Lecture 2 9/9

More examples of rings:
- variants of $\otimes$-alg;
- Ug.
- skew polynomial ring;
- division rings.

Modules.
- Free modules, IBN.
- Schur's Lemma
- Jordan-Hölder Theorem,
- acc/dcc conditions.

Last time: $S_A(M)$ sym. alg.
$A =$ comm ring
M : A. mod.

$$\bigoplus_{i \geq 0} \Lambda^i_A(M) = \Lambda_A(M) = \text{graded commutative}$$

$$a \in \Lambda^i_A(M), \quad b \in \Lambda^j_A(M)$$

$$\Rightarrow \quad a \cdot b = (-1)^{i \cdot j}\, b \cdot a.$$

$\boxed{\text{Deforming these examples:}}$

$k =$ field, $V =$ vector space$/k$.

$$S(V) = T(V) \Big/ \left(\underset{x,y \in V}{xy - yx} \right)$$

add lower order terms here.

<u>Ex.</u>

$$\to \underbrace{k\langle x, y\rangle}_{\substack{\parallel \\ T(V)}} \Big/ \left(xy - yx - 1 \right)$$

$V = k\cdot x \oplus k\cdot y$. $\leftarrow$ $\omega(x,y) = 1$.

In general, take alternating form.

$$\omega : V \times V \longrightarrow k$$

$$\omega(v, v) = 0.$$

$$Aut(V,\omega) \circlearrowright T(V) \Big/ \left(\underset{\forall\, x,y \in V.}{xy - yx - \omega(x,y)} \right)$$

<u>Weyl algebra</u> attached to (V, ω)

usual defn of Weyl alg:

$$A_n(k) = k\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle \Big/ \begin{pmatrix} x_i y_i - y_i x_i - 1 \\ x_i x_j - x_j x_i \\ y_i y_j - y_j y_i \\ x_i y_j - y_j x_i \\ (i \neq j) \end{pmatrix}$$

$\circlearrowright$ $Sp_{2n}(k)$

$$\left(\longleftrightarrow \begin{array}{l} V = k\cdot x_1 \oplus \cdots \oplus k\cdot y_n \\ \omega(x_i, y_j) = \delta_{ij} \\ \text{----} \end{array} \right)$$

Clifford Alg: V / k.

$$\Lambda(V) = T(V) \Big/ \left(\underset{\forall\, x \in V}{x \cdot x} \right)$$

add constant to it.

$q : V \longrightarrow k$ quadratic form.

$$Cl(V,q) = T(V) \Big/ \Big(\underbrace{\boxed{x \cdot x} - q(x)}_{\forall x \in V} \Big)$$

Clifford algebra.

$$\dim_k Cl(V,q) = 2^n \qquad (n = \dim_k V)$$

(Spin groups).

$(\mathfrak{g}, [\cdot,\cdot])$ Lie algebra/k.

$$U(\mathfrak{g}) = T(\mathfrak{g}) \Big/ \Big(\underbrace{\boxed{xy - yx} - \boxed{[x,y]}}_{\forall x, y \in \mathfrak{g}} \Big)$$

$[x,y] \in \mathfrak{g}$.

universal enveloping algebra of $\mathfrak{g}$.

also a deformation of $S(\mathfrak{g})$.

$$(U\mathfrak{g})_{\le n} = \text{image of } \bigoplus_{0 \le i \le n} \mathfrak{g}^{\otimes i} \subset T(\mathfrak{g})$$

in $U\mathfrak{g}$.

$$k = (U\mathfrak{g})_{\le 0} \subset (U\mathfrak{g})_{\le 1} \subset (U\mathfrak{g})_{\le 2} \subset \cdots \subset U\mathfrak{g}.$$

$$(U\mathfrak{g})_{\le i} / (U\mathfrak{g})_{\le i-1} \xrightarrow{\sim} S^i(\mathfrak{g}).$$

Same for Weyl algebra/ Clifford alg.

graded alg $\rightsquigarrow$ filtered algebra

$$R = \oplus R_i \longrightarrow R' = \bigcup R'_{\le i}$$

$$R_i \xleftarrow{\sim} R'_{\le i} / R'_{\le i-1}$$

$$\underline{R'_{\le i} \cdot R'_{\le j} \subset R'_{\le i+j}}$$

$\boxed{\text{Skew polynomial ring}}$ (Hilbert)

R ring.

