Lecture 9 10/5

- Characters
- Krull-Schmidt Theorem. (Preparation)

Characters: k-algebra R.

$R \circlearrowright M$, $\dim_k M < \infty$.

$$\begin{aligned} \chi_M : R &\longrightarrow k \\ a &\longmapsto \underline{Tr(a|_M)}. \end{aligned}$$

- M has a composition series.

$$S_1 \Big) \; S_2 \Big) \; \cdots \; S_n \Big)$$

S_i are simple comp. factors.

$$Tr(a|_M) = \sum_i Tr(a|_{S_i})$$

$$\chi_M = \sum_i \chi_{S_i}$$

χ_M depends only on the <u>semisimplification</u> of M

$$M^{ss} = \bigoplus_i S_i$$

- $a \in J(R) \Rightarrow a$ acts by 0 on each S_i

$$\Rightarrow \chi_M(a) = 0 \quad \forall \text{ f.d } M.$$

$$\chi_M : R/J(R) \longrightarrow k.$$

- $\mathrm{Tr}\left([a,b]|_M\right) = 0 \quad \forall\ a, b \in R.$

$[R, R] = k$-span of $[a,b] = ab - ba$
$\forall\ a, b \in R.$

(not an ideal).

e.g. $R = M_n(k)$, $\quad [R,R] = \{ A \in R \mid \mathrm{Tr}(A) = 0 \}.$

$$\Rightarrow \chi_M : R/J(R) + [R,R] \longrightarrow k.$$

. If R is $\boxed{\text{f.d.}/k.}$ ($\Rightarrow R$ is left & right artinian)

$$\overline{R} = R/J(R) \cong \prod_i M_{n_i}(D_i)$$

$$[D_i : k] < \infty.$$

$$R/J(R) + [R,R] = \overline{R}/[\overline{R}, \overline{R}].$$

$$M_{n_i}(D_i)/[M_{n_i}(D_i), M_{n_i}(D_i)] \xrightarrow[\text{"}\sum a_{ii}\text{"}]{} D_i/[D_i, D_i]$$

easy case: $k = \bar{k}$. all $D_i = \bar{k}$.

$$\bar{R} = \prod_{i\in I} M_{n_i}(k) \xrightarrow{Tr}$$

$$\bar{R}/[\bar{R},\bar{R}] = \prod_{i\in I} k$$

Characters of R factor through $\prod_{i\in I} k$.

each simple $S_i \simeq k^{n_i}$ $(i \in I)$

$$\chi_{S_i} : \bar{R}/[\bar{R},\bar{R}] = \prod_{i\in I} k \xrightarrow[p_i]{} k$$

$\{\chi_S\}_{S \in \text{simple } R\text{-mod}/\cong}$ forms a k-basis for $\left(\bar{R}/[\bar{R},\bar{R}]\right)^*$

<u>Thm</u> (linear independence of characters) $k = \bar{k}$. (or $\mathrm{char}(k)=0$).
$R = k$-algebra.

$\{\chi_S\}_{S \in \text{simple, f.d. } R\text{-mod}/\cong}$ are k-linearly independent functions on R (or on $\bar{R}/[\bar{R},\bar{R}]$)

<u>Pf</u>: Suppose $\sum_{i=1}^{n} c_i \underline{\underline{\chi_{S_i}}} = 0$.

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, two heads] & \prod_{i=1}^{n} \mathrm{End}_k(S_i) \\ R_1 \arrow[ur, hook] & \end{tikzcd}$$

$R_1 = \text{image}(\alpha)$, f.d. $/k$.

S_i are simple modules for R_1.

(see discussion in the case of ~~f.d. k-alg.~~)

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without $k = \bar{k}$. $ch(k) = p$.

$R = D \circlearrowright D$. $[D:k] = p^2$.

$\underline{\chi_D : D \longrightarrow k}$ is 0.

