Lecture 22 11/30

Localization

Right Ore set: $S \subset R$.

- S is multiplicative $(1 \in S)$.
- If $a \in R$, s.t. $sa = 0$ for some $s \in S$, then $as' = 0$ for some $s' \in S$.
- $\forall a \in R$, $\forall s \in S$.

$$sR \cap aS \neq \emptyset.$$

(can find $b \in R$, $t \in S$. s.t.

$$sb = a\cancel{t}.$$

$$\text{"}\ s^{-1}a = bt^{-1}\ \text{"}$$).

Theorem (Ore) $S \subset R$ right Ore set.

$\exists$ $\varphi : R \longrightarrow R'$ ring homo. s.t.

1) $\varphi(S)$ are invertible ;
2) R' is universal under condition 1).
3) every elt in R' has form $\varphi(a)\varphi(s)^{-1}$ some $a \in R$, $s \in S$.

4) $\ker \varphi = \{ a \in R \mid as = 0 \text{ for some } s \in S \}$

Pf: <u>Step 1</u>. Construct R' as an R-mod.

$$R' = \varinjlim_{\Sigma} R \cdot s^{-1}$$

$$\Sigma = \begin{cases} \text{obj:} & S \\ \text{morphisms:} & \mathrm{Hom}_{\Sigma}(s, s') = \{ x \mid sx = s' \} \end{cases}$$

filtered: •

$$\begin{tikzcd} s_1 \arrow[dr, "t"] & \\ & s \\ s_2 \arrow[ur, "a"] & \end{tikzcd}$$

s = common (right) multiple of s_1, s_2.

why? $s_1 S \cap s_2 R \neq \emptyset$.

$$S \ni s_1 \underset{\substack{\cap \\ S}}{t} = s_2 \underset{\substack{\cap \\ R}}{a}.$$

•

$$\begin{tikzcd} s_1 \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & s_2 \arrow[r, dashed, "z"] & s. \end{tikzcd}$$

st. $z \circ x = z \circ y \in \mathrm{Hom}(s_1, s)$.

why? $\begin{cases} s_1 x = s_2 \\ s_1 y = s_2 \end{cases}$

$s_1(x-y) = 0$.

$(x-y)\, t = 0$. some $t \in S$.

$t = z$, $s = s_2 \cdot z \in S$.

$$s \xrightarrow{x} s'$$

$$\begin{tikzcd} R\cdot s^{-1} \arrow[r] \arrow[d, equal] & R(s')^{-1} \arrow[d, equal] \\ R \arrow[r, "x"] & R \end{tikzcd}$$

$$a s^{-1} = (a x)\underset{\substack{\shortparallel \\ s'}}{(s x)^{-1}}$$

Every elt in $\varinjlim_{\Sigma} \underset{\sim\sim}{R\cdot s^{-1}}$ lies in the image of some $R\cdot s^{-1}$

write such elt as $a\cdot s^{-1}$. $\in R'$

$$a\cdot s^{-1} = (ax)\cdot(sx)^{-1} \in R'$$

$\forall\ x\in R$ s.t. $sx\in S$.

$$R' = (R\times S)\Big/ \underset{\substack{a\in R,\, s\in S.\\ \forall\ x\in R \text{ s.t. } sx\in S.}}{(a,s)\sim(ax,sx)}$$

<u>Step 2</u>. $S\subset R \circlearrowright R'$

Want: $\forall\ s\in S$. $R' \xrightarrow{s\cdot(-)} R'$
is bijective.

injective. $s\cdot(a\cdot t^{-1}) = 0 \in R'$. $s,t\in S$.

$(sa)\cdot t^{-1} = 0$.

$\exists\ x$ s.t. $tx\in S$.

$$s\underline{a x} = 0.$$

$$\Rightarrow \exists\ s' \in S.$$

$$a x s' = 0.$$

$$\Rightarrow \quad a t^{-1} = (a x s') \cdot (t x s')^{-1} = 0 \in R'.$$

(same argument shows

$$\ker\left(\begin{array}{c} R \longrightarrow R' \\ a \longmapsto a \cdot 1^{-1} \end{array} \right) = \left\{ a \in R \,\middle|\, \begin{array}{l} as = 0 \\ \text{some } s \in S \end{array} \right\}.$$

)

