

Noncommutative algebras in

- Rep Theory
- Alg Geom
- Number Theory.

Rep Theory:

G finite gps
 modular reps $k = \bar{k}$ char(k) = p ,
 $p \mid \#G$.
 $k[G]$ not semisimple.

infinite groups

e.g. $GL_n(\mathbb{Q}_p)$, $GL_n(\mathbb{F}_q((t)))$, $GL_n(\mathbb{C})$, $GL_n(\mathbb{R})$.
 ↙ topological ring / field.
 ↘ p-adic groups

automorphic repn theory studies reps of

$$\underline{G(\mathbb{A}_{\mathbb{Q}})} = \prod_P' G(\mathbb{Q}_p) \times G(\mathbb{R})$$

$$\mathbb{C} [GL_n(\mathbb{Q}_p)] - \text{mod}$$

\cong (abstract) \mathbb{C} -reps of $GL_n(\mathbb{Q}_p)$.

$GL_n(\mathbb{Q}_p)$ has topology.

(g_{ij}) is close to I_n

if $v_p(g_{ij} - \delta_{ij}) \gg 0$.

care about continuous reps

\mathbb{C} -v.s.
possibly ∞ -dim

$$\int: GL_n(\mathbb{Q}_p) \longrightarrow GL(V)$$

discrete.

continuous \iff each vector $v \in V$
is fixed by an open subgrp
 $\subset GL_n(\mathbb{Q}_p)$.

open subgps

$$GL_n(\mathbb{Q}_p) \supset GL_n(\mathbb{Z}_p) \supset K_1 \supset K_2 \supset \dots$$

$$\supset g GL_n(\mathbb{Z}_p) g^{-1}$$

compact
open.

nbhd basis
of I_n .

$$K_i = \{ g \equiv I_n \pmod{p^i} \}$$

$$V = \bigcup V^K$$

$K \subset GL_n(\mathbb{Q}_p)$
open
compact.

$$G = GL_n(\mathbb{Q}_p)$$

$$H_{G,K} \curvearrowright V^K$$

$$H_{G,K} = \left\{ f: G \longrightarrow \mathbb{C} \right. \\ \left. \begin{array}{l} \text{compactly supp} \\ f(k_1 g k_2) = f(g) \quad \forall \begin{array}{l} g \in G \\ k_1, k_2 \in K \end{array} \end{array} \right\}$$

$$\begin{array}{c} f * v \\ \Downarrow \\ V^K \end{array} = \sum_{g \in G/K} f(g) \cdot g v \in V^K$$

(finite sum)

$$\text{Rep}(G) \xrightarrow{(-)^K} \underline{H_{G,K}}\text{-mod}$$

$$K = GL_n(\mathbb{Z}_p)$$

$H_{G,K}$ is commutative.

$$GL_n(\mathbb{Z}_p) \backslash GL_n(\mathbb{Q}_p) / GL_n(\mathbb{Z}_p) \longrightarrow \mathbb{C}$$

$$\begin{pmatrix} p^{a_1} & & & \\ & p^{a_2} & & \\ & & \ddots & \\ & & & p^{a_n} \end{pmatrix}$$

$$\underline{a_1 \geq a_2 \geq \dots \geq a_n} \\ \in \mathbb{Z}$$

$$I = \left\{ g \in GL_n(\mathbb{Z}_p), g \text{ mod } p \text{ is upper triang.} \right\}$$

Iwahori
Subgp

$$n=2. \quad \begin{pmatrix} * & * \\ p* & * \end{pmatrix} \quad * \in \mathbb{Z}_p.$$

$H_{G, I}$

$$I \backslash GL_n(\mathbb{O}_p) / I$$

$$\left\{ \begin{pmatrix} p^{a_1} & & \\ & \ddots & \\ & & p^{a_n} \end{pmatrix} \cdot w \right\}$$

$$(a_1, \dots, a_n) \in \mathbb{Z}^n; w \in \underline{S_n}$$

$Z(H_{G, I})$ is large:

$H_{G, I}$ is f.g. over $Z(H_{G, I})$

$$R = \textcircled{Z(R)}$$

Spec $Z(R)$

R lives over $\text{Spec } Z(R)$ as a sheaf of algebre.

