of based root systems). For each $\beta \in {}_{\mathcal{L}}\underline{W}_{\epsilon\mathcal{L}}$, w^β gives a bijection of based root systems $w^\beta : \Phi_{\epsilon\mathcal{L}} \to \Phi_{\mathcal{L}}$. The composition $w^\beta \circ \epsilon$ is an automorphism of the based root system $(\Phi_{\mathcal{L}}, \Delta_{\mathcal{L}})$.

Fix a lifting $\dot{w}^\beta$ for each $\beta \in {}_{\mathcal{L}}\underline{W}_{\epsilon\mathcal{L}}$. In Situation (A), there is a unique $\mathbb{F}_q$-Frobenius structure $\sigma_{\beta\epsilon} : H \to H$ preserving (T, B_H), inducing $w^\beta \circ \epsilon$ on the root system of H (which is identified with $\Phi_{\mathcal{L}}$), and such that $(\sigma_{\beta\epsilon}, \mathrm{Ad}(\dot{w}^\beta) \circ \epsilon)$ is compatible with the relative pinning in the sense that the following diagram is commutative for each simple root of α of H:

$$\begin{tikzcd}
H_\alpha \arrow[r, "\iota_\alpha"] \arrow[d, "\sigma_{\beta\epsilon}"] & G_\alpha \arrow[d, "\mathrm{Ad}(\dot{w}^\beta)\circ\epsilon"] \\
H_{w^\beta\epsilon(\alpha)} \arrow[r, "\iota_{w^\beta\epsilon(\alpha)}"] & G_{w^\beta\epsilon(\alpha)}
\end{tikzcd}$$

Note that the construction of $\sigma_{\beta\epsilon}$ depends on the choice of the lifting $\dot{w}^\beta$. Similarly, in Situation (B), there is a unique finite-order automorphism $\sigma_{\beta\epsilon} : H \to H$ preserving (T, B_H), inducing $w^\beta \circ \epsilon$ on the root system of H, and such that $(\sigma_{\beta\epsilon}, \mathrm{Ad}(\dot{w}^\beta) \circ \epsilon)$ is compatible with the relative pinning in the above sense.

EXAMPLE 12.2. Suppose ϵ is the Frobenius map for the split $\mathbb{F}_q$-structure of G. In this case, $\epsilon\mathcal{L} = \mathcal{L}^{1/q}$ (note that the order of $\mathcal{L}$ is always prime to p). Assume that $W_{\mathcal{L}} = \{1\}$ (we always assume $\mathfrak{o}$ be stable under ϵ). In this case, $H = T$, and there is a unique $w \in W$ such that $w\mathcal{L}^{1/q} = \mathcal{L}$ (the blocks are singletons, so w can be viewed as a block). Then $\sigma_{w\epsilon} : T \to T$ is the Frobenius map for the $\mathbb{F}_q$-form of T given by the W-conjugacy class of w (note that the conjugacy classes of maximal tori of the split G defined over $\mathbb{F}_q$ are classified by conjugacy classes in W).

12.3. Character sheaves on disconnected groups as a twisted center. Recall that we fix a W-orbit $\mathfrak{o} \subset \mathrm{Ch}(T)$ stable under ϵ, and also fix $\mathcal{L} \in \mathfrak{o}$. Let $\mathfrak{c} \subset W \times \mathfrak{o}$ be a two-sided cell that is stable under ϵ. As in Section 11.4, we may write $\mathfrak{c} = [\mathbf{c}]$ for some two-sided cell $\mathbf{c} \subset W^\circ_{\mathcal{L}}$, which is well defined up to the action of $\Omega_{\mathcal{L}}$.

