

- Category theory
- Morita equivalence.

### Adjoint functors

$$F: \mathcal{C} \rightleftarrows \mathcal{D} : G$$

Adjunction data:

$$\forall x \in \mathcal{C}, y \in \mathcal{D}.$$

$$\alpha_{x,y} : \text{Hom}_{\mathcal{D}}(F(x), y) \cong \text{Hom}_{\mathcal{C}}(x, G(y)).$$

+ comp. with maps between  $X$ 's &  $Y$ 's.

Get natural transf

$$\text{id}_{\mathcal{C}} \xRightarrow{\lambda} G \circ F : \mathcal{C} \rightarrow \mathcal{C}$$

$$\forall x \in \mathcal{C}.$$

$$\left( x \longrightarrow \textcircled{G} F(x) \right)$$

$$\uparrow \alpha_{x, F(x)}$$

$$\left( F(x) \xlongequal{\text{id}} F(x) \right)$$

Similarly,  $F \circ G \xRightarrow{\mu} id_{\mathcal{D}}$

$$\left( \textcircled{F} \circ G(Y) \longrightarrow Y \right)$$



$$\left( G(Y) \xRightarrow{id} G(Y) \right)$$

$$G \circ F \circ G \xrightleftharpoons[G \circ \mu]{G \circ \lambda} G$$

$$G \xRightarrow{\lambda \circ G} GFG \xRightarrow{G \circ \mu} G$$

equality of nat trans.

$$\xrightleftharpoons[id]{id} G$$

$$F \xRightarrow{F \circ \lambda} FGF \xRightarrow{\mu \circ F} F$$

$$\xrightleftharpoons[id]{id} F$$

adjunction data  $\alpha_{x,y}$  can be recovered from

$$id_{\mathcal{C}} \Rightarrow GF$$

$$FG \Rightarrow id_{\mathcal{D}}$$

Alternatively: An adjunction is a pairing

$$h: \mathcal{C}^{op} \times \mathcal{D} \longrightarrow \underline{Set}$$

(contrav. in  $\mathcal{C}$ , covariant in  $\mathcal{D}$ ).

satisfying

- $\forall Y \in \mathcal{D}$ ,

$$h(-, Y) : \mathcal{C}^{\text{op}} \longrightarrow \underline{\text{Set}}$$

is representable by an obj in  $\mathcal{C}$ .

(recall Yoneda

$$\mathcal{C} \xleftrightarrow{\quad} \text{Fun}(\mathcal{C}^{\text{op}}, \underline{\text{Set}})$$

$$X \mapsto h_X = \text{Hom}_{\mathcal{C}}(-, X).$$

~~any~~  
a

functor  $\varphi \in \text{Fun}(\mathcal{C}^{\text{op}}, \underline{\text{Set}})$

is called representable,

if  $\varphi \cong h_X$  for some  $X \in \mathcal{C}$ .

$\uparrow$   
nat. transf. that is an isom.)

- $\forall X \in \mathcal{C}$ .

$$h(X, -) : \mathcal{D} \longrightarrow \underline{\text{Set}}$$

is corepresentable by an obj in  $\mathcal{D}$ .

(i.e., this functor is  $\cong$  to

something in the image of

$$\mathcal{D}^{\text{op}} \xleftrightarrow{\quad} \text{Fun}(\mathcal{D}, \underline{\text{Set}}).$$

$$Y \mapsto h_Y = \text{Hom}_{\mathcal{D}}(Y, -)$$

$\uparrow$   
corep. by  $Y$ .

To recover  $F: \mathcal{C} \rightarrow \mathcal{D}$   
from the pairing.

$\forall x \in \mathcal{C}$ .

$$h(x, -) \cong \overset{\tau}{Y} h$$

for some  $Y \in \mathcal{D}$ .

$$F(x) := Y.$$

check:  $(Y, \tau)$  is unique up to unique isom.

i.e.,  $(Y_1, \tau_1)$

$(Y_2, \tau_2)$

$$Y_1 h \xrightarrow[\tau_1]{\cong} h(x, -) \xleftarrow[\tau_2]{\cong} Y_2 h$$

then  $\exists!$  isom  $Y_1 \xrightarrow[\cong]{\alpha} Y_2$

s.t.

