# Lecture 23



- Semiprime rings
- Uniform modules, Goldie rank
- Goldie's Theorem

**Thm** (Goldie) Let $R$ be a semiprime, right Noetherian ring.

- $S$ = regular elts in $R$. is a right Ore set.
- Then $Q(R) = R_S$ is a semisimple ring.

$$\left( Q(R) \simeq \prod_{i=1}^{r} M_{n_i}(D_i) \right).$$

Special case: $R$ = domain, right Noetherian, then $Q(R)$ = division ring.

**Prime rings**

Comm: $A$ is a domain $\iff$ $(0)$ is a prime ideal.

$$\boxed{\mathfrak{p} \subset A \text{ prime}: \quad ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}.}$$

Noncomm

same def gives the notion of "completely prime ideals".

$\mathfrak{p}$ = completely prime ideal.
$\Longleftrightarrow$ $R/\mathfrak{p}$ is a domain.

Prime rings will include $\underline{M_n(R)}$   $R$ = domain.

Def. $R$ is a prime ring if $\forall$ ideals $I_1, I_2 \subset R$

$I_1 I_2 = 0 \implies$ either $I_1 = 0$ or $I_2 = 0$.

e.g. $R$ = simple. $\implies$ $R$ is prime.

$R$ = domain $\implies$ $R$ is prime.
$\not\Longleftarrow$

Equivalent formulation ( $R$ prime).

- $\boxed{\forall a, b \in R, \ \underbrace{aRb}_{(aR)\cdot(R\cdot b)} = \{0\} \implies a = 0 \text{ or } b = 0.}$
- $\forall$ $I_1$ right ideal, $I_2$ left ideal.
  $I_1 I_2 = \{0\} \implies I_1 = 0$ or $I_2 = 0$.

This implies $R$ prime: $I_1 I_2 = 0$.
$\forall a \in I_1$, $I_1 \supset aR$.
$\forall b \in I_2$ $I_2 \supset Rb$.
$0 = I_1 I_2 \supset aRb \implies a = 0$ or $b = 0$.

e.g. $R$ is prime, so is $\underline{M_n(R)}$.
ideals are $M_n(I)$.

$R$ is prime, so is $R[x]$.

$R[x_1, \dots, x_n]$.

**Def.** $\mathfrak{p} \subset R$ is a <u>prime ideal</u> if $R/\mathfrak{p}$ is a prime ring.

( $\Longleftrightarrow$ if $a, b \in R$. s.t. $aRb \subset \mathfrak{p}$.
then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.).

**Ex.** Primitive ideal:

$R \circlearrowright M =$ simple $R$-module.

$\underbrace{I}_{\text{such ideals are called primitive}} = \mathrm{Ann}_R(M) = \{a \in R \mid aM = 0\}$.

primitive $\Longrightarrow$ prime.

**Pf.** $\bar{R} = R/I \circlearrowright M =$ simple $\bar{R}$-mod.

↑ faithful mod.

If $\bar{a}, \bar{b} \in \bar{R}$ s.t. $\bar{a}\bar{R}\bar{b} = 0$.

If $\bar{b} \neq 0$. $\bar{R}\underbrace{\bar{b}\bar{M}}_{\text{nonzero}} = \bar{M}$ ($\because \bar{M}$ is simple)

apply $\bar{a}$

$\underbrace{\bar{a}\bar{R}\bar{b}}_{=0}\bar{M} = 0$

$\Longrightarrow \bar{a}$ acts by $0$ on $\bar{M}$ $\Rightarrow$ $\bar{a} = 0$.

| NC | Comm |
|---|---|
| Prime rings | domains |
| $\cup$ | |
| domains | |

| NC | Comm |
|---|---|
| Semiprime | reduced |
| $\cup$ | |
| reduced | |

**Def** A <u>Semiprime ring</u> $R$ is one in which there is no nonzero nilpotent ideal.

Equivalently,

- $I \subset R$ ideal, $I^2 = 0 \Rightarrow I = 0$.
- if $a \in R$. s.t. $aRa = 0 \Rightarrow a = 0$.

(This implies $I^2 = 0 \Rightarrow I = 0$ ?

$\forall a \in I$. $0 = I^2 \supset aRa = 0$

$\Rightarrow a = 0$.).

- If $I \subset R$ is a left ideal, s.t. $I^2 = 0$ then $I = 0$.