\downarrow

\mathfrak{p} .

$$R_{\mathfrak{p}}, \underline{R}_{\mathfrak{p}}R, \underline{R \otimes k(\mathfrak{p})}$$

Hecke alg. appears more generally.

$$\textcircled{H} \subset G.$$

ρ , irrep of H

$$\text{End}_G(\text{Ind}_H^G(\rho))$$

$$\parallel$$

$$\text{Hom}_H(\rho, \text{Res}_H^G \text{Ind}_H^G(\rho))$$

$$H \backslash G / H \longrightarrow \boxed{\text{End}(\rho)}$$

$$\rho = 1. \quad \mathcal{H}_{G,H}$$

e.g. $G = GL_n(\mathbb{F}_q).$

$$H = B(\mathbb{F}_q) \text{ upper triang.}$$

$$\mathcal{H}_{G,H} = \mathbb{C} \left[\underbrace{B(\mathbb{F}_q) \backslash GL_n(\mathbb{F}_q) / B(\mathbb{F}_q)} \right]$$

"deformed" S_n

$$\mathbb{C}[S_n]$$

abstractly, $\underline{\mathcal{H}_{G,H}} \cong \underline{\mathbb{C}[S_n]}$

$$\text{Ind}_{B(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) = \mathbb{C} \left[\underbrace{Fl_n(\mathbb{F}_q)} \right].$$

$$0 \subset V_1 \subset \dots \subset V_n = \mathbb{F}_q^n$$

$GL_n(\mathbb{C})$ $GL_n(\mathbb{R})$

\mathfrak{g} .

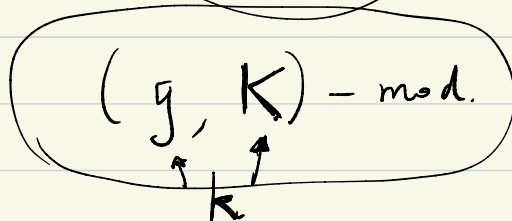
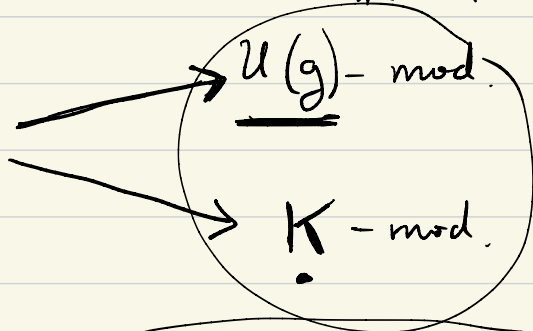
Compact ^{connected} Lie gps K e.g. U_n .

$$\text{Irrep}(K) \longrightarrow \text{Irrep}(\mathfrak{g})$$

This is a bijection
for $K = \text{simply connected}$. s.t. SU_n
 $\mathfrak{g} = (\text{Lie } K) \otimes \mathbb{C}$.

$$\text{Rep}_{fd}(K) \underset{SU_n}{\parallel} \text{Rep}_{fd}(K_{\mathbb{C}}) \underset{GL_n(\mathbb{C})}{\parallel} \text{Rep}_{fd}(\mathfrak{g}).$$

G - mod
 U
 $K = \text{max. cpt.}$



Similar to $\underline{\mathcal{H}}_{G,K} - \text{mod}$.

$$G \underset{GL_n(\mathbb{C})}{\parallel} K \underset{U_n}{\parallel}$$

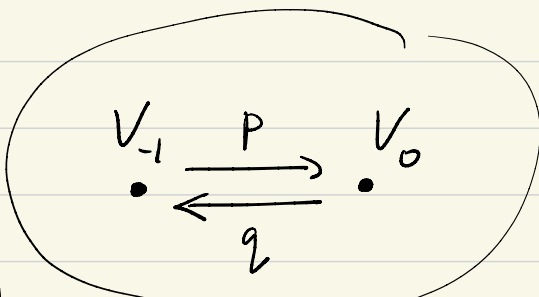
$$GL_n(\mathbb{R}) \supset O_n.$$

$GL_n(\mathbb{C})$ (\mathfrak{g}, K) -mod

\rightsquigarrow certain ^{sub}category of $U(\mathfrak{g})$ -mod.