Let $\underline{\mathcal{CS}}^{\mathfrak{c}}_{\mathfrak{o}}(G; \epsilon)$ be the semisimple Abelian category whose objects are finite direct sums of character sheaves on $G \cdot \epsilon$ with semisimple parameter $\mathfrak{o}$ and belonging to the cell $\mathfrak{c}$ (see [23, Section 6.1]). In Situation (A), the G-conjugation action on $G \cdot \epsilon$ is transitive by Lang's theorem, with the stabilizer of $1 \cdot \epsilon$ equal to the finite group G^ϵ. Therefore we have an equivalence

$$\underline{\mathcal{CS}}^{\mathfrak{c}}_{\mathfrak{o}}(G; \epsilon) \cong \mathrm{Rep}^{\mathfrak{c}}_{\mathfrak{o}}(G^\epsilon), \tag{12.1}$$

the latter being the semisimple Abelian category of $\overline{\mathbb{Q}}_\ell$-representations of the finite group G^ϵ whose semisimple parameter is $\mathfrak{o}$ and are finite direct sums of irreducible representations belonging to the cell $\mathfrak{c}$.

The following theorem proved in [23] is a common generalization of Theorems 11.3, 11.6 and 11.8.

THEOREM 12.4 [23, Theorem 7.3]. *Under the above assumptions (in particular, $\mathfrak{o}$ and $\mathfrak{c}$ are stable under ϵ), there is a canonical equivalence of categories*

$$\underline{\mathcal{CS}}^{\mathfrak{c}}_{\mathfrak{o}}(G;\epsilon)\xrightarrow{\sim}\mathcal{Z}(\underline{\mathcal{S}}^{\mathfrak{c}}_{\mathfrak{o}};\epsilon).$$

12.5. More notations We need some more notation to state the next theorem. Fix a two-sided cell $\mathbf{c}$ of $W^{\circ}_{\mathcal{L}}$ contained in $\mathfrak{c}\cap W^{\circ}_{\mathcal{L}}$. Then $\mathfrak{c}\cap W^{\circ}_{\mathcal{L}}$ is the union of two-sided cells that are in the same $\Omega_{\mathcal{L}}$-orbit of $\mathbf{c}$.

For $\beta\in{}_{\mathcal{L}}\underline{W}_{\epsilon\mathcal{L}}$, $w^{\beta}\circ\epsilon\mathcal{L}=\mathcal{L}$; hence $w^{\beta}\circ\epsilon$ acts on $W_{\mathcal{L}}$, $W^{\circ}_{\mathcal{L}}$ and on $\Omega_{\mathcal{L}}$, and permutes the cells in $W^{\circ}_{\mathcal{L}}$ that belong to $\mathfrak{c}$. Let

$$\mathfrak{B}_{\mathbf{c}}=\{\beta\in{}_{\mathcal{L}}\underline{W}_{\epsilon\mathcal{L}}|w^{\beta}\circ\epsilon\text{ preserves the cell }\mathbf{c}\text{ of }W^{\circ}_{\mathcal{L}}\}.$$

The left translation action of $\Omega_{\mathcal{L}}$ on ${}_{\mathcal{L}}\underline{W}_{\epsilon\mathcal{L}}$ (using the multiplication of blocks defined in Section 4.1) restricts to an action of $\Omega_{\mathbf{c}}=\mathrm{Stab}_{\Omega_{\mathcal{L}}}(\mathbf{c})$ on $\mathfrak{B}_{\mathbf{c}}$, making it a $\Omega_{\mathbf{c}}$-torsor. Similarly, the right translation action of $\gamma\in\Omega_{\mathcal{L}}$ on ${}_{\mathcal{L}}\underline{W}_{\epsilon\mathcal{L}}$ by $\beta\mapsto\beta\epsilon(\gamma)$ makes $\mathfrak{B}_{\mathbf{c}}$ a right $\Omega_{\mathbf{c}}$-torsor. Combining the two actions, we get a twisted conjugation action of $\Omega_{\mathcal{L}}$ on ${}_{\mathcal{L}}\underline{W}_{\epsilon\mathcal{L}}$:

$$\mathrm{Ad}_{\epsilon}(\gamma)(\beta)=\gamma\beta\epsilon(\gamma)^{-1},\quad\gamma\in\Omega_{\mathcal{L}},\beta\in{}_{\mathcal{L}}\underline{W}_{\epsilon\mathcal{L}}.$$