**Def** $I \subset R$ is a <u>semiprime ideal</u> if $R/I$ is semiprime.

$R$ is reduced $\Longrightarrow$ $R$ is semiprime

**Ex.** $R$ semiprime $\Rightarrow$ $R[x]$ is semiprime.

— " — $\Rightarrow$ $M_n(R)$ is semiprime.

$R_1, R_2$ semiprime $\Rightarrow$ $R_1 \times R_2$ is semiprime.

Recall: $J(R)$ contains all $\boxed{\text{nil-ideals}}$ (ideals consisting of nilp elts).

If $J(R) = 0 \Rightarrow$ no $\overset{\text{nonzero}}{\text{nilp ideal}} \Rightarrow R$ is semiprime.

## Uniform Modules

**Def**: $M \circlearrowleft R$ is called uniform, if any two nonzero submodules have a nonzero intersection.

$$\left( \Longleftrightarrow \quad \underset{\neq 0}{M_1} \oplus \underset{\neq 0}{M_2} \not\subset M \right).$$

**Ex**.
- $A =$ domain.
  $A$ as an $A$-mod is uniform.
  $(a) \cap (b) \supset (ab) \neq 0$.
- $A = k[x] \circlearrowright M = k[x]/(x^n)$.

$x$ acts as $\begin{pmatrix} \boxed{0} & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$ on $M$.

$0 \neq \overset{x \circlearrowright}{M'} \subset M \Rightarrow M'$ contains $x^{n-1}$

$\Rightarrow M$ is uniform.

f.g. $k[x]$-mod $M$ is uniform

$\Longleftrightarrow \begin{cases} M \simeq k[x]. \\ M \simeq k[x]/f(x)^n \end{cases}$

$f(x)$ irred. poly.

$M$ is uniform $\Longrightarrow$ any submod of $M$ is uniform.

$\not\Longrightarrow$ quotient mod ~~are uniform~~.

$$k[x]/(x(x-1)) = k[x]/x \oplus k[x]/(x-1)$$

- $R = k\langle x, \partial\rangle \circlearrowright k[x]$. Simple modules

$R = k\langle x, \partial\rangle \circlearrowright R$ is a uniform module

$R \circlearrowright R$ is uniform.

$\Longleftrightarrow 0 \neq I_1, I_2 \subset R$ left ideals $I_1 \cap I_2 \neq 0$.

$\Longleftrightarrow 0 \neq a, b \in R$. then $Ra \cap Rb \neq 0$.

$$\boxed{\begin{array}{l} R = \text{domain. then} \\ \quad S = R \setminus \{0\} \text{ is right Ore} \\ \Longleftrightarrow \quad R \circlearrowleft R \text{ is uniform.} \end{array}}$$

$aS \cap sR \neq \emptyset$.
$aR \cap sR \neq \{0\}$.

Goldie rank.

Essential inclusion: $M' \subseteq_e M$.

if any nonzero submod $N \subset M$

$\Rightarrow M' \cap N \neq 0$.

$(\Longleftrightarrow M' \oplus N \not\subset M)$.

**Def** $M \circlearrowleft R$ has <u>finite Goldie rank</u>, if it doesn't contain $\infty$ direct sum of submod.

( $M$ is noetherian $\Longrightarrow$ $M$ has finite Goldie rk).

**Def** $M \circlearrowleft R$.

$$\mathrm{grk}(M) = \sup\{ n \mid \underset{\neq 0}{M_1} \oplus \underset{\neq 0}{M_2} \oplus \cdots \oplus \underset{\neq 0}{M_n} \subset M \}$$

$\mathrm{grk}(M) < \infty \Longrightarrow M$ has finite Goldie rank.

$\overset{?}{\Longleftarrow}$ will prove in next prop.

**Ex** $\mathrm{grk}(M) = 0 \Longleftrightarrow M = 0$.

$\mathrm{grk}(M) = 1 \Longleftrightarrow M$ is uniform.

**Prop** ① Suppose $M$ has finite Goldie rank. then $\exists\ M_1 \oplus \cdots \oplus M_n \subseteq_e M$.

$M_i$ uniform

② If $M_1 \oplus \cdots \oplus M_n \subseteq_e M$
$M_i$ uniform
$\Longrightarrow n = \mathrm{grk}(M)$.

