Lecture 3

9/14

- Finish J-H.
- Semisimple modules

Thm (Jordan-Hölder) $R \circlearrowright M$

Comp. series:

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_a = M$$

$$0 \subset M_1' \subset M_2' \subset \cdots \subset M_b' = M$$

Then

- $a = b$ (weak)
- $\exists$ canonical bijection

$$\sigma : \{1, 2, \ldots, a\} \xrightarrow{\sim} \{1, 2, \ldots, b\}$$

and Canon. isom (of R-mod)

$$M_i / M_{i-1} \xrightarrow{\sim} M'_{\sigma(i)} / M'_{\sigma(i)-1}.$$

Pf.

$$S_{i,j} = \frac{M_i \cap M_j'}{(M_i \cap M'_{j-1}) + (M_{i-1} \cap M_j')}$$

$$1 \le i \le a, \qquad 1 \le j \le b.$$

<u>Claim 1</u>: Fix i, $\exists!\ \underset{\sigma(i)}{\underset{\|}{j}}$ s.t. $S_{i,j} \neq 0$.

$$\begin{array}{llll}
M_1 \cap M_1' & M_1 \cap M_2' & M_1 \cap M_3' & \cdots \\
M_2 \cap M_1' & M_2 \cap M_2' & & \\
M_3 \cap M_1' & & &
\end{array}$$

double filtration on M.

1^{st} row: gives a filtration of $M_1 =$ simple.

$\exists!\ j = \sigma(1)$ s.t.
$$M_1 \cap M'_{\sigma(1)} = M_1$$
$$M_1 \cap M'_{\sigma(1)-1} = 0.$$

$$S_{1,\sigma(1)} = \frac{M_1 \cap M'_{\sigma(1)}}{\underset{0}{\underset{\|}{(M_0 \cap M'_{\sigma(1)})}} + \underset{0}{\underset{\|}{(M_1 \cap M'_{\sigma(1)-1})}}}$$

$$\simeq M_1.$$

$j \neq \sigma(1)$ $\quad S_{1,j} = 0.$

Fix i. i^{th} rows gives a filt. of M_i

But M_i / M_{i-1} is simple.

$\rightsquigarrow$ filtration of M_i / M_{i-1}

$$0 \subset \frac{M_i \cap M_1'}{M_{i-1} \cap M_1'} \hookrightarrow \frac{M_i \cap M_2'}{M_{i-1} \cap M_2'} \subset \frac{M_i \cap M_3'}{M_{i-1} \cap M_3'} \subset \cdots \rightsquigarrow \subset \boxed{\frac{M_i}{M_{i-1}}}$$

simple.

$\Rightarrow$ unique $j = \sigma(i)$ s.t.

$$\underbrace{\frac{M_i \cap M_{j-1}'}{M_{i-1} \cap M_{j-1}'} \subsetneq \frac{M_i \cap M_j'}{M_{i-1} \cap M_j'}}_{\text{quotient} = S_{i,j}}$$

$$\Rightarrow \quad S_{i,\sigma(i)} \underset{\text{canon.}}{\cong} M_i / M_{i-1}.$$

<u>Claim 2</u>. $\forall j$. $\exists!$ $i = \tau(j)$ s.t. $S_{i,j} \neq 0$.

and $S_{\tau(j),j} \simeq M_j' / M_{j-1}'$.

$$\{1, \ldots, a\} \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \{1, 2, \ldots, b\}$$

$$S_{i,\sigma(i)} \neq 0.$$

$$\Downarrow$$

$$i = \underline{\tau\sigma(i)}.$$

$\Rightarrow$ σ, τ are inverse to each other.

$$M'_{\sigma(i)} / M'_{\sigma(i)-1} \cong S_{i,\sigma(i)} \cong M_i / M_{i-1}$$

canon.

$\blacksquare$

Semisimple modules

Def. $R \circlearrowright M$ is ss. if

$$M \cong \bigoplus_{i \in I} S_i$$

where S_i are simple.

Ex. $R = k$ (= field) any mod is ss.

writing M as $\oplus$ of simples

$\Longleftrightarrow$ choosing a basis of M (up to scalar)

Canonical form for ss module:

gp together isomorphic simple factors.