$$\begin{array}{ccc}
 Y_1 h & \xleftarrow{(-) \cdot \alpha} & Y_2 h \\
 \downarrow \wr & \text{G} & \downarrow \wr \\
 & h(x, -) & 
 \end{array}$$

# Equivalence of categories.

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

Say: "F is an equiv."

$$\mathcal{D} \xrightarrow{G} \mathcal{C}$$

$$FG \cong \overset{\lambda}{\leftarrow} id_{\mathcal{D}}$$

$$GF \cong \overset{\mu}{\leftarrow} id_{\mathcal{C}}$$

$$\lambda \circ id_{\mathcal{C}} : \underline{(FG)F} \Rightarrow \underline{id_{\mathcal{D}} \circ F}$$

$$\parallel$$

$$F$$

+ conditions

$$F \xrightarrow{\lambda^{-1}} FG F \xrightarrow[\sim]{\lambda \circ id_{\mathcal{C}} = F\mu} F$$

$$G F G \xrightarrow[G\lambda = \mu G]{} G$$

$$\left( \Rightarrow F: \mathcal{C} \rightleftharpoons \mathcal{D} : G$$

$$G: \mathcal{D} \rightleftharpoons \mathcal{C} : F \quad \text{both are adjunctions}$$

Lemma:  $F: \mathcal{C} \rightarrow \mathcal{D}$

is part of an equivalence.

$$\Leftrightarrow F \text{ is } \left\{ \begin{array}{l} \text{fully faithful} \\ \text{essentially surjective} \end{array} \right.$$

$\cong$  ( $\cong$  on Hom).

i.e.,  $\forall Y \in \mathcal{D}$ .

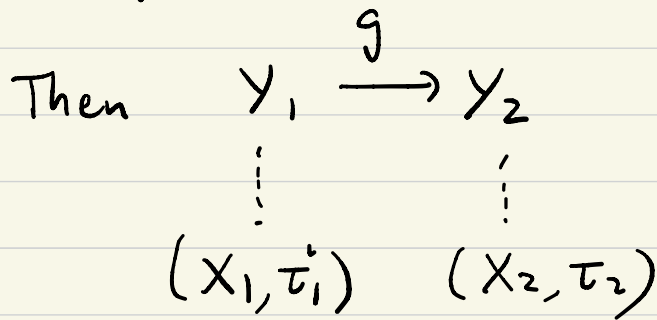
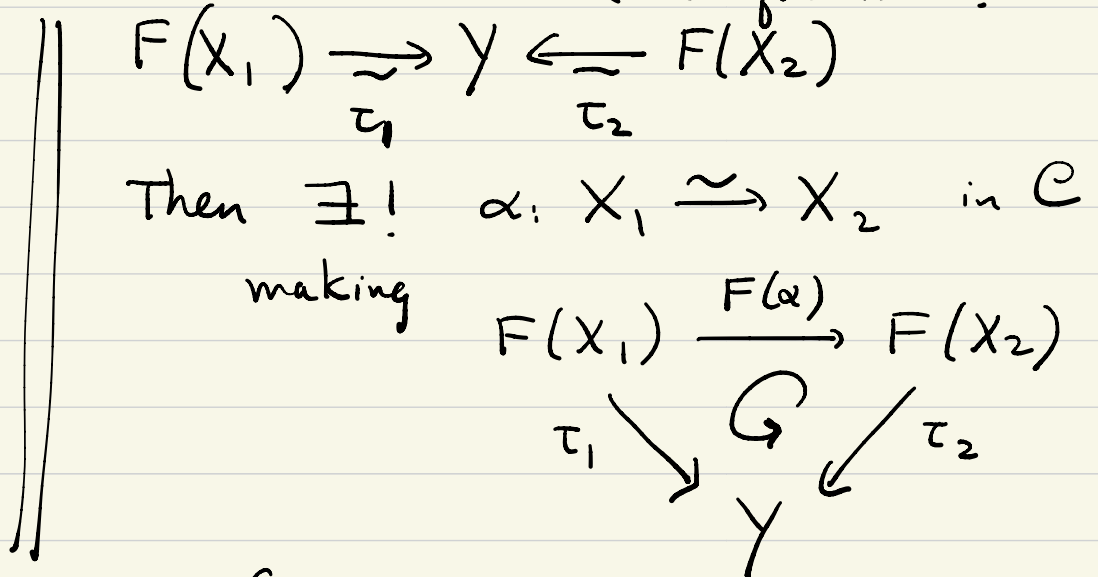
$$Y \cong F(X) \text{ for some } X \in \mathcal{C}.$$

