Lecture 5   9/21

- Simple rings
- Artinian condition / Noetherian
- Wedderburn - Artin Thm.

Def A ring R is simple, if the only 2-sided ideal of R is 0.

(convention : ideals can't be R)

Ex.
- $M_n(D)$. $\quad D =$ division ring.
- Weyl algebra / $k =$ field of char 0.

$$A_n(k) = k\langle \underbrace{x_1, x_2, \dots, x_n}_{\text{comm}}, \overbrace{\partial_1, \dots, \partial_n}^{\text{comm}} \rangle \Big/ \left(\begin{matrix} \partial_i x_j - x_j \partial_i \\ = \delta_{ij} \end{matrix} \right)$$

$$= \left\{ \begin{matrix} \text{differential operators} \\ \text{acting on } k[x_1, \dots, x_n] \end{matrix} \right\}$$

$A_n(k)$ is a simple ring.

- k, $\operatorname{char}(k) = p$.
diff. operators on $k[x_1, x_2, \dots, x_n]$. ?

Approach 1: Copy defn of $A_n(k)$.

Approach 2: diff. op. on $k[x_1, \dots, x_n]$
operators generated by
- mult. by $f(x_1, \dots, x_n)$
- partials ∂_i.

<u>Fact</u>: App 2 defines a (smaller) quotient of Appr. 1.

$$\partial_i^p (f) = 0$$

$$\partial_i^p (x_i^n) = \underbrace{n(n-1)\cdots(n-p+1)}_{\text{div. by } p} x_i^{n-p} = 0.$$

but $\partial_i^p \neq 0$ in $A_n(k)$.

$$A_n(k) \circlearrowright k[x_1, x_2, \dots, x_n]$$

not faithful.

Approach 3. Include d.o. with <u>divided powers</u>

$$\text{"} \frac{\partial_i^p}{p!} \text{"}$$ makes sense as an k-linear endo. of $k[x_1, \dots, x_n]$.

$$\frac{\partial_i^p}{p!} (x_i^n) = \binom{n}{p} x_i^{n-p}.$$

$$(\text{app } 1) = A_n(k) \twoheadrightarrow (\text{app } 2) \hookrightarrow (\text{app } 3)$$

not simple. ∂_i^p

Compute the center of $A_n(k)$. $ch(k) = p$.

<u>Ex</u>. $V = k$-v.s. $\dim_k V = \infty$.

$\mathrm{End}_k(V)$ (column-finite $\infty \times \infty$ matrices)

Simple ?

$I = \{ \text{finite rank } T : V \to V \} \subsetneq \mathrm{End}_k(V)$.

k-subspace.

$\mathrm{Im}(T \circ A) \subseteq \mathrm{Im}(T)$

$\Rightarrow T \circ A \in I$.

$\mathrm{Im}(B \circ T) = B(\underbrace{\mathrm{Im}\, T}_{\text{fin.dim.}})$

$\Rightarrow B \circ T \in I$.

2-sided ideal.

$\Rightarrow \mathrm{End}_k(V)$ is not simple.

Artinian / Noetherian.

- Modules. $R \curvearrowright M$.

M is an <u>artinian</u> R-module if

$\forall$ descending chain of submod

$$M_1 \supset M_2 \supset M_3 \supset \cdots$$

stabilizes $(\exists\, n.\ M_n = M_{n+1} = M_{n+2} = \cdots)$

M is <u>noetherian</u> R-mod if

$\forall$ ascending chain of submod

$$M_1 \subset M_2 \subset M_3 \subset \cdots$$

stabilizes.

<u>Ex.</u> Non-noetherian, non-artinian.

$$R^{\oplus I} \qquad |I| = \infty.$$

<u>Lemma.</u> A noetherian R-mod is finitely generated.

Why? $x_1, x_2, \cdots$ generators.

$$Rx_1 \subset Rx_1 + Rx_2 \subset Rx_1 + Rx_2 + Rx_3 \quad \left(\begin{matrix}\text{union of all}\\ = M\end{matrix}\right)$$

stabilizes $\underset{\text{at step } n}{\Longrightarrow}$ $x_1, x_2, \cdots, x_n$ generate M.

Ex. Noetherian but not artinian module.

$R = \mathbb{Z} \circlearrowright M = \mathbb{Z}$.

submodules of M: $\quad n\mathbb{Z}$

$n_1\mathbb{Z} \subset n_2\mathbb{Z} \subset \cdots$

$n_2 \mid n_1, \quad n_3 \mid n_2, \quad \ldots$ stabilizes ✓

$\Rightarrow M$ is noeth

$n_1\mathbb{Z} \supset n_2\mathbb{Z} \supset n_3\mathbb{Z} \supset \cdots$ doesn't stab.

$n_1 \mid n_2 \mid n_3 \mid \cdots \Rightarrow$ not artinian.

Ex. Artinian but not noetherian module

$R = \mathbb{Z} \circlearrowright M = \frac{1}{2^\infty}\mathbb{Z}/\mathbb{Z}$

submodules. $\begin{cases} \frac{1}{2^n}\mathbb{Z}/\mathbb{Z} & n \in \mathbb{N}. \\ M. \end{cases}$

Not Noetherian:

$$\frac{1}{2}\mathbb{Z}/\mathbb{Z} \subset \frac{1}{2^2}\mathbb{Z}/\mathbb{Z} \subset \frac{1}{2^3}\mathbb{Z}/\mathbb{Z} \subset \cdots.$$

Any descending chain.

$$M \underset{\neq}{\supset} \underbrace{\frac{1}{2^n}\mathbb{Z}/\mathbb{Z}}_{\text{finite}} \supset \cdots \text{ has to stabilize.}$$

$\Rightarrow M$ is Artinian

$$\mathbb{Q}/\mathbb{Z} = \bigoplus_p \frac{1}{p^\infty}\mathbb{Z}/\mathbb{Z} \quad \text{not artinian.}$$

Rings

Def. R is <u>left noetherian</u>, if ${}_R R$ is a noeth R-mod.
<u>left artinian</u> —— " —— artinian ——

(right noeth
right art.)