Recall Schur's lemma

$\boxed{D_i := \mathrm{End}_R(S_i)}$ is a division ring.

D_i-mod. $=$ D_i-vector spaces.

(basis. dim., ...).

Goal: write M as

$$\bigoplus_i \underline{V_i} \otimes_{D_i} S_i$$

V_i → right D_i-vector space.

Convention:

$$D_i \circlearrowright S_i$$

$f, g \in \mathrm{End}_R(S_i)$. $\rightsquigarrow$ $f \circ g$

naturally gives a left action of $\mathrm{End}_R(S_i)$ on S_i

$\Rightarrow$ $D_i \circlearrowright S_i$

$\Rightarrow$ $S_i \circlearrowleft D_i^{op}$

} same data.

$\otimes$ of modules.

$$M \circlearrowleft R \circlearrowright N.$$

$M \otimes_R N$ is defined. (just an abelian group)

$$= \{ \sum m_i \otimes n_i \} \Big/ \begin{pmatrix} \underline{m} r \otimes n \\ - m \otimes r \underline{n} \end{pmatrix}$$

$$M \overset{?}{=} \bigoplus_{S_i} \underline{\underline{V_i \otimes_{D_i} S_i}}$$

S_i — isom. classes of simple mod.

$$V_i = \cancel{\mathrm{Hom}_R(M, S_i)} \leftarrow \text{left } D_i\text{-mod.}$$

or $\boxed{\mathrm{Hom}_R(S_i, M)} \circlearrowleft D_i$ precomposing with D_i-action on S_i.

$$\mathrm{Hom}_R(S_i, M) \otimes_{D_i} S_i \xrightarrow{\text{eval}} M$$

$$((\varphi : S_i \to M), (x \in S_i)) \longmapsto \varphi(x).$$

D_i-bilinear.

Sum up:

$$\bigoplus V_i \otimes_{D_i} S_i \longrightarrow M.$$

<u>Claim</u>: This is an isom.

Pf: $M = \bigoplus$ simple.

$$= \bigoplus_{S_i} \underline{S_i^{\oplus I_i}}$$

S_i-isotypical summand.

set.

$$V_i = \mathrm{Hom}_R(S_i, S_i^{\oplus I_i})$$

$= D_i$-vec. sp. with basis I_i

<u>Check</u>: $V_i \otimes_{D_i} S_i \xrightarrow[\text{eval}]{\sim} S_i^{\oplus I_i}$

direct sum over $i \Rightarrow$ claim ∎

Canonical Form of a ss mod M

$$M \cong \bigoplus_{S_i} \underbrace{\mathrm{Hom}_R(S_i, M) \otimes_{D_i} S_i}$$

Functoriality: $M \longrightarrow N$ (both ss)

$\rightsquigarrow \mathrm{Hom}_R(S_i, M) \longrightarrow \mathrm{Hom}_R(S_i, N)$.

<u>Cor</u> A submod of a ss mod is ss.
same for quotient mod.

<u>Pf</u>. $\underset{\text{submod}}{N} \subset M = ss$.

We will use the lemma

<u>Lemma</u>: $\forall$ R-mod N, the evaluation map

$$\mathrm{eval}_N : \bigoplus_{S_i} \mathrm{Hom}_R(S_i, N) \otimes_{D_i} S_i \longrightarrow N$$

is injective.

(Proof is given after the proof of the Cor.)

Let $V_i = \mathrm{Hom}_R(S_i, M)$

$U_i = \mathrm{Hom}_R(S_i, N)$

$W_i = \mathrm{Hom}_R(S_i, M/N)$

Then $0 \to U_i \to V_i \to W_i$. (left exact)

$$\Rightarrow \quad 0 \to U_i \otimes_{D_i} S_i \longrightarrow V_i \otimes_{D_i} S_i \longrightarrow W_i \otimes_{D_i} S_i$$

