Lecture 14 10/21

- Category theory
- Morita equivalence.

Adjoint functors

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

Adjunction data:

$\forall\ X \in \mathcal{C},\ Y \in \mathcal{D}$.

$$\alpha_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, G(Y)).$$

+ comp. with maps between X's & Y's.

Get natural transf

$$\mathrm{id}_{\mathcal{C}} \xRightarrow{\lambda} G \circ F : \mathcal{C} \to \mathcal{C}$$

$\forall\ X \in \mathcal{C}$.

$$\begin{array}{c} (X \longrightarrow GF(X)) \\ \updownarrow \alpha_{X,F(X)} \\ (F(X) \overset{\mathrm{id}}{=\!=} F(X)) \end{array}$$

Similarly, $F \circ G \overset{\mu}{\Longrightarrow} id_{\mathcal{D}}$

$$\left(F\, G(Y) \longrightarrow Y \right)$$

$$\Updownarrow$$

$$\left(G(Y) \overset{id}{=\!=} G(Y) \right)$$

$$\begin{tikzcd} G \circ F \circ G \arrow[r, Rightarrow, bend left, "G \circ \mu"] & G \arrow[l, Rightarrow, bend left, "\lambda \circ G"] \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, Rightarrow, "\lambda \circ G"] \arrow[rr, equal, bend right, "id"'] & GFG \arrow[r, Rightarrow, "G \circ \mu"] & G \end{tikzcd}$$

equality of nat trans.

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F \circ \lambda"] \arrow[rr, equal, bend right, "id"'] & FGF \arrow[r, Rightarrow, "\mu \circ F"] & F \end{tikzcd}$$

adjunction data $\alpha_{X,Y}$ can be recovered from

$id_{\mathcal{C}} \Rightarrow GF$

$FG \Rightarrow id_{\mathcal{D}}$.

Alternatively: An adjunction is a pairing

$$h: \mathcal{C}^{op} \times \mathcal{D} \longrightarrow \underline{Set}$$

(contrav. in $\mathcal{C}$, covariant in $\mathcal{D}$).

satisfying

- $\forall\ Y \in \mathcal{D}$,

$$h(-, Y) : \mathcal{C}^{op} \longrightarrow \underline{Set}$$

is representable by an obj in $\mathcal{C}$.

(recall Yoneda

$$\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \underline{Set})$$

$$X \mapsto h_X = \mathrm{Hom}_{\mathcal{C}}(-, X).$$

~~any~~ a functor $\varphi \in \mathrm{Fun}(\mathcal{C}^{op}, \underline{Set})$ is called representable, if $\varphi \simeq h_X$ for some $X \in \mathcal{C}$.

nat. transf. that is an isom.)

- $\forall\ X \in \mathcal{C}$.

$$h(X, -) : \mathcal{D} \longrightarrow \underline{Set}$$

is corepresentable by an obj in $\mathcal{D}$.

(i.e, this functor is $\cong$ to something in the image of

$$\mathcal{D}^{op} \hookrightarrow \mathrm{Fun}(\mathcal{D}, \underline{Set}).$$

$$Y \mapsto {}_Y h = \mathrm{Hom}_{\mathcal{D}}(Y, -)$$

corep. by Y.)

To recover $F : \mathcal{C} \longrightarrow \mathcal{D}$ from the pairing.

$\forall \; X \in \mathcal{C}$.

$$h(X,-) \overset{\tau}{\cong} {}_{Y}h$$

for some $Y \in \mathcal{D}$.

$$F(X) := Y.$$

__Check__: (Y, τ) is unique up to __unique__ isom.

i.e., (Y_1, τ_1)

(Y_2, τ_2)

$${}_{Y_1}h \xrightarrow[\tau_1]{\simeq} h(X,-) \xleftarrow[\tau_2]{\simeq} {}_{Y_2}h$$

then $\exists!$ isom $Y_1 \xrightarrow[\sim]{\alpha} Y_2$

s.t.

$$\begin{tikzcd}
{}_{Y_1}h \arrow[dr, "\wr"'] & & {}_{Y_2}h \arrow[ll, "(-)\circ\alpha"'] \arrow[dl, "\wr"] \arrow[dll, phantom, "\circlearrowright"] \\
& h(X,-) &
\end{tikzcd}$$

Equivalence of categories.

