Lecture 25

12/9

Noncommutative algebras in
- Rep Theory
- Alg Geom
- Number Theory.

Rep Theory:

G finite gps

modular reps $k = \bar{k}$ char$(k) = p$.

$p \mid \# G$.

$k[G]$. not semisimple.

infinite groups

e.g. $GL_n(\mathbb{Q}_p)$. ← topological ring / field.

$GL_n(\mathbb{F}_q((t)))$.

$\Big\}$ p-adic groups

$GL_n(\mathbb{C})$

$GL_n(\mathbb{R})$.

automorphic repn theory studies reps of

$$G(\mathbb{A}_{\mathbb{Q}}) = \prod_p{}' G(\mathbb{Q}_p) \times G(\mathbb{R}).$$

$\mathbb{C}[GL_n(\mathbb{Q}_p)]$-mod

$\equiv$ (abstract) $\mathbb{C}$-reps of $GL_n(\mathbb{Q}_p)$.

$GL_n(\mathbb{Q}_p)$ has topology.

(g_{ij}) is close to I_n

if $v_p(g_{ij} - \delta_{ij}) >> 0$.

care about continuous reps

$\mathbb{C}$-v.s. possibly ∞-dim

$$\rho : GL_n(\mathbb{Q}_p) \longrightarrow GL(\mathbf{V})$$

discrete.

continuous $\iff$ each vector $v \in V$ is fixed by an open subgp $\subset GL_n(\mathbb{Q}_p)$.

open subgps

$$GL_n(\mathbb{Q}_p) \supset \underbrace{GL_n(\mathbb{Z}_p) \supset K_1 \supset K_2 \supset}_{\text{compact open.}} \cdots$$

$\supset g\, GL_n(\mathbb{Z}_p)\, g^{-1}$

nbhd basis of I_n.

$$K_i = \{ g \equiv I_n \bmod p^i \}.$$

$$V = \bigcup_{\substack{K \subset GL_n(\mathbb{Q}_p) \\ \text{open} \\ \text{compact.}}} \mathbf{V}^K$$

$G = GL_n(\mathbb{Q}_p)$.

$\mathcal{H}_{G,K} \circlearrowright V^K$

$$\mathcal{H}_{G,K} = \left\{ f : G \longrightarrow \mathbb{C} \quad \begin{array}{l} \text{compactly supp} \\ f(k_1 g k_2) = f(g) \ \forall \begin{array}{l} g \in G \\ k_1, k_2 \in K \end{array} \end{array} \right\}$$

$\cup$

$$\underset{\substack{\cap \\ V^K}}{f * v} = \sum_{g \in G/K} f(g) \cdot gv \in V^K$$

(finite sum)

$$\mathrm{Rep}(G) \xrightarrow{(-)^K} \underline{\mathcal{H}_{G,K}}\text{-mod}.$$

$K = GL_n(\mathbb{Z}_p)$.

$\mathcal{H}_{G,K}$ <u>is commutative</u>.

$$GL_n(\mathbb{Z}_p) \backslash GL_n(\mathbb{Q}_p) / GL_n(\mathbb{Z}_p) \longrightarrow \mathbb{C}.$$

$$\begin{pmatrix} p^{a_1} & & & \\ & p^{a_2} & & \\ & & \ddots & \\ & & & p^{a_n} \end{pmatrix} \qquad \underline{\underline{a_1 \geq a_2 \geq \cdots \geq a_n}} \in \mathbb{Z}$$

$$I = \{ g \in GL_n(\mathbb{Z}_p),\ g \bmod p \text{ is upper triang} \}$$

Iwahori subgp

$n=2.$ $\begin{pmatrix} * & * \\ p* & * \end{pmatrix}$ $* \in \mathbb{Z}_p.$

$\mathcal{H}_{G,I}$ $\qquad I \backslash GL_n(\mathbb{Q}_p) / I$

$$\updownarrow$$

$$\left\{ \begin{pmatrix} p^{a_1} & & \\ & \ddots & \\ & & p^{a_n} \end{pmatrix} \cdot w \quad (a_1, \dots, a_n) \in \mathbb{Z}^n;\ w \in \underline{S_n} \right\}$$

$Z(\mathcal{H}_{G,I})$ is large:

$\mathcal{H}_{G,I}$ is $\underline{\text{f.g.}}$ over $Z(\mathcal{H}_{G,I})$

$$R \supset Z(R)$$

$\underline{\operatorname{Spec} Z(R)}$.