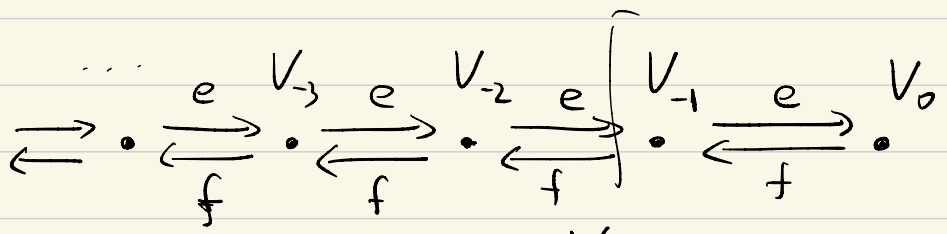
(category \mathcal{O} : Beilinson
Bernstein
Gelfand)

$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$
cat. \mathcal{O} :
(finitely many simple objects)

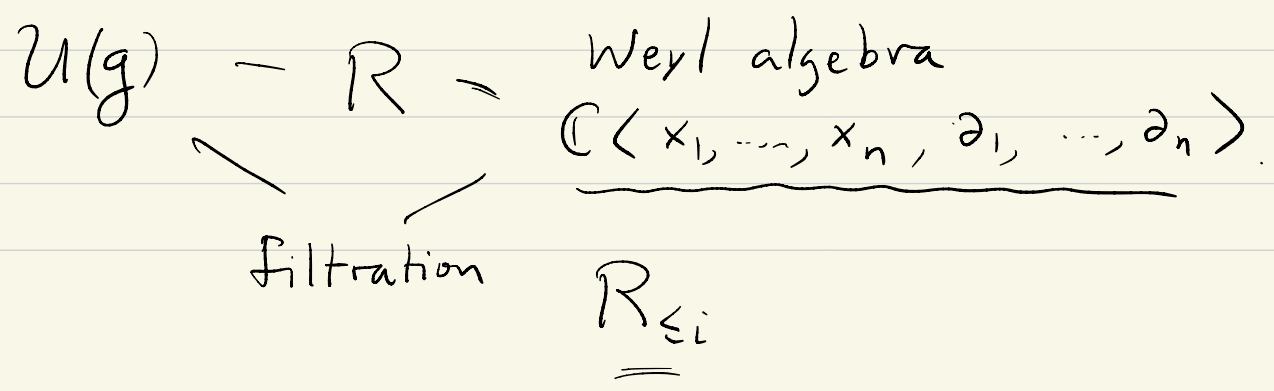
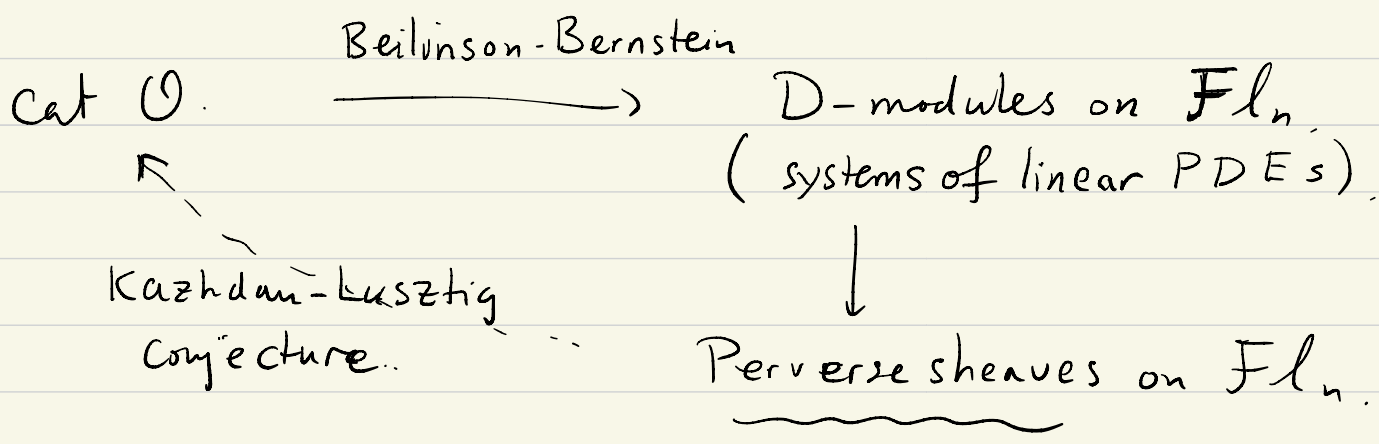


$Pq = 0$.