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

Say: "F is an equiv."

$$\mathcal{D} \xrightarrow{G} \mathcal{C}$$

$$FG \overset{\lambda}{\cong} id_{\mathcal{D}}$$

$$GF \overset{\mu}{\cong} id_{\mathcal{C}}.$$

$\lambda \circ id_F : (FG)F \Rightarrow id_D \circ F = F$

$\cancel{\lambda F}$:

+ conditions

$$F \xRightarrow{\lambda^{-1}} FGF \xRightarrow[\sim]{\lambda \circ id_F = F\mu} F$$

$$GFG \xRightarrow[G\lambda = \mu G]{} G$$

($\Rightarrow$ $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$

$G : \mathcal{D} \rightleftarrows \mathcal{C} : F$ both are adjunctions)

Lemma: $F : \mathcal{C} \longrightarrow \mathcal{D}$

is part of an equivalence.

$\Longleftrightarrow$ F is fully faithful ($\cong$ on Hom).
essentially surjective
i.e., $\forall\ Y \in \mathcal{D}$.
$Y \cong F(X)$. for some $X \in \mathcal{C}$.

$\Rightarrow$ ✓

$\Leftarrow$ $\forall\, Y.$ $\quad Y \overset{\tau}{\cong} F(X).$ $\quad (X \in \mathcal{C})$

want to define $G(Y) := X.$

Need to check: (X, τ) is unique up to a unique isom.

$$F(X_1) \underset{\tau_1}{\xrightarrow{\sim}} Y \underset{\tau_2}{\xleftarrow{\sim}} F(X_2)$$

Then $\exists!\ \alpha : X_1 \xrightarrow{\sim} X_2$ in $\mathcal{C}$

making

$$\begin{tikzcd} F(X_1) \arrow[rr, "F(\alpha)"] \arrow[dr, "\tau_1"'] & \circlearrowleft & F(X_2) \arrow[dl, "\tau_2"] \\ & Y & \end{tikzcd}$$

Then

$$\begin{tikzcd} Y_1 \arrow[r, "g"] \arrow[d, dashed, no head] & Y_2 \arrow[d, dashed, no head] \\ (X_1, \tau_1) & (X_2, \tau_2) \end{tikzcd}$$

Want to define $X_1 \overset{f}{\dashrightarrow} X_2$

$\Updownarrow$ $F(f)$ is a bijection.

$$\begin{tikzcd} F(X_1) \arrow[r, dashed, "\exists!\, F(f)"'] \arrow[d, "\tau_1"', "\wr"] & F(X_2) \arrow[d, "\tau_2"', "\wr"] \\ Y_1 \arrow[r, "g"'] \arrow[ur, phantom, "\circlearrowleft"] & Y_2 \end{tikzcd}$$

Rk. Inverse to F G
(together with $GF \overset{\lambda}{\cong} id$, $FG \overset{\mu}{\cong} id$).
is unique up to a unique nat. transf.

$$FG \underset{\sim}{\overset{\lambda}{\Longrightarrow}} id_D$$

$$GF \underset{\sim}{\overset{\mu}{\Longrightarrow}} id_C$$

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\lambda^{-1}"] \arrow[rr, Rightarrow, bend right, "id"'] & FGF \arrow[r, Rightarrow, "\mu"] & F \end{tikzcd}$$

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\lambda^{-1}"'] \arrow[rr, Rightarrow, bend left=40, "id"] & FGF \arrow[r, Rightarrow, bend left, "\lambda"] \arrow[r, Rightarrow, bend right, "\mu"'] & F \end{tikzcd}$$

Morita equivalence.