R lives over $\operatorname{Spec} Z(R)$ as a sheaf of algebra.

$$\begin{matrix} \operatorname{Spec} Z(R) \\ \Psi \\ \mathfrak{p}. \end{matrix}$$

$R_{\mathfrak{p}}$, $\underline{\underline{R/\mathfrak{p}R}}$. $\underline{R \otimes k(\mathfrak{p})}$.

Hecke alg. appears more generally.

$$H \subset G.$$

ρ: irrep of H

$$\underset{\sim\sim\sim}{\mathrm{End}_G\left(\mathrm{Ind}_H^G(\rho)\right)}$$

$$\|$$

$$\mathrm{Hom}_H\left(\rho, \underbrace{\mathrm{Res}_H^G \mathrm{Ind}_H^G(\rho)}\right).$$

$$H \backslash G / H \rightsquigarrow \boxed{\mathrm{End}(\rho)}$$

$$\rho = 1. \qquad \mathcal{H}_{G,H}.$$

e.g. $\quad G = GL_n(\mathbb{F}_q).$

$\quad H = B(\mathbb{F}_q)$ upper triang.

$$\mathcal{H}_{G,H} = \mathbb{C}\Big[\underbrace{B(\mathbb{F}_q)\backslash GL_n(\mathbb{F}_q)/B(\mathbb{F}_q)}_{S_n}\Big]$$

"deformed" $\mathbb{C}[S_n]$.

abstractly. $\underline{\underline{\mathcal{H}_{G,H}}} \cong \underline{\underline{\mathbb{C}[S_n]}}$.

$$\mathrm{Ind}_{B(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) = \mathbb{C}\Big[\underbrace{Fl_n(\mathbb{F}_q)}\Big].$$

$$0 \subset V_1 \subset \cdots \subset V_n = \mathbb{F}_q^n$$

$GL_n(\mathbb{C})$. $\quad GL_n(\mathbb{R})$.

$\mathfrak{g}$.

Compact $\overbrace{\text{Lie}}^{\text{connected}}$ gps K e.g. U_n.

$$\mathrm{Irrep}(K) \longrightarrow \mathrm{Irrep}(\mathfrak{g})$$

$\mathfrak{g} = (\mathrm{Lie}\, K) \otimes \mathbb{C}$.

This is a bijection
for K = <u>simply connected</u>. s.t. SU_n

$$\underset{\substack{\| \\ SU_n}}{\mathrm{Rep}_{fd}(K)} \simeq \underset{\substack{\| \\ GL_n(\mathbb{C})}}{\mathrm{Rep}_{fd}(K_{\mathbb{C}})} \simeq \mathrm{Rep}_{fd}(\mathfrak{g}).$$

G - mod $\longrightarrow$ <u>$\mathcal{U}(\mathfrak{g})$</u> - mod.

$\cup$

K = max. cpt. $\longrightarrow$ K - mod.

$(\mathfrak{g}, K)$ - mod.

Similar to <u>$\mathcal{H}_{G,K}$</u> - mod.

$$\begin{matrix} G & & K \\ \| & & \| \\ GL_n(\mathbb{C}) & \supset & U_n \end{matrix}$$

$GL_n(\mathbb{R}) \supset O_n$.

$GL_n(\mathbb{C})$. $\qquad (\mathfrak{g}, K)$- mod

$\rightsquigarrow$ certain sub category of $U(\mathfrak{g})$-mod.

(category $\mathcal{O}$: Beilinson
Bernstein
Gelfand)

$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

cat. $\mathcal{O}$:

(finitely many simple objects)

$$\begin{tikzcd} \underset{V_{-1}}{\bullet} \arrow[r, shift left, "p"] & \underset{V_0}{\bullet} \arrow[l, shift left, "q"] \end{tikzcd}$$

$$pq = 0.$$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$$\begin{tikzcd} \cdots \arrow[r, shift left] & \underset{V_{-3}}{\bullet} \arrow[l, shift left] \arrow[r, shift left, "e"] & \underset{V_{-2}}{\bullet} \arrow[l, shift left, "f"] \arrow[r, shift left, "e"] & \underset{V_{-1}}{\bullet} \arrow[l, shift left, "f"] \arrow[r, shift left, "e"] & \underset{V_0}{\bullet} \arrow[l, shift left, "f"] \end{tikzcd}$$

$ef = 0$ on V_0.