It restricts to an action of $\Omega_{\mathbf{c}}$ on $\mathfrak{B}_{\mathbf{c}}$, which we still denote by Ad_{ϵ}. For $\beta\in{}_{\mathcal{L}}\underline{W}_{\epsilon\mathcal{L}}$, let $\Omega_{\beta}\subset\Omega_{\mathcal{L}}$ be its stabilizer under $\Omega_{\mathcal{L}}$; let $\Omega_{\mathbf{c},\beta}=\Omega_{\mathbf{c}}\cap\Omega_{\beta}$. Since $\Omega_{\mathcal{L}}$ is Abelian, the groups Ω_{β}, $\Omega_{\mathbf{c}}$ and $\Omega_{\mathbf{c},\beta}$ are independent of the choices of $\mathbf{c}$ and β.

When $\beta\in\mathfrak{B}_{\mathbf{c}}$, we can define the semisimple Abelian category $\underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H;\sigma_{\beta\epsilon})$ consisting of finite direct sums of unipotent character sheaves on $H\cdot\sigma_{\beta\epsilon}$ belonging to the cell $\mathbf{c}$. In Situation (A), we have an equivalence

$$\underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H;\sigma_{\beta\epsilon})\cong\mathrm{Rep}^{\mathbf{c}}_{u}(H^{\sigma_{\beta\epsilon}}),\tag{12.2}$$

the latter being the semisimple Abelian category of unipotent $\overline{\mathbb{Q}}_{\ell}$-representations of the finite group $H^{\sigma_{\beta\epsilon}}$ belonging to the cell $\mathbf{c}$.

For $\beta\in{}_{\mathcal{L}}\underline{W}_{\epsilon\mathcal{L}}$, we introduce a twisted analogue of the cocycle Λ_{β} defined in (11.1). For $\gamma\in\Omega_{\beta}\subset\Omega_{\mathcal{L}}$, define

$$\Lambda_{\beta\epsilon}(\gamma)=\mathrm{Hom}(\xi_{\gamma},\xi_{\beta}\star\epsilon_{*}\xi_{\gamma}\star\xi_{\beta}^{-1}).$$

This is canonically independent of the choices of $\xi_{\gamma}\in{}_{\mathcal{L}}\mathfrak{P}^{\gamma}_{\mathcal{L}}$ and $\xi_{\beta}\in{}_{\mathcal{L}}\mathfrak{P}^{\gamma}_{F_{\epsilon}\mathcal{L}}$, and it defines a normalized 1-cocycle $\Lambda_{\beta\epsilon}$ of lines on Ω_{β}.

Finally, using the group ${}_{\mathcal{L}}\mathfrak{H}_{\mathcal{L}}$ with neutral component H and group of components equal to $\Omega_{\mathcal{L}}$, there is a canonical action of Ω_{β} on $\underline{\mathcal{CS}}_{u}(H;\sigma_{\beta\epsilon})$, defined in the same way as discussed in Section 11.9. This restricts to an action of $\Omega_{\mathbf{c},\beta}$ on $\underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H;\sigma_{\beta\epsilon})$. It therefore makes sense to form the category $\underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H;\sigma_{\beta\epsilon})^{(\Omega_{\mathbf{c},\beta},\Lambda_{\beta\epsilon})}$ of objects in $\underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H;\sigma_{\beta\epsilon})$ equipped with $\Omega_{\mathbf{c},\beta}$-equivariant structures twisted by the cocycle $\Lambda_{\beta\epsilon}$ (restricted to $\Omega_{\mathbf{c},\beta}$), in the sense discussed in Section 11.9.

The following result gives a relationship between character sheaves for a disconnected group with a fixed semisimple parameter and unipotent character sheaves on its endoscopic groups, generalizing Theorem 11.10.