Surjectivity: $\forall \quad a \cdot t^{-1} \in R'.$

want to write

$$s \cdot (b \sigma^{-1}) = a \quad (\sigma \in S)$$

want $s^{-1} a = b\sigma^{-1}$

$$\left(\begin{array}{c} s \cdot (\underbrace{b \sigma^{-1} t^{-1}}) = a t^{-1} \\ \| \\ b (t\sigma)^{-1} \end{array} \right)$$

$$s R \cap a S \neq \emptyset.$$

$sb = a\sigma$ for some $b \in R$, $\sigma \in S$.

$$s(b\sigma^{-1}) = (sb)\sigma^{-1} = a \in R'.$$

$\Rightarrow \quad s^{-1} : R' \longrightarrow R'.$ is defined, $\forall\ s \in S.$

$$(st)^{-1} = t^{-1} \cdot s^{-1}.$$

<u>Step 3</u>. Define multiplication on R'.

$$(a\cdot s^{-1})\cdot(b\cdot t^{-1}) \overset{\text{def}}{=} a\cdot \underbrace{s^{-1}(bt^{-1})}_{\in R'}.$$

$$\begin{array}{ccc} R & \longrightarrow & \mathrm{End}_{\mathbb{Z}}(R') \\ \cup & & \cup \\ S & \longrightarrow & \text{invertible} \end{array}$$

$$R[S^{-1}] \longrightarrow \mathrm{End}_{\mathbb{Z}}(R').$$

R' is a cyclic $R[S^{-1}]$-mod.

Associativity:

$x, y \in R'$.

$$\begin{tikzcd} R' \arrow[r, "(-)x"] \arrow[rr, bend right, "(-)(xy)"'] & R' \arrow[r, "(-)y"] & R' \end{tikzcd}$$

~~both~~ all maps $\in \mathrm{End}_R(R')$

<u>Claim</u>: $\mathrm{Hom}_R(R', R') \hookrightarrow R'$

$$\varphi \longmapsto \varphi(1).$$

Assuming Claim, check asso, only need.

$$(1\cdot x)\, y = 1\,(xy). \checkmark$$

Pf of claim: $\quad \varphi(a \cdot s^{-1}) = a \boxed{\varphi(s^{-1})}$

$$S \cdot \varphi(s^{-1}) = \varphi(1) = \text{given.}$$

s is bijective on R'.

$\Rightarrow$ $\varphi(s^{-1})$ is uniquely determined by $\varphi(1)$.

$$\varphi(s^{-1}) = s^{-1}(\varphi(1)).$$

$\Rightarrow$ R' is a ring.

Universality:

$$\begin{tikzcd} R \arrow[r, "\psi"] \arrow[d, "\varphi"'] & R'' \\ R' = \varinjlim R \cdot s^{-1} \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

$\psi(S)$ are inv.

First construct R-linear

$$\tilde{\psi} : \varinjlim R \cdot s^{-1} \longrightarrow R''$$

$$a \cdot s^{-1} \longmapsto \psi(a) \cdot \psi(s)^{-1}.$$

($\tilde{\psi}$ is the unique R-linear map extending ψ)

Check $\tilde{\Psi}$ is a ring homo.

$$(a \cdot \boxed{s^{-1}) \cdot (b \cdot} t^{-1}) \longmapsto \Psi(a) \boxed{\Psi(s)^{-1} \Psi(b)} \Psi(t)^{-1}$$

$$\| \qquad\qquad\qquad\qquad \|?$$

$$a \cdot \boxed{\beta \sigma^{-1} \cdot} t^{-1} \longmapsto \Psi(a) \boxed{\Psi(\beta) \Psi(\sigma)^{-1}} \Psi(t)^{-1}$$

$b\sigma = s\beta$.

$\Downarrow$

$\Psi(b)\Psi(\sigma) = \Psi(s)\Psi(\beta)$

Ex. $R = k\langle x, \partial \rangle$. (domain)

① $S = \{x^n\}_{n \geq 0}$ is right Ore set.

Need: $x^n R \cap f(x, \partial) \cdot S \neq \emptyset$.

i.e, given any $f(x, \partial) \in R$, and $n \in \mathbb{N}$.

want to find $m \gg n$

$$f(x, \partial) x^m \stackrel{?}{=} x^n (\cdots)$$

$f(x, \partial) = \partial^i \qquad m > i$

$$\partial^i x^m = x^{m-i}(\cdots).$$

② $S = R \setminus \{0\}$

(in general. may consider any domain R.