$R[x] \leftarrow \qquad a x \neq x a \qquad a \in R$, x

give rule :

$$x \underset{\substack{\uparrow \\ R}}{a} = \underset{\substack{\uparrow \\ R}}{\varphi(a)}\, x$$

$\varphi : R \longrightarrow R$. ring homo.

$$R\langle x ; \varphi \rangle = \left\{ a_0 + a_1 x + \cdots + a_n x^n \;\middle|\; \begin{array}{l} n \geq 0. \\ a_i \in R \end{array} \right\}.$$

$$\begin{aligned}
&(a \cdot x^n)(b x^m) \\
&= a(x^n \cdot b) x^m \\
&= a \quad x^{n-1} \cdot (x b) \; x^m \\
&= a \left(x^{n-1} \cdot \varphi(b)\right) x^{m+1} \\
&= a \left(x^{n-2} \cdot \varphi(\varphi(b))\right) \cdot x^{m+2} \\
&\;\;\vdots \\
&= \left(a \cdot \varphi^n(b)\right) x^{m+n}.
\end{aligned}$$

Example: $\mathrm{char}(k) = p$.

$\varphi : k \longrightarrow k$

$\boxed{x \longmapsto x^p}$. (field endo)

$k\langle x;\varphi\rangle$

$$(ax^n)(bx^m) = a\cdot b^{p^n}\cdot x^{n+m}.$$

<u>Ex</u>. $k\langle x,y\rangle/(xy-qyx)$ $(q\in k^\times)$

"quantum plane".

$(q=1.$ $\underline{k[x,y]}$ $)$

$R=k[x]$

$\varphi: R\longrightarrow R$

$x\longmapsto q^{-1}x$

$$R\langle y;\varphi\rangle = k\langle x,y\rangle/(xy-qyx).$$

<u>related</u>: $A_1(k)=k\langle x,y\rangle/(xy-yx-1)$.

$R=k[x]$.

$A_1(k)=R\langle y;\delta\rangle$ see below.

<u>Variant</u>: R, add variable x.

$$xa = ax+\underset{\substack{\cap\\ R}}{\delta(a)}.$$

$\delta: R\longrightarrow R.$

$a, b \in R.$

$$x(ab) = (ax + \delta(a))\cdot b$$
$$= axb + \delta(a)b$$
$$= abx + \underbrace{\boxed{a\,\delta(b) + \delta(a)b}}_{= \delta(ab)}$$

δ should satisfy:

$$\delta(ab) = a\,\delta(b) + \delta(a)\,b. \quad (\text{Leibniz})$$
$$\forall\ a, b \in R.$$

(a derivation on R)

$R\langle x\,;\,\delta\rangle$

Weyl algebra: $R = k[x]$.

add variable y

$$\delta : \underset{k[x]}{\overset{\parallel}{R}} \longrightarrow \underset{k[x]}{\overset{\parallel}{R}}.$$
$$f(x) \longmapsto f'(x).$$

$R\langle y\,;\,\delta\rangle = A_1(k).$

Combine φ & δ: $R\langle x\,;\,\varphi, \delta\rangle$

$$xa = \varphi(a)x + \delta(a).$$

δ should be a φ-derivation.

$\boxed{\text{Division rings:}}$

R is a division ring if $\forall\ r \in R \setminus \{0\}$ is invertible.

$a \in R$.

Left invertible: $\exists\, b \in R$, s.t. $ba = 1$.
Right inv. $\exists\, b \in R$. st $ab = 1$.

left inv $\not\Leftrightarrow$ right inv.

invertible: both left & right invertible.

$$b_\ell a = 1, \qquad a b_r = 1.$$

$$b_\ell = b_\ell (a b_r) = (b_\ell \cdot a) \cdot b_r = b_r.$$

Ex: Comm. div rings = fields.

non-comm. ex: $\mathbb{H} = \mathbb{C} \oplus \mathbb{C} \cdot j$

$$\begin{cases} j^2 = -1 \\ aj = j\bar{a} \qquad \forall\, a \in \mathbb{C}. \end{cases}$$

$$\mathbb{R} \oplus \mathbb{R} i \oplus \mathbb{R} j \oplus \mathbb{R} k$$

$$ij = k = -ji, \quad \text{etc.}$$

Ex: k, $a, b \in k^\times$.

quaternion algebra (central simple alg/k).

$$k \oplus k \cdot x \oplus k \cdot y \oplus k \cdot z$$

$$x^2 = a, \quad y^2 = b, \quad z^2 = -ab$$

$$xy = z = -yx, \quad \ldots,$$

$\boxed{\text{Modules}}$. $R \circlearrowright M$. ← Notation for left R-mod

simple module: only submodules are 0, M.

$\underline{k[G]}$-mod $=$ representations of G in k-vector spaces.