(reduced trace $\neq 0$)

$\chi_D = p \cdot (\text{reduced trace of } D)$.

$$D \underset{k}{\otimes} \bar{k} \simeq M_p(\bar{k}) \xrightarrow[\substack{\text{red trace} \\ \parallel \\ \text{usual trace}}]{} \bar{k}$$

$$M_p(\bar{k}) \circlearrowright M_p(\bar{k}) = k^p \oplus \cdots \oplus k^p.$$

Thm. $\boxed{\text{Char}(k) = 0.}$

R: k-alg. $\leftarrow$

$R \circlearrowright M$, $\dim_k M < \infty$.

$\Longrightarrow$ χ_M determines the composition factors of M.

($\Longleftrightarrow$ χ_M determines M^{ss}).

$\text{char}(k) = p$. $\chi_{S^{\oplus p}} = 0$.

Pf: $M^{ss} = \bigoplus_i S_i^{\oplus n_i}$; $(M')^{ss} = \bigoplus S_i^{\oplus n_i'}$

$\chi_M = \sum_i n_i \chi_{S_i}$; $\chi_{M'} = \sum_i n_i' \chi_{S_i}$

- $\chi_M = \chi_{M'} \iff n_i = n_i' \ \forall i.$

  by linear indep. of $\{\chi_{S_i}\}$ $\Rightarrow n_i - n_i' = 0 \in k.$

- another proof: may reduce to R f.d./k

  $\rightarrow$ reduced to R ss.

$$R = \prod_i \underbrace{M_{n_i}(D_i)}_{} \qquad S_i \circlearrowleft \underline{D_i}$$

$$= \prod_i \mathrm{End}_{D_i}(S_i)$$

idempotents $e_i \in R$ corresp. to i^{th} factor.

$$(0, \dots, 0, \underset{\uparrow}{1}, \dots, 0) \in R$$

$$M_{n_i}(D_i) = End_{D_i}(S_i).$$

$$e_i \underset{0}{\circlearrowright} S_j \quad j \neq i$$

$$e_i \underset{id}{\circlearrowright} S_i$$

$$\chi_M(e_i) = n_i \, \chi_{S_i}(e_i) = n_i \cdot \dim_k S_i$$

$$\chi_{M'}(e_i) = \quad \dots\dots \quad = n_i' \dim_k S_i$$

$$\chi_M = \chi_{M'} \implies n_i = n_i'.$$

(need to divide by $\underline{\dim_k S_i}$)

Indecomposable modules.

<u>Def.</u> $R \circlearrowright M$ is indecomp

if $M \neq M_1 \oplus M_2$ (as R-mod)

$M_1, M_2 \neq 0$.

Simple modules are indecomp.

e.g. $R = k[x]/x^2$, $\quad R \circlearrowright R$ is indecomp but not simple.

e.g. $\bullet \longrightarrow \bullet$

$$\left.\begin{array}{l} k \longrightarrow 0 \\ 0 \longrightarrow k \end{array}\right\} \text{simple}$$

$$k \xrightarrow{\sim} k$$

e.g. $\bullet \longrightarrow \bullet \longrightarrow \bullet$

$$\begin{array}{ccccc}
k & & 0 & & 0 \\
0 & & k & & 0 \\
0 & & 0 & & k \\
k & = & k & & 0 \\
0 & & k & = & k \\
k & = & k & = & k \\
\cancel{k} & & \cancel{0} & & \cancel{k}
\end{array}$$

e.g.

$$\begin{tikzcd}
 & \bullet \arrow[d] & \\
\bullet \arrow[r] & \bullet & \bullet \arrow[l] \\
 & \bullet \arrow[u] &
\end{tikzcd}$$

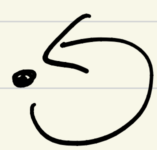
$\dim_k V = 2$

$$\begin{tikzcd} & L_1 \arrow[d, hook] & \\ L_2 \arrow[r, hook] & V & L_4 \arrow[l, hook'] \\ & L_3 \arrow[u, hook] & \end{tikzcd}$$

assume $\{L_i\}$ distinct lines.

$$4 \text{ lines} \xrightarrow[\text{cross ratio}]{} k \setminus \{0,1\}.$$

$\rightsquigarrow$ "continuous" family of indecomposable R_Q-mod.