THEOREM 12.6. *Choose a representative for each* $\mathrm{Ad}_{\epsilon}(\Omega_{\mathbf{c}})$*-orbit of* $\mathfrak{B}_{\mathbf{c}}$*, and denote this set of representatives by* $\dot{\mathfrak{B}}_{\mathbf{c}}$*. There is an equivalence of semisimple Abelian categories:*

$$\underline{\mathcal{CS}}^{\mathbf{c}}_{\mathfrak{o}}(G;\epsilon)\cong\bigoplus_{\beta\in\dot{\mathfrak{B}}_{\mathbf{c}}}\underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H;\sigma_{\beta\epsilon})^{(\Omega_{\mathbf{c},\beta},\Lambda_{\beta\epsilon})}.$$

Using (12.1) and (12.2), we get the following corollary.

COROLLARY 12.7. *In Situation (A), under the same notations as Theorem 12.6, there is an equivalence of semisimple Abelian categories:*

$$\mathrm{Rep}^{\mathbf{c}}_{\mathfrak{o}}(G^{\epsilon})\cong\bigoplus_{\beta\in\dot{\mathfrak{B}}_{\mathbf{c}}}\mathrm{Rep}^{\mathbf{c}}_{u}(H^{\sigma_{\beta\epsilon}})^{(\Omega_{\mathbf{c},\beta},\Lambda_{\beta\epsilon})}.$$

EXAMPLE 12.8. Consider $G=\mathrm{SL}_n$, and ϵ is the Frobenius map for the split $\mathbb{F}_q$-structure on G. Let $\mathcal{K}\in\mathrm{Ch}(\mathbb{G}_m)$ be of order n. For a rational number a whose denominator is prime to n, it makes sense to take the tensor power $\mathcal{K}^a$. Let

$$\mathcal{L}=\mathcal{K}\boxtimes\mathcal{K}^2\boxtimes\cdots\boxtimes\mathcal{K}^n\in\mathrm{Ch}(\mathbb{G}_m^n).$$

Restricting $\mathcal{L}$ to the diagonal torus T of G (identified with the subtorus $T\subset\mathbb{G}_m^n$ with product equal to 1), we denote it still by $\mathcal{L}\in\mathrm{Ch}(T)$. The Weyl group S_n acts on $\mathrm{Ch}(\mathbb{G}_m^n)$ by $w(\mathcal{L}_1\boxtimes\cdots\boxtimes\mathcal{L}_n)=\mathcal{L}_{w^{-1}(1)}\boxtimes\cdots\boxtimes\mathcal{L}_{w^{-1}(n)}$, and it restricts to an action on $\mathrm{Ch}(T)$.

Let $\mathfrak{o}$ be the W-orbit of $\mathcal{L}$. In this case, $W^{\circ}_{\mathcal{L}}=\{1\}$ but $W_{\mathcal{L}}=\Omega_{\mathcal{L}}\cong\mathbb{Z}/n\mathbb{Z}$ can be identified with the group generated by the cyclic permutation $c:i\mapsto i+1$ in S_n. We have $H=T$. Since there is only one cell $\mathbf{c}$ for T with any semisimple parameter, we will omit $\mathbf{c}$ from the notation.

We have $\epsilon_*\mathcal{L}=\mathcal{K}^{1/q}\boxtimes\mathcal{K}^{2/q}\cdots\boxtimes\mathcal{K}^{n/q}$. Let $w(i)=i/q\mod n$, viewed as an element in S_n; then $\mathfrak{B}={}_{\mathcal{L}}W_{\epsilon\mathcal{L}}=\{c^iw|i\in\mathbb{Z}/n\mathbb{Z}\}$. For $\beta\in\mathfrak{B}$, we have $\mathrm{Ad}_{\epsilon}(c)(\beta)=c\beta\epsilon(c)^{-1}=c\beta c^{-1}$. Direct calculation shows that $\mathrm{Ad}_{\epsilon}(c)$ sends $c^iw\in\mathfrak{B}$ to $c^{i+(q-1)/q}w\in\mathfrak{B}$. Let $d=\gcd(n,q-1)$; then the Ad_{ϵ}-action of

$\Omega_{\mathcal{L}} = \mathbb{Z}/n\mathbb{Z}$ on $\mathfrak{B}$ has d orbits, and the stabilizers are isomorphic to $\mathbb{Z}/d\mathbb{Z}$. By Corollary 12.7, $\mathrm{SL}_n(\mathbb{F}_q)$ has d^2 irreducible representations with semisimple parameter $\mathfrak{o}$ (the twistings $\Lambda_{\beta\epsilon}$ can be trivialized since $\Omega_{\mathcal{L}}$ is cyclic).