R is left noeth $\iff$ $\forall$ ascending chain of left ideals stabilizes.

Left noetherian / artinian $\not\Longrightarrow$ right noeth / artin.

Fact. Left artinian rings are left noetherian.

Ex. $k[x_1, x_2, \dots, x_n]$ noetherian.
(Hilbert basis theorem).

$k\langle x_1, x_2, \dots, x_n\rangle$? not noetherian.

$R = k\langle x, y\rangle$.

$$Ry \subset R(y, yx) \subset \underbrace{R(y, yx, yx^2)}_{\sum (\ )yx^{\leq 2}} \subset \cdots$$

$$\sum (\ )yx^{\leq 2} \subsetneq \sum (-)yx^{\leq 3}$$

$$\cancel{\ni} \quad yx^3 \in \sum (-)yx^{\leq 3}$$

Ex. $\overset{D}{k}$ = field. R : finite-dim'l $\overset{D}{k}$-alg
$\Rightarrow$ R is left and right artinian.

R_0 artinian. $R \supseteq R_0$, finitely gen'd as left R_0-mod
$\Rightarrow$ R is artinian R_0-mod.
$\Rightarrow$ R is artinian R-mod.
$\Rightarrow$ R is left artinian.

Lemma 2 R is left noeth, $R \curvearrowright M$ is f.g. R-mod.
$\Rightarrow$ M is a noeth R-mod.
same for artinian.

Lemma 1 $0 \to M' \to M \to M'' \to 0$.
(R-mods).
Then M is noeth $\iff$ M' & M'' are noeth.
M is artin $\iff$ M' & M'' are artin.

Lemma 1 $\Rightarrow$ Lemma 2
R=noeth, M is f.g. $\Rightarrow$ $R^n \twoheadrightarrow M$.
noetherian $\Rightarrow$ M is noeth.
Same argument for artinian.

Fact: R = left artinian, M = f. g. R-mod.

$\Rightarrow$ M has finite length (M has a composition series).

Pf will use R = noetherian.

Lemma 2 $\Rightarrow$ M is both noeth. and artin module.

M_1 = a minimal nonzero submod.

(existence $\Leftarrow$ artinian.)

($N_1 \supsetneq N_2 \supsetneq N_3 \supsetneq \cdots$ (nonzero). stabilize $\Rightarrow$ get minimal submod)

Then M_1 is simple.

$\overline{M} = M/M_1$ artin. and noeth mod.

$\Rightarrow$ $\overline{M_2}$ = minimal submod of $\overline{M}$.

M_2 = preimage of $\overline{M_2}$ in M

$$\underbrace{M_1 \subsetneq M_2}_{\text{simple } \overline{M_2}} \subsetneq \cdots$$

keep doing this has to stop since M is noetherian

Thm (Wedderburn - Artin)

TFAE for ring R:

invt under left $\Leftrightarrow$ right

① R is a simple (left) artinian ring;

② $\boxed{R \simeq M_n(D)}$. (D = division ring)

③ R is semisimple with only one isom class of simple modules.

④ R, as left R-mod, is semisimple and isotypical ($S^{\oplus m}$).

⑤ R is (left) artinian, and has a faithful simple module. (primitive).

(same for right version, and left $\Longleftrightarrow$ right).

Pf. $$\underbrace{① \Rightarrow ⑤ \Rightarrow ④ \Rightarrow ③ \Rightarrow ② \Rightarrow ①}_{\text{"easy"}}$$

use Wedderburn's Thm.

① $\Longrightarrow$ ⑤

artin simple

artin faithful.

$R \circlearrowright M =$ any simple module

$R \longrightarrow \mathrm{End}_{\mathbb{Z}}(M)$ ring homo.

R simple $\Rightarrow$ this map is injective.

$\Rightarrow M$ is a faithful R-mod.

⑤ $\Longrightarrow$ ④

R: artinian

$\exists$ faithful M.

${}_R R = M^{\oplus m}$.

<u>idea</u>: produce an injection

$$R \hookrightarrow M^{\oplus n}.$$

Such a map comes from acting on $x_1, \dots, x_n \in M$

$$R \longrightarrow M^{\oplus n}$$

$$r \longmapsto (rx_1, rx_2, \dots, rx_n)$$

$R \circlearrowright M$ faithful.

$x \in M$, $\mathrm{Ann}(x) = \{ r \in R \mid rx = 0 \}$.

left ideal $\subset R$.

$$\bigcap_{x \in M} \mathrm{Ann}(x) = \{0\}.$$

R artinian $\Longrightarrow \exists\ x_1, x_2, \dots, x_n \in M$. s.t.

$$\bigcap_{i=1}^{n} \mathrm{Ann}(x_i) = \{0\}.$$

$$\Longrightarrow \quad \begin{array}{ccc} R & \longrightarrow & M^{\oplus n} \\ r & \longmapsto & (rx_1, \dots, rx_n). \end{array} \quad \text{is injective.}$$