Def. R, S rings

$R \underset{\text{Morita}}{\sim} S$ if $\exists$ an equiv. of categories

$$F: R\text{-mod} \xrightarrow{\sim} S\text{-mod}.$$

Ex. $R \underset{\text{Morita}}{\sim} M_n(R)$

$$F: R\text{-mod} \longrightarrow M_n(R)\text{-mod}$$
$$M \longmapsto M^{\oplus n}$$

$$M_n(R) \circlearrowright \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in M^{\oplus n} \right\}.$$

$$G: M_n(R)\text{-mod} \longrightarrow R\text{-mod}.$$
$$N \longmapsto \underline{\underline{e_{11} N}}$$

$$e_{11} = \begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & \ddots \\ & & & 0 \end{pmatrix}.$$

check: $N \xmapsto{G} e_{11} N \xmapsto{F} (e_{11} N)^{\oplus n} \cong N$

$$N = e_{11}N \oplus e_{22}N \oplus \cdots \oplus e_{nn}N.$$

$$\left(\text{because} \quad \begin{array}{l} e_{ii} \text{ idempotent} \\ \sum e_{ii} = 1_n \end{array} \right)$$

need canonical

$$\begin{tikzcd} e_{ii}N \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & e_{jj}N \arrow[dl, "\sim"] \\ & e_{kk}N & \end{tikzcd}$$

$$e_{ji} : e_{ii}N \rightleftarrows e_{jj}N : e_{ij}.$$

Morita equiv. for comm rings

$$A \underset{\text{Morita}}{\sim} B \iff A \cong B.$$

$$A\text{-mod} \cong B\text{-mod}.$$

we can recover A from A-mod as the center

Def: $\mathcal{C}$ category

$$Z(\mathcal{C}) = \mathrm{End}(\mathrm{id}_{\mathcal{C}})$$

$$= \{ \text{natural transf } \mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}} \}$$

$$= \left\{ (z_X)_{X \in \mathcal{C}} \;\middle|\; \begin{array}{l} z_X : X \to X \text{ in } \mathcal{C}. \\ \text{s.t.} \end{array} \right.$$

$\forall \quad X \xrightarrow{f} Y$ in $\mathcal{C}$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "z_X"'] & Y \arrow[d, "z_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$Z(\mathcal{C})$ = comm. ring.

$z = \{z_X\}$, $z' = \{z'_X\}$

$$\begin{tikzcd} X \arrow[r, "z'_X"] \arrow[d, "z_X"'] & X \arrow[d, "z_X"] \\ X \arrow[r, "z'_X"] & X \end{tikzcd}$$

$(z \circ z')_X = (z' \circ z)_X$

$Z(\underline{\underline{A\text{-mod}}}) = A$. $\quad (A = \text{comm.})$.

$Z(R\text{-mod}) = \underline{\underline{Z(R)}}$.

$R \sim M_n(R)$

same center $Z(R)$.

R. M = left R-mod

$$R\text{-mod} \longrightarrow \{\text{abelian gps}\} = \underline{Ab}$$
$$N \longmapsto \mathrm{Hom}_R(M, N)$$

$\mathrm{Hom}_R(M, -)$ is a right module for $\mathrm{End}_R(M)$

The above functor lifts to:

$$ {}_M h : R\text{-mod} \longrightarrow \text{mod-}\mathrm{End}_R(M).$$
$$\cong$$
$$\mathrm{End}_R(M)^{op}\text{-mod}.$$

<u>Defn</u> A <u>progenerator</u> in R-mod is a f.g. projective module P s.t. $\exists$ surjection $P^{\oplus I} \twoheadrightarrow R$.

(can be chosen to be finite.)

<u>Thm</u> Let P be a progenerator of R-mod. $S = \mathrm{End}_R(P)^{op}$.

Then

$$\mathrm{Hom}_R(P,-) : R\text{-mod} \longrightarrow S\text{-mod}$$

is an equivalence.

<u>Thm</u>. Any morita equiv. arises this way.

$$R \sim S$$