EXAMPLE 12.9. Consider the case $G = \mathrm{SL}_n$, but ϵ is the Frobenius map corresponding to the special unitary group SU_n over $\mathbb{F}_q$. Its action on the diagonal torus is given by $(x_1, x_2, \ldots, x_n) \mapsto (x_n^{-q}, x_{n-1}^{-q}, \ldots, x_1^{-q})$. We consider the same $\mathcal{L} \in \mathrm{Ch}(T)$ as in Example 12.8. This time, $\epsilon\mathcal{L} = \mathcal{K}^{-n/q} \boxtimes \cdots \boxtimes \mathcal{K}^{-1/q}$. Let $w(i) = (i - n - 1)/q \mod n$, viewed as an element in S_n; then $\mathfrak{B} = {}_{\mathcal{L}}W_{\epsilon\mathcal{L}} = \{c^i w | i \in \mathbb{Z}/n\mathbb{Z}\}$. For $\beta \in \mathfrak{B}$, we have $\mathrm{Ad}_\epsilon(c)(\beta) = c\beta\epsilon(c)^{-1} = c\beta c$ because $\epsilon(c) = c^{-1}$. Then $\mathrm{Ad}_\epsilon(c)(c^i w) = c^{i+(q+1)/q} w$. Let $d' = \gcd(q + 1, n)$. Then as in the discussion in Example 12.8, $\mathrm{SU}_n(\mathbb{F}_q)$ has d'^2 irreducible representations with semisimple parameter $\mathfrak{o}$.

12.10. Sketch of proof of Theorem 12.6. Applying Theorem 12.4 to $\underline{\mathcal{CS}}^{\mathbf{c}}_{\mathfrak{o}}(G; \epsilon)$ and to $\underline{\mathcal{CS}}^{\mathbf{c}}_{u}(H; \sigma_{\beta\epsilon})$ separately, we reduce to showing that

$$\mathcal{Z}(\underline{\mathcal{S}}^{\mathbf{c}}_{\mathfrak{o}}; \epsilon) \cong \bigoplus_{\beta \in \dot{\mathfrak{B}}_{\mathbf{c}}} \mathcal{Z}(\underline{\mathcal{S}}^{\mathbf{c}}_{H}; \sigma_{\beta\epsilon})^{(\Omega_{\mathbf{c},\beta}, \Lambda_{\beta\epsilon})}.$$

By the equivalence in Theorem 9.2, we may replace $\underline{\mathcal{S}}^{\mathbf{c}}_{H}$ by ${}_{\mathcal{L}}\underline{\mathcal{S}}^{\circ,\mathbf{c}}_{\mathcal{L}}$ on the right side and reduce to showing that

$$\mathcal{Z}(\underline{\mathcal{S}}^{\mathbf{c}}_{\mathfrak{o}}; \epsilon) \cong \bigoplus_{\beta \in \dot{\mathfrak{B}}_{\mathcal{L},\mathbf{c}}} \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}^{\circ,\mathbf{c}}_{\mathcal{L}}; \beta \circ \epsilon_*)^{(\Omega_{\mathbf{c},\beta}, \Lambda_{\beta\epsilon})}. \tag{12.3}$$

Here the twisting $\beta \circ \epsilon_*$ that appears on the right side refers to the autoequivalence $\mathcal{F} \mapsto \xi_\beta \star \epsilon_*\mathcal{F} \star \xi_\beta^{-1}$ of ${}_{\mathcal{L}}\underline{\mathcal{S}}^{\circ,\mathbf{c}}_{\mathcal{L}}$.

