Lecture 4 9/16.

- Semisimple rings
- Wedderburn's theorem
- Density Theorem
- Primitive rings and ideals.

<u>Def</u>. R is (left) semisimple if every left R-mod is semisimple.

<u>Criterion</u>. to check R is left-ss it's enough to check R itself, as left R-mod is ss.

(If ${}_R R$ is ss.

M : any R-mod.

$${}_R R^{\oplus I} \twoheadrightarrow M$$

(by choosing a set of generators for M).

${}_R R^{\oplus I}$ is ss., M as a quot of ss mod is also ss.)

Ex $R = M_n(D)$ is ss.
$\llcorner$ division ring.

$$M_n(D) \circlearrowright \underbrace{M_n(D)}_{\substack{\| \text{ column decomp.} \\ \underbrace{D^n \oplus D^n \oplus \cdots \oplus D^n}_{n}}}$$

Claim D^n is a simple $M_n(D)$-mod.

Pf: $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \neq 0$ left mult E_{ii}

$\boxed{\text{Let } M = M_n(D)\text{-sub gen'd by } x}$

$$E_{ii} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ x_i \\ \vdots \\ 0 \end{bmatrix} \in M$$

Say $\underset{D.}{x_i \in} \neq 0$.

left mult by D

$$\Longrightarrow \begin{array}{c} \\ i^{th} \rightarrow \\ \\ \end{array} \begin{bmatrix} 0 \\ \vdots \\ d \\ \vdots \\ 0 \end{bmatrix} \in M \quad \forall d \in D.$$

$$E_{ji} \begin{bmatrix} 0 \\ \vdots \\ d \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ d \\ \vdots \\ 0 \end{bmatrix} \begin{array}{c} \\ \\ \leftarrow j^{th} \\ \\ \\ \end{array} \in M.$$

add up $D^n = M$. ∎

R_1, R_2 are ss rings

$\Rightarrow$ $R_1 \times R_2$ is ss.

(any module for $R_1 \times R_2$ is of the form $M_1 \overset{\oplus}{\times} M_2$)

$R_i \circlearrowright M_i$

<u>Theorem</u> (Wedderburn) Any left semisimple ring R is isom to $\prod_{i=1}^{r} Mat_{n_i}(D_i)$

where D_i are division rings.

<u>Cor</u> Left ss $\Longleftrightarrow$ Right ss.

(R left ss. $\Leftrightarrow$ R^{op} right ss

$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$

$R \simeq \prod \underline{M_{n_i}(D_i)} \Leftrightarrow R^{op} \simeq \prod \underline{M_{n_i}(D_i^{op})}$.

$\underline{M_{n_i}(D_i)} \xrightarrow{(-)^{op}} \underline{M_{n_i}(D_i^{op})}$

i.e, right version of Wedderburn Thm holds.)

$$M_n(R)^{op} \simeq M_n(R^{op})$$

$$A \longmapsto A^t$$

$$AB \longmapsto (AB)^t \neq B^t \cdot A^t$$

$(AB)^t = B^t \circ A^t$

using mult in R^{op}

Proof (Thm)

R left ss ring

$${}_R R = \bigoplus_{i \in I} S_i \quad (S_i \text{ simple } R\text{-mod})$$

Claim: This is a finite sum. $(|I| < \infty)$.

$$1 \in R \longrightarrow \sum_{i \in I_0} x_i$$

$I_0 \subset I$ finite subset.

Since 1 generates R as left R-mod.

$\Rightarrow$ $\sum_{i \in I_0} x_i$ generates $\bigoplus_{i \in I} S_i$.

but it at most generates $\bigoplus_{i \in I_0} S_i$

$\Rightarrow I_0 = I$.

$${}_R R = \bigoplus V_i \otimes_{D_i} S_i$$

S_i non-isom

V_i is finite-dim'l right D_i-v.s. $D_i = \mathrm{End}_R(S_i)$

$$= \bigoplus S_i^{\oplus n_i}$$

<u>Fact:</u> $\operatorname{End}_R({}_{\underline{R}}R) \underset{\text{ring}}{\cong} R^{op}$