$\Rightarrow$  ✓

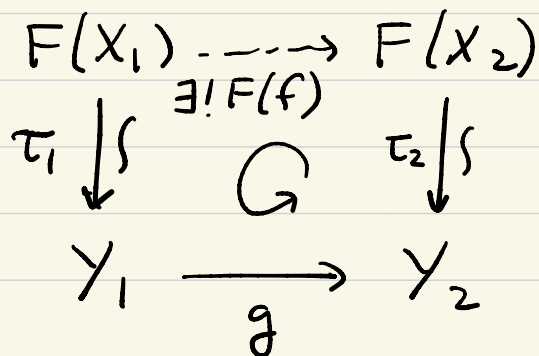
$$\Leftarrow \forall Y. \exists X. Y \cong F(X). \quad (X \in \mathcal{C})$$

want to define  $G(Y) := X$ .

Need to check:  $(X, \tau)$  is unique up to a unique isom.



Want to define  $X_1 \xrightarrow{f} X_2$   
 $\uparrow \downarrow F(f)$  is a bijection.



G

Rk.

Inverse to F  
(together with

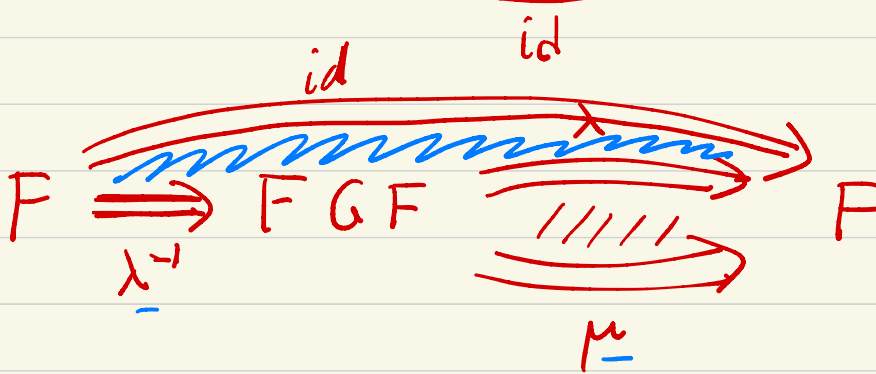
$$\left. \begin{array}{l} GF \stackrel{\lambda}{\cong} id \\ FG \stackrel{\mu}{\cong} id \end{array} \right\}$$

is unique up to a unique nat. transf.

$$FG \stackrel{\lambda}{\cong} id_D$$

$$GF \stackrel{\mu}{\cong} id_C$$

$$F \stackrel{\lambda^{-1}}{\cong} FG F \stackrel{\mu}{\cong} F$$



# Morita equivalence.

Def.  $R, S$  rings

$R \underset{\text{Morita}}{\sim} S$  if  $\exists$  an equiv. of categories

$F: R\text{-mod} \xrightarrow{\sim} S\text{-mod}.$

Ex.  $R \underset{\text{Morita}}{\sim} M_n(R)$

$F: R\text{-mod} \longrightarrow M_n(R)\text{-mod}$

$M \longmapsto M^{\oplus n}$

$$M_n(R) \hookrightarrow \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in M^{\oplus n} \right\}.$$

$G: M_n(R)\text{-mod} \longrightarrow R\text{-mod}.$

$N \longmapsto \underline{e_{11} N}$

$$e_{11} = \begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & \cdots & 0 \end{pmatrix}.$$

check:  $N \xrightarrow{G} e_{11} N \xrightarrow{F} (e_{11} N)^{\oplus n}$   
 $\searrow \text{SI}$   
 $N$



$$N = e_{11}N \oplus e_{22}N \oplus \dots \oplus e_{nn}N.$$

$$\left( \begin{array}{l} \text{because } e_{ii} \text{ idempotent} \\ \sum e_{ii} = 1_n \end{array} \right)$$