The argument for (12.3) is similar to that of Theorem 11.10, so we only sketch the main steps. We have a twisted analogue of (11.3): restriction gives an equivalence

$$\mathcal{Z}(\underline{\mathcal{S}}^{\mathbf{c}}_{\mathfrak{o}}; \epsilon) \xrightarrow{\sim} \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}^{\mathbf{c}}_{\epsilon\mathcal{L}}; \epsilon). \tag{12.4}$$

Here the right side contains objects $\mathcal{F} \in {}_{\mathcal{L}}\underline{\mathcal{S}}^{\mathbf{c}}_{\epsilon\mathcal{L}}$ together with functorial isomorphisms $\mathcal{F} \circ \epsilon_*\mathcal{G} \xrightarrow{\sim} \mathcal{G} \circ \mathcal{F}$ for $\mathcal{G} \in {}_{\mathcal{L}}\underline{\mathcal{S}}^{\mathbf{c}}_{\mathcal{L}}$ ($\circ$ denotes the truncated convolution). One can then write $\mathcal{F} \in \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}^{\mathbf{c}}_{\epsilon\mathcal{L}}; \epsilon)$ as a sum

$$\mathcal{F} = \bigoplus_{\mathbf{c}' \sim \mathbf{c}, \beta \in {}_{\mathcal{L}}\underline{W}_{\epsilon\mathcal{L}}} \mathcal{F}^{\mathbf{c}'}_{\beta} \star \xi_\beta$$

for $\mathcal{F}^{\mathbf{c}'}_{\beta} \in {}_{\mathcal{L}}\underline{\mathcal{S}}^{\circ,\mathbf{c}'}_{\mathcal{L}}$ (where $\mathbf{c}'$ runs over the $\Omega_{\mathcal{L}}$-orbit of $\mathbf{c}$). The ϵ-commutation with $\mathcal{G}$ in various cell subcategories of ${}_{\mathcal{L}}\underline{\mathcal{S}}^{\circ}_{\mathcal{L}}$ implies that if $\mathcal{F}^{\mathbf{c}}_{\beta} \neq 0$, then $w^\beta \circ \epsilon$

preserves $\mathbf{c}$, that is, $\beta \in \mathfrak{B}_{\mathbf{c}}$. The ϵ-commutation with $\mathcal{G} \in {}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ,\mathbf{c}}$ shows that $\mathcal{F}_{\beta}^{\mathbf{c}}$ carries a $\beta \circ \epsilon_*$-twisted central structure. The ϵ-commutation with $\mathcal{G} = \xi_{\gamma}$ for $\gamma \in \Omega_{\mathcal{L}}$ then shows that $\mathcal{F}_{\beta}^{\mathbf{c}}$ determines $\mathcal{F}_{\gamma \cdot_{\epsilon} \beta}^{\gamma \cdot \mathbf{c}}$, and that $\mathcal{F}_{\beta}^{\mathbf{c}}$ (for $\beta \in \mathfrak{B}_{\mathbf{c}}$) is equipped with an $(\Omega_{\mathbf{c},\beta}, \Lambda_{\beta\epsilon})$-equivariant structure. Sending $\mathcal{F} \in \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}_{\epsilon\mathcal{L}}^{\mathbf{c}}; \epsilon)$ to $\{\mathcal{F}_{\beta}^{\mathbf{c}}\}_{\beta \in \dot{\mathfrak{B}}_{\mathbf{c}}}$ then induces an equivalence

$$\mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}_{\epsilon\mathcal{L}}^{\mathbf{c}}; \epsilon) \cong \bigoplus_{\beta \in \dot{\mathfrak{B}}_{\mathbf{c}}} \mathcal{Z}({}_{\mathcal{L}}\underline{\mathcal{S}}_{\mathcal{L}}^{\circ,\mathbf{c}}; \beta \circ \epsilon_*)^{(\Omega_{\mathbf{c},\beta}, \Lambda_{\beta\epsilon})}.$$

Combining this with (12.4), we get (12.3), proving the theorem. □

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