

Lecture 13

10/19

- Injective hulls
- Indecomposable injectives.
- category theory.

Def. $0 \neq M \hookrightarrow N$ is essential (ess. extn).
if \forall every nonzero $N' \subset N$, $N' \cap M \neq 0$.

Def. $M: R\text{-mod}$.
An injective hull of M is an essential
 $M \hookrightarrow I$
where I is an injective $R\text{-mod}$.

Existence of I_M :

$M \hookrightarrow I' = \text{injective } R\text{-mod.}$

Consider all $\left\{ \begin{array}{c} E \\ M \subset E \subset I' \\ \uparrow \\ \text{ess.} \end{array} \right\}$

Take an max. elt from this set, denoted E_{\max} .

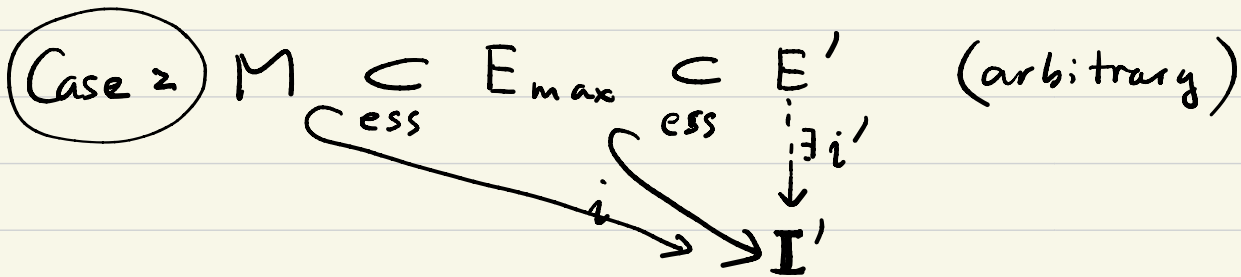
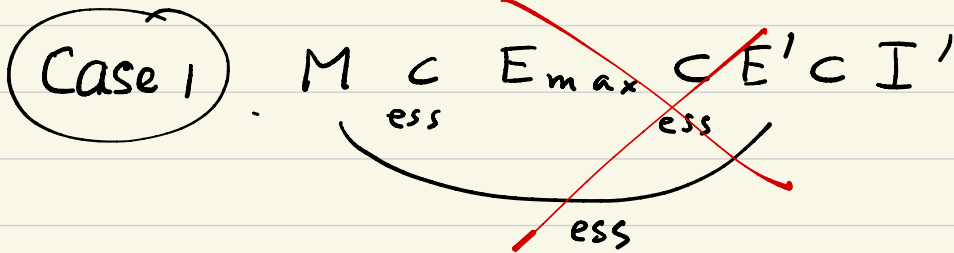
$\left(\begin{array}{c} M \subset E_1 \subset I' \\ \uparrow \\ \text{ess.} \end{array} \right), \text{ then } M \subset E_1 + E_2 \subset I'$
 $\left(\begin{array}{c} M \subset E_2 \subset I' \\ \uparrow \\ \text{ess.} \end{array} \right)$

We will show E_{\max} is an injective $R\text{-mod}$.

Recall: An R -mod I is injective

$\Leftrightarrow I$ has no nontrivial essential extn.
($I \subset_{\text{ess}} J \Rightarrow I=J$)

Suffices to show E_{\max} has no ess. extn.



$\text{Ker}(i') \subset E'$ submod.

$M \subset_{\text{ess}} E'$

if $\text{Ker}(i') \neq 0, \Rightarrow \text{Ker}(i') \cap M \neq 0$.

contradiction: $i'|_M = i|_M = \text{injective}$.

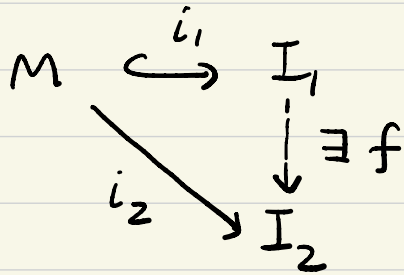
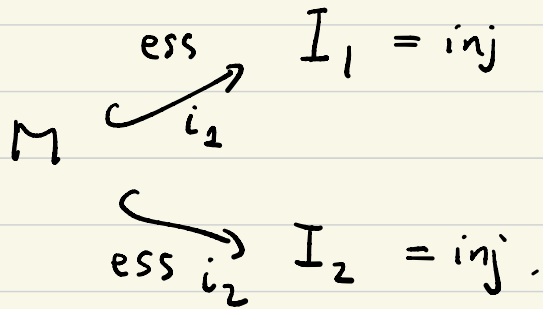
$\Rightarrow \text{Ker}(i') = 0$.

Can view E' as a submod of I' .

back to Case 1.

$\Rightarrow E_{\max}$ is injective.