 $R = k[t]$ indecomposables $\updownarrow$ (λ, n). with $\lambda \in k$, $n \in \mathbb{N}$.

Can we write any R-mod M into $\oplus$ indecomp.
(uniqueness ?).

<u>**Theorem**</u> (Krull-Schmidt) R is <u>left artinian</u>.
M = f.g. R-mod.

Then M can be written as a finite direct sum of indecomp. R-modules, and for each indecomp N. its multiplicity in M is well-defined.

$$M = N_1 \oplus N_2 \oplus \cdots \oplus N_a$$
$$= N_1' \oplus N_2' \oplus \cdots \oplus N_b'.$$
$$\Rightarrow a = b \text{ and } \{N_i\} = \{N_j'\}.$$

<u>Fittings lemma</u>: R any ring

$R \circlearrowright M$ with finite length.

$T : M \longrightarrow M$ (R-linear)

$\Rightarrow \exists!\ M = M_0 \oplus M_1$ (T-stable R-submod.)

s.t. $T|_{M_0}$ is nilpotent

$T|_{M_1}$ is an $\cong$.

Special case: $R = k$; $T \circlearrowright M = V$.

$$V = V_0 \oplus V_1$$

V_0: generalized eigensp. with eval 0.

V_1: sum of other gen. eigensp.

<u>Pf</u> (Lemma) Uniqueness clear

$M_0 = \ker(T^n)$ $n >> 0$.

$M_1 = \mathrm{Im}(T^n)$ $n >> 0$.

$\ker(T) \subset \ker(T^2) \subset \ker(T^3) \subset \cdots$ stabilizes

$\mathrm{Im}(T) \supset \mathrm{Im}(T^2) \supset \mathrm{Im}(T^3) \supset \cdots$ stabilizes.

$$\begin{tikzcd} 0 \arrow[r] & M_0 \arrow[r] & M \arrow[r, "T^n"] & M_1 \arrow[r] \arrow[l, bend left, "i_1 = \text{inclusion}"] & 0 \end{tikzcd}$$

$$T^n|_{M_1} = T^n \circ i_1 : M_1 \longrightarrow M_1 \qquad (n \gg 0).$$

If we show $T^n|_{M_1}$ is an $\cong$.

then $\quad M_0 \oplus i_1(M_1) = M.$

- $T^n|_{M_1}$ is surjective: $\quad x \in M_1 = \operatorname{Im}(T^n) = \operatorname{Im}(T^{2n})$

$$\Rightarrow \quad x = T^{2n} y = T^n(\underbrace{T^n y}_{\in M_1})$$

- $T^n|_{M_1}$ is injective: $\quad x \in M_1.$

$$x = T^n y. \quad \left.\begin{array}{l} T^n(x) = 0. \\ y \in M. \end{array}\right\} \Rightarrow T^{2n} y = 0.$$

$$\ker(T^n) = \ker(T^{2n})$$

$$\Rightarrow \quad T^n y = 0. \Rightarrow \quad x = 0.$$

- $T|_{M_0}$ nilp. since $T^n|_{M_0} = 0.$

$T|_{M_1}$ is $\cong. \quad \Longleftarrow \quad T^n|_{M_1}$ is $\cong.$

<u>Cor</u> R any ring
M : finite length R-mod
and indecomp.

$\Rightarrow$ any endo $T \in \mathrm{End}_R(M)$
is either nilpotent or an $\cong$.

$\Longleftrightarrow$ $\mathrm{End}_R(M)$ is a local ring.

$\mathrm{End}_R(M) / J(\cdots)$ = division ring.

Pf of K-S, next time.