need canonical

$$e_{ii}N \xrightarrow{\sim} e_{jj}N$$

$$\searrow \quad \swarrow$$

$$e_{kk}N$$

$$e_{ji} : e_{ii}N \xrightarrow{\sim} e_{jj}N : e_{ij}.$$

Morita equiv. for comm rings

$$A \underset{\text{Morita}}{\sim} B \iff A \cong B.$$

$$A\text{-mod} \cong B\text{-mod}.$$

We can recover  $A$  from  $A\text{-mod}$  as the center

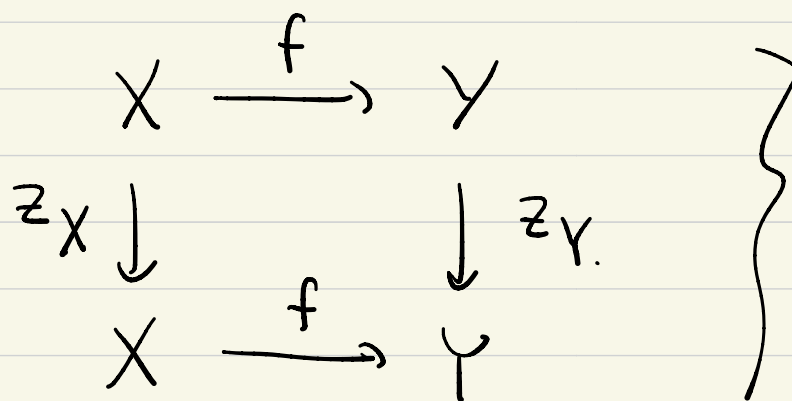
Def :  $\mathcal{C}$  category

$$Z(\mathcal{C}) = \text{End}(\text{id}_{\mathcal{C}})$$

$$= \{ \text{natural transf } \text{id}_{\mathcal{C}} \Rightarrow \text{id}_{\mathcal{C}} \}$$

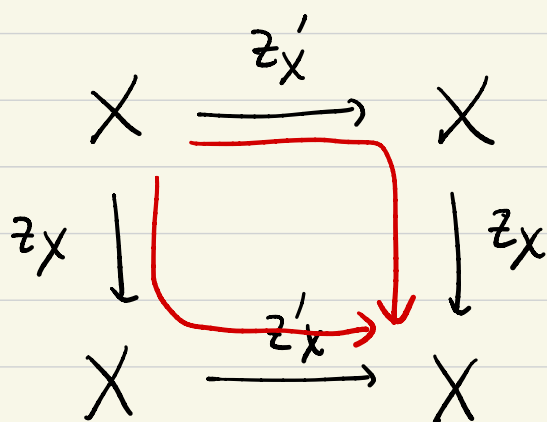
$$= \left\{ (z_x)_{x \in \mathcal{C}} \mid \begin{array}{l} z_x : X \rightarrow X \text{ in } \mathcal{C}. \\ \text{s.t.} \end{array} \right.$$

$\forall X \xrightarrow{f} Y \text{ in } \mathcal{C}.$



$Z(\mathcal{C}) = \text{comm. ring.}$

$z = \{z_X\}, z' = \{z'_X\}$



$$(z \circ z')_X = (z' \circ z)_X$$

$Z(\underline{A\text{-mod}}) = A. \quad (A = \text{comm.})$

$Z(R\text{-mod}) = \underline{Z(R)}.$

$R \sim M_n(R)$

same center  $Z(R).$

$R$ .  $M = \text{left } R\text{-mod}$

$$R\text{-mod} \longrightarrow \{\text{abelian gps}\} = \underline{Ab}$$

$$N \longmapsto \text{Hom}_R(M, N)$$

$\text{Hom}_R(M, -)$  is a right module for  $\text{End}_R(M)$

The above functor lifts to:

$$\begin{array}{ccc} M^h: & R\text{-mod} & \longrightarrow & \text{mod} - \text{End}_R(M). \\ & & & \downarrow S \\ & & & \text{End}_R(M)^{\text{op}} - \text{mod}. \end{array}$$

Defn A progenerator in  $R\text{-mod}$  is a f.g. projective module  $P$

s.t.  $\exists$  surjection  $P^{\oplus I} \longrightarrow R$ .

(can be chosen to be finite.)

Thm Let  $P$  be a progenerator of  $R\text{-mod}$ .

$$S = \text{End}_R(P)^{\text{op}}.$$

Then

$$\text{Hom}_R(P, -): R\text{-mod} \longrightarrow S\text{-mod}$$

is an equivalence.

Thm. Any Morita equiv. arises this way.

$$R \sim S$$