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$
 \parallel e, h, f \parallel
 \parallel
 $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



$ef = 0$ on V_0 .
 $V_i = \text{eigensp of } h, \text{ eigenvalue } 2i$.



$$\text{gr } R = \bigoplus R_{\leq i} / R_{\leq i-1}$$

is commutative,

(and \cong polynomial ring).

R is a quantization of $\text{gr } R$.

$D\text{-mod on } X$ $\xrightarrow[\text{quantization}]{\text{classical limit}}$ (coh) sheaves on T^*X .

$$(R, R_{\leq i})$$

$$\begin{matrix} \mathcal{R} \\ (M, F) \end{matrix}$$

$$\text{gr } R$$

$$\mathcal{R} \\ \text{gr } M$$

has a support in $\text{Spec}(\text{gr } R)$

$D_X\text{-mod } M \longrightarrow$ subset of T^*X
(singular supp.)

Alg. Geom. $X =$ projective smooth / \mathbb{C} .

$D^b \text{Coh}(X)$ "complexes of v.b. on X "

controlled by a "small" non-comm ring.

e.g. $X = \mathbb{P}^1$.

$$Q = \left(\begin{matrix} V & \longrightarrow & W \\ \bullet & \longrightarrow & \bullet \end{matrix} \right)$$

$$D^b \text{Rep}(Q) \cong D^b \text{Coh}(\mathbb{P}^1)$$

$$\mathcal{O}, \mathcal{O}(1)$$

$$R_Q \cong \underline{\text{End}(\mathcal{O} \oplus \mathcal{O}(1))}$$

Number Theory / Arithmetic

Central simple alg. / \mathbb{Q}_p .

$$\text{Br}(\mathbb{Q}_p) \xrightarrow{\sim} \underline{\mathbb{Q}/\mathbb{Z}}$$

Local class field theory.

abelian extns of F (e.g. $F = \mathbb{Q}_p$).

$$\iff \underline{\text{Gal}(\bar{F}/F)^{\text{ab}}}$$

$$\begin{array}{ccc} \text{Gal}(\bar{F}/F)^{\text{ab}} & \longleftrightarrow & F^* \\ \downarrow & & \downarrow \text{val} \\ \hat{\mathbb{Z}} & \longleftrightarrow & \mathbb{Z} \end{array}$$

H_1

$$\text{Br}(F) = H^2(\text{Gal}(\bar{F}/F), \bar{F}^*) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

"fundamental class".

Semi-linear algebra

in studying geometry / char p
or p -adic.

$$\sigma \curvearrowright k \quad \text{e.g. } \text{char}(k) = p. \quad \sigma(x) = x^p.$$

$$\underline{k\langle x; \sigma \rangle} \curvearrowright V$$

$$\Leftrightarrow V \xrightarrow{\varphi} V$$

$$\varphi(av) = \sigma(a)\varphi(v)$$

Formal groups / p-div gps

Dieudonné theory →

modules over
Dieudonné ring

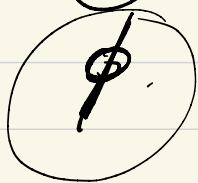
X/k , $\text{char}(k)=p$.

$H_{\text{crys}}^*(X/W(k))$

module over
 $W(k)$

(Witt ring of k)

$$k = \mathbb{F}_p \quad W(k) = \mathbb{Z}_p$$



Dieudonné-Manin classification
of modules over

$$\underline{W(\overline{\mathbb{F}}_p) \langle x; \sigma \rangle \left[\frac{1}{p} \right]}$$