$(\varphi_a : r \mapsto ra) \longleftarrow\!\shortmid \; a$

$$R \circlearrowright \underline{\underline{R}} \circlearrowleft R$$

$$\operatorname{End}_R({}_R R) = \left\{ \varphi : R \to R \;\middle|\; \varphi \text{ comm w/ left mult} \right\}$$

$$= \{\text{right mult of } R \text{ on } R\}.$$

$${}_R R \simeq \bigoplus_{i=1}^{r} S_i^{\oplus n_i} \quad (\text{left } R\text{-mod})$$

$$\begin{array}{ccc} \operatorname{End}_R({}_R R) & \simeq & \operatorname{End}_R\left(\bigoplus_{i=1}^{r} S_i^{\oplus n_i}\right) \\ \wr\| & & \| \\ R^{op} & & \prod_{i=1}^{r} \operatorname{End}_R(S_i^{\oplus n_i}) \\ & & \| \\ & & \prod_{i=1}^{r} M_{n_i}(\underbrace{\operatorname{End}_R(S_i)}_{D_i}) \end{array}$$

$$R^{op} \cong \prod_{i=1}^{r} M_{n_i}(D_i).$$

$$\Rightarrow \quad R \cong \prod_{i=1}^{r} M_{n_i}(D_i^{op}) \qquad \blacksquare$$

Canonicity of the decomposition.

$$\begin{array}{c} \text{matrix ring factors in } R \\ \Big\updownarrow \ 1:1 \\ \text{isom classes of simple modules that appear in } {}_R R. \\ \Big\updownarrow \\ \text{isom classes of simple module.} \end{array}$$

($\forall$ simple S, ${}_R R \twoheadrightarrow S$).

Conversely, any decomp

$$R \cong \prod_{i=1}^{s} M_{m_i}(D_i')$$

D_i' div. ring.

Then simple modules of $R/\cong$ are: (column modules)

$$S_1 = (D_1')^{\oplus m_1}, \ \ldots, \ S_s = (D_s')^{\oplus m_s}$$

$\Rightarrow$ $s = \#$ of simple modules$/\cong$.

$$\mathrm{End}_R(S_i) = (D_i')^{op} \qquad d : \begin{bmatrix} x_1 \\ \vdots \\ x_{m_i} \end{bmatrix} \mapsto \begin{bmatrix} x_1 d \\ \vdots \\ x_{m_i} d \end{bmatrix}.$$

i.e., $D_i' = \mathrm{End}_R(S_i)^{op}$.

$$m_i = \dim_{(D_i')^{op}}(S_i)$$

$\Rightarrow$ s, m_i, D_i' are invariants of R.

$R \circlearrowright M$ = simple module

$$\boxed{R \longrightarrow \mathrm{End}_D(M)} \qquad D = \mathrm{End}_R(M). \text{ (div. ring)}$$

$$R \times D \circlearrowright M.$$

If R is ss. $\quad R \simeq \prod M_{n_i}(D_i)$

$$M \simeq D_i^{\oplus n_i} \qquad \mathrm{End}_R(M) = D_i^{op}$$

$$\begin{tikzcd} R \arrow[r, "\text{surj.}"] \arrow[dr] & \mathrm{End}_{D_i^{op}}(M) \\ & M_{n_i}(D_i) \arrow[u, "\wr"'] \end{tikzcd}$$

<u>Thm</u> (Density) $\quad R \circlearrowright M$ = simple module.

$D = \mathrm{End}_R(M)$.

1) Suppose $\dim_D M < \infty$.

then $R \longrightarrow \mathrm{End}_D(M)$ is surjective.

$$\mathrm{End}_D(M) \simeq M_n(D^{op}). \qquad n = \dim_D M.$$

2) In general, for any $x_1, \dots, x_n \in M$ linearly indep. over D.
any $y_1, \dots, y_n \in M$.

$\exists \; r \in R$ s.t. $r x_i = y_i$, $i = 1, 2, \dots, n$.
(density).

<u>Pf</u>. 2) $\Rightarrow$ 1). Take $x_1, \dots, x_n$ to be a D-basis for M.