$M \longleftrightarrow M$

simple $k[G]$-mod $\longleftrightarrow$ irred reps.

<u>Thm</u> (Schur's Lemma): M: simple R-mod.

then $\mathrm{End}_R(M)$ is a division ring.

<u>Pf</u>. Need to show: any $\varphi : M \xrightarrow[\text{(R-linear)}]{} M$ <u>nonzero</u>

is invertible. i.e., $\begin{cases} \mathrm{Ker}(\varphi) = 0 \\ \mathrm{Im}(\varphi) = M. \end{cases}$

$\mathrm{Im}(\varphi) \subset M$ submod. $\neq 0$.

M simple $\Rightarrow$ $\mathrm{Im}(\varphi) = M$. ∎

$R = k$-algebra. $k =$ field.

$\mathrm{End}_R(\underbrace{M}_{\text{simple}}) =$ k-algebra, division ring.

When can we conclude $\mathrm{End}_R(M) = \underline{k}$.

$\boxed{k = \text{alg. closed.}}$

(still has transcendental extn)

$\underline{\text{Ex}}$:

$G =$ finite

$k =$ alg. closed.

M: irred G-rep.

$$\Longrightarrow \quad \underbrace{\mathrm{End}_{k[G]}(M)}_{\text{finite-dim'l } k\text{-v.s.}} = k.$$

$D = \mathrm{End}_{k[G]}(M)$ div. alg$/k$

$\dim_k D < \infty$.

Claim: $D = k$.

$x \in D \setminus k. \Rightarrow f(x) = 0$ for some poly. f, $\deg f > 1$.

$$f(x) = \prod \underbrace{(x - a_i)}_{\text{one of them } = 0}$$

$$\Longrightarrow \quad \mathrm{End}_{k[G]}(M) = k.$$

$G =$ countably infinite. gp

$k = \mathbb{C}$.

M: irred G-rep $/\mathbb{C}$

$\underline{\text{Claim}}$: $\mathrm{End}_{\mathbb{C}[G]}(M) = \mathbb{C}$

(finite) Composition series of M.

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M.$$

M_i / M_{i-1} is simple R-mod.

$i = 1, \ldots, n.$

$R = \mathbb{Z}$, $M = \mathbb{Z}$, no comp. series.

$$\cdots \subset pq\mathbb{Z} \subset \underbrace{p\mathbb{Z} \subset \mathbb{Z}}$$

M has comp. series $\Longleftrightarrow$ M has finite length

$$\boxed{\mathrm{leng}(M) = n.}$$

__Thm__ (Jordan-Hölder) Two comp. series

$$\begin{array}{c} 0 \subset M_1 \subset M_2 \subset \cdots \subset M_a = M \\ \| \\ 0 \subset M_1' \subset M_2' \subset \cdots \subset M_b' = M. \end{array}$$

__weak version__:

- $a = b$.
- $\{M_i / M_{i-1}\} = \{M_i' / M_{i-1}'\}$

as multi-sets of elements in isom. classes of simple R-mod.

Strong version:

$\exists$ canonical bijection

$$\sigma : \{1, 2, \dots, a\} \xrightarrow{\sim} \{1, 2, \dots, b\}.$$

and canon. isom

$$M_i / M_{i-1} \xrightarrow{\sim} M'_{\sigma(i)} / M'_{\sigma(i)-1}.$$

Special case: $R = k$.

(flags) $\quad 0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$

$0 \subset M'_1 \subset \cdots \subset M'_{n-1} \subset M'_n = M.$

$\rightsquigarrow \quad \sigma \in S_n$ (rel. pos. of the two flags)

$\dim M = 2$

$0 \subset M_1 \subset M$

$0 \subset M'_1 \subset M$

$M_1 = M'_1 \implies \sigma = id$

$M_1 \neq M'_1 \implies \sigma = (12).$

Pf:

$$S_{i,j} = \frac{M_i \cap M'_j}{M_i \cap M'_{j-1} + M_{i-1} \cap M'_j}$$

$1 \leq i \leq a, \quad 1 \leq j \leq b.$

Claim. - Fix i, there's a unique j s.t. $S_{i,j} \neq 0$

- Fix j. —— " —— i s.t. $S_{i,j} \neq 0$

define $\sigma(i)$ to be unique j s.t. $S_{i,j} \neq 0$.

Claim

$$M_i / M_{i-1} \xleftarrow[\text{can}]{\sim} S_{i,\sigma(i)} \xrightarrow[\text{can}]{\sim} M'_{\sigma(i)} / M'_{\sigma(i)-1}$$