V_i = eigensp of h, eigenvalue $2i$.

$$\begin{tikzcd} \text{Cat } \mathcal{O}. \arrow[r, "\text{Beilinson-Bernstein}"] & \text{D-modules on } \mathcal{F}l_n \\ & \text{(systems of linear PDEs)} \arrow[d] \\ & \text{Perverse sheaves on } \mathcal{F}l_n. \arrow[uul, dashed, "\text{Kazhdan-Lusztig conjecture}"] \end{tikzcd}$$

$U(\mathfrak{g})$ — R — Weyl algebra

$\mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$.

filtration

$R_{\leq i}$

$$\mathrm{gr}\, R = \bigoplus \underbrace{R_{\le i}/R_{\le i-1}}$$

is commutative.
(and $\cong$ polynomial ring).

R is a quantization of $\mathrm{gr}\, R$.

$$D\text{-mod on } X. \quad \underset{\text{quantization}}{\overset{\text{classical limit}}{\rightleftarrows}} \quad \text{(coh) sheaves on } T^*X.$$

$$\begin{array}{ccc} (R, R_{\le i}) & & \mathrm{gr}\, R \\ \circlearrowright & & \circlearrowright \\ (\underline{M}, F) & & \underline{\underline{\mathrm{gr}\, M}} \end{array}$$

has a support in $\mathrm{Spec}(\mathrm{gr}\, R)$

D_X-mod M $\longrightarrow$ subset of T^*X.
(singular supp.)

Alg. Geom. X = projective smooth $/\mathbb{C}$.

$\underline{D^b \mathrm{Coh}}(X)$. "complexes of v.b. on X".

controlled by a "small" non-comm ring.

e.g. $X = \mathbb{P}^1$.

$$Q = \left(\overset{V}{\bullet} \rightrightarrows \overset{W}{\bullet} \right).$$

$$D^b\, \underline{\mathrm{Rep}(Q)} \simeq \underline{D^b\, \boxed{\mathrm{Coh}(\mathbb{P}^1)}}$$

$$\mathcal{O}, \ \mathcal{O}(1)$$

$$R_Q \simeq \underline{\mathrm{End}(\mathcal{O} \oplus \mathcal{O}(1))}.$$

Number Theory / Arithmetic

Central simple alg. / $\mathbb{Q}_p$.

$$\mathrm{Br}(\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

Local class field theory.

abelian extns of F (e.g. $F = \mathbb{Q}_p$).

$$\Longleftrightarrow \quad \mathrm{Gal}(\bar{F}/F)^{ab}.$$

H_1

$$\begin{tikzcd}
\mathrm{Gal}(\bar{F}/F)^{ab} \arrow[d] & F^* \arrow[l, hook'] \arrow[d, "\mathrm{val}"] \\
\hat{\mathbb{Z}} & \mathbb{Z} \arrow[l, hook']
\end{tikzcd}$$

$$\mathrm{Br}(F) = H^2(\mathrm{Gal}(\bar{F}/F), \bar{F}^*). \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

"fundamental class".

Semi-linear algebra
in studying geometry / char p
or p-adic.

$\sigma \circlearrowright k$ e.g. $\mathrm{char}(k) = p$. $\sigma(x) = x^p$.

$$k\langle x ; \sigma \rangle \circlearrowright V$$

$$\Longleftrightarrow \quad V \xrightarrow{\varphi} V$$

$$\varphi(av) = \sigma(a)\varphi(v).$$

Formal groups / p-div gps

Dieudonné theory $\longrightarrow$ modules over Dieudonné ring.

X/k, $\mathrm{char}(k)=p$.

$H^*_{crys}(X/W(k))$ module over $W(k)$ (Witt ring of k)

ϕ

$k=\mathbb{F}_p$. $W(k)=\mathbb{Z}_p$

Dieudonné–Manin classification of modules over

$$W(\bar{\mathbb{F}}_p)\langle x;\sigma\rangle[\tfrac{1}{p}].$$