**Properties of $\subseteq_e$**

- $M_1 \subseteq_e M_2 \subseteq_e M_3 \Longrightarrow M_1 \subseteq_e M_3$

• $M_i' \subseteq_e M_i \qquad i=1,\dots,n.$

$$\Rightarrow \quad \bigoplus M_i' \subseteq_e \bigoplus M_i$$

• $$\begin{array}{ccc} f: M & \longrightarrow & N. \\ \cup_e & \Longleftarrow & \cup_e \\ \underline{f^{-1}(N')} & & N' \end{array} \qquad (R\text{-linear})$$

• $M$ has no essential proper submod.

$\Leftrightarrow$ $M$ is semisimple.

( $\Leftarrow$: $M' \subsetneq M$. then $M' \oplus M'' = M$.

$\Rightarrow$ $M'$ not essential in $M$.

$\Rightarrow$ $M' \subsetneq M$. any submod.

$\{N \subset M \mid N \cap M' = 0\}$.

has a max. element. $N$.

$\underline{M' \oplus N} \subseteq_e M \Rightarrow M' \oplus N = M$. )

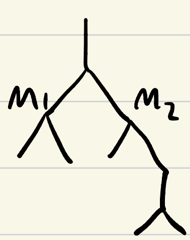
Pf of prop

① . $M$ has finite Goldie rank.

$M$ not unif $\Rightarrow$ $M \supset \boxed{M_1 \oplus M_2}$

$\cup$

$M_1' \oplus M_1''$




Stop $\quad M \supset \underbrace{M_1 \oplus \cdots \oplus M_n}_{\text{uniform.}}$

if not ess, can find $M_{n+1}$.

$$M_1 \oplus \cdots \oplus M_n \oplus M_{n+1} \subset M.$$

$$\underbrace{M_1 \oplus \cdots \oplus M_n \oplus M_{n+1} \oplus \cdots \oplus M_{n+k}}_{\text{unif}} \subset M.$$

$$\vdots$$

$$M_1 \oplus \cdots \oplus M_N \subseteq_e M.$$

(2). $\underbrace{M_1 \oplus \cdots \oplus M_n}_{\text{unif}} \subseteq_e M.$

$$\|$$

$$N_1 \oplus \boxed{\cdots \oplus N_\ell} \subset M.$$

want to show $\ell \le n$.

$N^1$ not essential in $M$.

Claim: $N^1 \cap M_j = 0$. some $1 \le j \le n$.

Pf. Otherwise. $0 \ne N^1 \cap M_j \subseteq_e M_j$

$$\bigoplus_j N^1 \cap M_j \subseteq_e \bigoplus_j M_j \subseteq_e M$$

ess in $M$.

but $\bigoplus_j N^1 \cap M_j \subset N^1$. not ess. in $M$. $\times$

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Rename this $M_j$ by $M_1$.

i.e, $N^1 \cap M_1 = 0$

$$\boxed{M_1} \oplus \overbrace{M_2 \oplus \cdots \oplus M_n}$$

$$N_1 \oplus \boxed{N_2 \oplus \underbrace{\boxed{\cdots}}_{N^2}}$$

<u>Claim</u>: $N^2 \cap M_j = 0$, for some $2 \le j \le n$

Pf. otherwise

$$N^2 \supset \bigoplus_{j=2}^{n} N^2 \cap M_j \subseteq_e \bigoplus_{j=2}^{n} M_j$$

$$\boxed{N^2 \oplus M_1 \subseteq_e M}$$

$$\downarrow \qquad \downarrow$$

$$N^2 \subseteq_e M/M_1$$

impossible $N_2 \cap (N^2 \oplus M_1)$. $\Rightarrow$ claim.

<u>Repeat</u>

$$M_1 \oplus M_2 \oplus \cancel{M_3 \oplus \cdots}$$

$$N_1 \oplus N_2 \oplus \boxed{N_3 \oplus N^3}$$

If $\ell > n$. $M_1 \oplus \cdots \oplus M_n$

$$N_1 \oplus \cdots \oplus N_n \oplus (\underbrace{N_{n+1} \oplus \cdots}_{N^n})$$

$$N^n \cap (M_1 \oplus \cdots \oplus M_n) = 0.$$

$\Rightarrow$ $M_1 \oplus \cdots \oplus M_n$ not ess. in $M$. $\times$.

∎