2): $\varphi : {}_R R \longrightarrow M^{\oplus n}$ (R-mod map)
$r \longmapsto (r x_1, r x_2, \dots, r x_n)$.

want φ to be surjective.
If not surj.
$\mathrm{coker}(\varphi)$ is a quot of $M^{\oplus n}$
$\Rightarrow \mathrm{coker}(\varphi) \cong M^{\oplus m}$ $(m \le n)$

can choose a surjection $\mathrm{coker}(\varphi) \overset{\pi}{\twoheadrightarrow} M$.

$$\begin{tikzcd} {}_R R \arrow[r, "\varphi"'] \arrow[rrr, bend right, "0"'] & M^{\oplus n} \arrow[r, two heads] & \mathrm{coker}(\varphi) \arrow[r, two heads, "\pi"] & M \end{tikzcd}$$

$${}_R R \xrightarrow[\varphi]{} M^{\oplus n} \xrightarrow{\neq 0} M$$
$(y_1, \dots, y_n) \longmapsto d_1 y_1 + d_2 y_2 + \dots + d_n y_n$
where $d_i \in D$.

$r \longmapsto (r x_1, \dots, r x_n) \longmapsto d_1 r x_1 + \dots + d_n r x_n$
$= 0$.
$\forall r$.

$1 \longmapsto (x_1, \dots, x_n) \longmapsto d_1 x_1 + \dots + d_n x_n$
$= 0$

Contradicting x_i are D_i-linearly indep.

∎

Def A ring R is <u>primitive</u>, if R has a faithful simple module.

(i.e., $\exists$ simple R-mod M. s.t.
$R \longrightarrow \mathrm{End}(M)$ is injective.)

Cor A ring R is primitive
$\iff$ R is a <u>dense</u> subring of $\mathrm{End}_D(V)$
for some division ring D and V a D-v.s.

Pf $\Rightarrow$ $V = M$ simple <u>faithful</u> module.
$R \subset \mathrm{End}_D(V)$. $D = \mathrm{End}_R(V)$.
Previous thm $\Rightarrow$ density of R.

$\Leftarrow$ $R \circlearrowright V$ (commuting with D)
faithful ✓
Simple? : if $0 \neq V' \subsetneq V$ is R-submod.
$0 \neq x \in V'$, $y \in V \setminus V'$.
density $\Rightarrow$ $\exists$ $r \in R$ s.t. $\underline{r \cdot x = y}$.
contradiction!
$\Rightarrow$ V is simple.

∎

Def <u>Primitive ideals</u>: are annihilators of simple R-mod.

$R \circlearrowright M =$ simple. $\mathrm{Ann}(M) = \ker(R \to \mathrm{End}(M))$
2-sided ideal.

P : primitive ideal in R

$\overline{R} = R/P$ is a primitive ring (if $P = \mathrm{Ann}(M)$. $\Rightarrow M$ is a faithful simple mod of R/P)

$$\{\text{Simple (left) } R\text{-mod}\}/\cong \twoheadrightarrow \{\text{Primitive ideals in } R\}$$

<u>Lemma</u>. Any simple R-mod is of the form R/I where I = max. left ideal in R ($\neq R$).

<u>Pf</u>. • If I is max left ideal.

if R/I not simple. $\exists$ submod $0 \subsetneq J/I \subsetneq R/I$

$$I \subsetneq J \subsetneq R.$$

• M simple mod.

take $x \in M$. nonzero.

$$R \twoheadrightarrow M = R/I$$
$$r \mapsto rx$$

$I = \mathrm{Ann}(x)$.

$= \{a \in R \mid ax = 0\}$.

left ideal.

I is maximal : if not $I \subsetneq J \subsetneq R$

then $0 \subsetneq J/I \subsetneq R/I$ = simple.

Warning: different left max ideals may give rise to isomorphic simple modules.

e.g. $M_n(D) \supset I_i = \left\{ \begin{array}{l} \text{matrices with the} \\ i^{th} \text{ column totally } 0 \end{array} \right\}$. $\quad i = 1, 2, \cdots, n.$

$M_n(D)/I_i \simeq D^n$ (column $\overset{\text{simple}}{\text{module}}$)

---

any isom $R/I \xrightarrow[\sim]{\neq 0} R/J$ (I, J max left)

$1 \longmapsto r$

$0 = I \longrightarrow \boxed{I \cdot r = J}$.

$\{\text{max left ideals in } R\} / \text{right translation} \overset{1:1}{\longleftrightarrow} \{\text{simple } \overset{\text{left}}{\text{modules}}\} / \cong$