$S = R \setminus \{0\}$.

If S is a right Ore set, R is called an (right) Ore domain)

Need to check: $\forall\ a, b \in R \setminus 0$.

$$\underline{\underline{a}}\,R \cap \underline{\underline{b}}\,R \neq \{0\}.$$

(equivalently, any two nonzero right ideals have $\neq 0$ intersection).

$f, g \in k\langle x, \partial\rangle$. nonzero.

what if

$$f R \cap g R = 0. \quad ?$$

$$\boxed{f R \oplus g R \subset R}.$$

R is filtered by total deg of x, ∂.

$\underline{\deg f \leq n}$. well-defined.

$$\boxed{\partial x - x \partial = 1.}$$

$\deg f = a$.
$\deg g = b$.

$$f(-) : R_{\leq n-a} \hookrightarrow R_{\leq n}.$$

$$g(-) : R_{\leq n-b} \hookrightarrow R_{\leq n}.$$

$$f \cdot R_{\leq n-a} \oplus g \cdot R_{\leq n-b} \subset R_{\leq n}.$$

$\dim R_{\leq n} = \dim k[x, y]_{\leq n} \sim \frac{1}{2} n^2$

$(\because \quad gr_\bullet R \simeq k[x, y])$.

$$\frac{1}{2} n^2 + \frac{1}{2} n^2 \cancel{\leq} \frac{1}{2} n^2$$

Question: R domain.

Can we find an embedding

$$R \hookrightarrow D = \text{division ring}\,?$$

- $\exists$ R domain, not embeddable to any div. ring.

- $\exists$ R domain, there are different minimal embeddings

$$\begin{tikzcd} R \arrow[r, hook] \arrow[dr, hook] & D \arrow[d, "\times" marking] \\ & D' \end{tikzcd}$$

Thm (Goldie) R is a right Noetherian domain. Then $S = R \setminus \{0\}$ is a right Ore set, and $Q(R) =$ loc. of R wrt S is a division ring.

Malcev's Example.

$$R = \frac{\mathbb{Q}\left\langle \begin{matrix} a, b, c, d \\ x, y, u, v \end{matrix} \right\rangle}{\left(\begin{matrix} ax - by \\ cx - dy \\ au - bv \end{matrix} \right)}$$

missing $\boxed{cu = dv}$.

$cu = dv$ does not hold in $R = \bigoplus_{n \geq 0} R_n$

Fact: R is a domain.

If $R \hookrightarrow D$ = div. ring.

$a, \cdots u, v \longmapsto D^\times$

$cu = dv$ would be implied by the other 3 equations. $\times$.

Ex. $k\langle x, y \rangle$.

$R_\sigma = k(t)\langle x; \sigma \rangle$. $\quad \sigma : k(t) \to k(t)$. (field homo)

$xa = \sigma(a)x$. $\quad \sigma(t) = t^n$.
$\forall\, a \in k(t)$.

domain, Euclidean algorithm on one side.

$\Rightarrow R_\sigma$ is a principal left ideal domain. (left noeth).

$$k\langle x, y \rangle \xrightarrow{\varphi_\sigma} R_\sigma \subset Q(R_\sigma) = D_\sigma \text{ div. ring.}$$

$x \longmapsto x$

$y \longmapsto tx$

$\sigma(t) = t^n$ $(n > 1)$. $\quad \varphi_\sigma$ is injective.

$$k\langle x,y\rangle \xrightarrow{\varphi_\sigma} D_\sigma \text{ is minimal.}$$

$$x \longmapsto x$$
$$y \longmapsto tx$$

$$t = \varphi_\sigma(y)\,\varphi_\sigma(x)^{-1}.$$

$$\varphi(x)\,t = t^n \varphi(x).$$

$\rightsquigarrow$ eqn of $\varphi(x)$ and $g(y)$.

different n give diff ~~?~~ $\varphi_\sigma : k\langle x,y\rangle \longrightarrow D_\sigma$.

<u>Thm.</u> (Goldie) R = semiprime (analogue of reduced rings in comm alg.) right Noetherian.

Then S = regular elts in R.

(s is reg. if
$as = 0 \Rightarrow a = 0$
$sb = 0 \Rightarrow b = 0$).

~~?~~ is an right Ore set.

and $Q(R) = R_S$ is a semisimple ring.

$$\left(\sim \prod_{i=1}^{r} M_{n_i}(D_i) \right).$$