(left exact) ($- \otimes_{D_i} S_i$ is exact)

get comm diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus U_i \otimes_{D_i} S_i \arrow[r] \arrow[d, "\mathrm{eval}_N"] & \bigoplus V_i \otimes_{D_i} S_i \arrow[r] \arrow[d, "\mathrm{eval}_M", "\cong"'] & \bigoplus W_i \otimes_{D_i} S_i \arrow[d, "\mathrm{eval}_{M/N}"] & \\
0 \arrow[r] & N \arrow[r] & M \arrow[r] & M/N \arrow[r] & 0
\end{tikzcd}$$

eval_M is $\cong$

Both eval_N, $\mathrm{eval}_{M/N}$ are injective
bottom row is exact

$\Rightarrow$ eval_N and $\mathrm{eval}_{M/N}$ are both $\cong$.

this proves that N and M/N are both ss. ∎

<u>Lemma</u> $\forall$ R-mod N

$$\bigoplus_{S_i} \mathrm{Hom}_R(S_i, N) \otimes_{D_i} S_i \xrightarrow{\mathrm{eval}_N} N$$

is injective.

<u>Pf</u>: It's enough to show each

$$\mathrm{Hom}_R(\underline{\underline{S_i}}, N) \otimes_{D_i} S_i \hookrightarrow N.$$

Let $N' = \underline{\text{sum of } S_i\text{-isotypical}}$ submod of $\underline{N}$.
$=$ ss R-mod, S_i-isotypical.

$$\begin{array}{ccc} \mathrm{Hom}_R(S_i, N') \otimes_{D_i} S_i & \xrightarrow{\sim} & N' \subset N. \\ \| & & \\ \mathrm{Hom}_R(S_i, N) \otimes_{D_i} S_i & \xrightarrow{\text{injective}} & N \end{array}$$

∎

<u>Def</u>. The <u>socle</u> of an R-mod M is the maximal ss submod of M.
$=$ image(eval_M).

Cor M ss R-mod.

$N \subset M$ submod has a complement.

$(\exists\ N' \subset M,\ N \oplus N' = M)$.

Pf.

$$\left.\begin{aligned} V_i &= \mathrm{Hom}_R(S_i, M) \\ U_i &= \mathrm{Hom}_R(S_i, N) \end{aligned}\right\} D_i\text{-v.s.}$$

$U_i \subset V_i$

Let $W_i \subset V_i$ be a complem. to U_i.
(D_i-v.s.)

Let $N' = \bigoplus_i W_i \otimes_{D_i} S_i \subset \bigoplus_i V_i \otimes_{D_i} S_i \cong M$.

$N \oplus N' = M$.

M is ss $\iff$ every submod of M has a compl.

$\Leftarrow$: Take $M' =$ max. ss submod.
if $M' \neq M$. then $M' \oplus M'' = M$.

Warning: $R \circlearrowright M$.
M may not contain a simple submodule.
(i.e.. $\mathrm{soc}(M)$ may be 0).

$R = \mathbb{Z} \circlearrowright \mathbb{Z} = M$.

simple mods of $\mathbb{Z}$: $\underline{\mathbb{Z}/p}$

Prop

Every submod of M has a complement $\iff$ M is semisimple

Pf: previous Cor gives $\Longleftarrow$

$\Longrightarrow$: Let $M' \subset M$ be the maximal ss submod. (i.e., $M' = soc(M)$)

Then $M' \oplus M'' = M$ for some submod M''.

If $M'' \neq 0$, we need to find a simple submod $S \subset M''$ to get contradiction with maximality of M'.

Take $x \in M''$, $x \neq 0$.

$R/I \simeq R \cdot x \subset M''$ where I is a left ideal.

$\Rightarrow I \subset J = $ a max left ideal. (Zorn)

i.e., $Rx \twoheadrightarrow R/J$ = simple module $= S$.

Let $N = \ker(R \cdot x \to S) \subset R \cdot x \subset M'' \subset M$.

By assumption, $N \oplus N' = M$ for some N'.

$R \cdot x$ also has a complement L.

$R \cdot x \oplus L = M$.

Then $N' \hookrightarrow M \xrightarrow[\text{projection}]{} R \cdot x$

The image gives a complement of N in $R\cdot x$.
i.e., a copy of S.

$\Rightarrow$ S is a direct summand of $R\cdot x \subset M''$.
contradicting the maximality of M'. $\blacksquare$