Uniqueness (up to non-trivial isom).

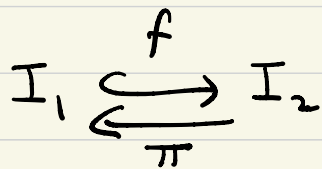


Claim: f is injective.

otherwise: $\text{Ker}(f) \subset I_1$

$\text{Ker}(f) \cap M \neq 0$.

contradict $f|_M = i_2 \text{ inj.}$



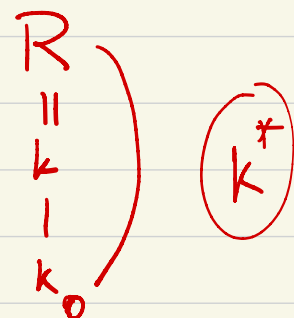
$$\Rightarrow I_1 \oplus I_3 = I_2.$$

If $I_3 \neq 0$, would have $M \oplus I_3 \subset I_2$.
 contradict that $M \subset_{\text{ess}} I_2$.

$\Rightarrow I_3 = 0$, f is an isom. ▣

Ex. $\dim_k R < \infty$.

Recall: $R \twoheadrightarrow \bar{R} = R/J(R)$
 is a projective cover.



$$(\bar{R})^* = \text{Hom}_k(\bar{R}, k) \xrightarrow{\quad} R^* = \text{Hom}_k(R, k).$$

$\text{soc}(R^*) = (\bar{R})^*$ \longleftrightarrow R^* is an injective left R -mod.
 $R \rightarrow \bar{R} = \text{max. ss quotient of } R. \text{ as } R\text{-mod.}$
 \uparrow this is essential.

General fact: $R = \text{left artinian}$ \leftarrow
 $M = R\text{-mod.}$

$\text{soc}(M) = \text{sum of simple submodules in } M$

$\text{soc}(M) \subset M$ is essential.

Pf: $0 \neq M' \subset M$

may assume $M' = \boxed{R \cdot x} \subset M$.

has finite length ($R = \text{art.}$)

\Rightarrow has a simple submod. S

$\Rightarrow M' \cap \text{soc}(M) = S$. \blacksquare

$\dim_k(R) < \infty$.

$\bar{R} = \bigoplus S^{\oplus m_S}$

\searrow right simple R -mod.

$\bar{R}^* = \bigoplus (S^*)^{\oplus m_S}$

\searrow simple left R -mod.

$S^* \hookrightarrow I_{S^*} = \text{inj. hull of } S^*$.

$\bar{R}^* = \bigoplus (S^*)^{\oplus m_S} \hookrightarrow \bigoplus (I_{S^*})^{\oplus m_S} = I_{\bar{R}^*}$

\uparrow \bigoplus ess extn = ess extn.

$\bar{R}^* \hookrightarrow R^*$ is also an injective hull.

$\Rightarrow R^* \cong I_{\bar{R}^*} \cong \bigoplus (I_{S^*})^{\oplus m_S}$.

Prop. $R =$ left artinian.

$$S \longleftrightarrow I_S$$

$$\left\{ \text{simple } R\text{-mods} \right\} / \cong \longleftrightarrow \left\{ \text{indecomp injective } R\text{-mods} \right\} / \cong$$

Pf. ① I_S is indecomp.

$$S \subset_{\text{ess}} I_S = \underset{\neq 0}{I_1} \oplus \underset{\neq 0}{I_2}$$

$$I_i \cap S \neq 0 \Rightarrow I_i \supset S, \quad i=1,2$$

$$\Rightarrow I_1 \cap I_2 \supseteq S. \quad \times$$

② I any indecomp injective.

$R =$ artinian $\Rightarrow I$ contains a simple mod S .

$$\begin{array}{ccc} S & \xrightarrow{\text{ess}} & I_S \\ & \searrow & \vdots \\ & & I \end{array} \quad f \quad \ker(f) = 0.$$

$$f: I_S \xrightarrow{\cong} I$$

$$I = I_S \oplus (\dots) \Rightarrow I \underset{f}{\cong} I_S.$$

(indecomp.)
must be 0.

③ $I_S \cong I_{S'}$

$$\text{If } S \neq S', \quad S \oplus S' \subset I_S \cong I_{S'}$$

$\Rightarrow S \subset I_S$ is not essential. ~~X~~

$R = \text{artinian}$.

Warning: I_S may not be f.g.

(Lam, lectures on rings and modules)
 $\sim 3F, 3G$.

Category theory:

$$\mathcal{C} = \left(\underbrace{\text{Obj } \mathcal{C}}_{\text{vertices}}, \underbrace{\left\{ \text{Hom}_{\mathcal{C}}(X, Y) \right\}}_{\text{set}} \right)_{X, Y \in \text{Obj } \mathcal{C}}$$

vertices arrows

axioms:

- composition of arrows

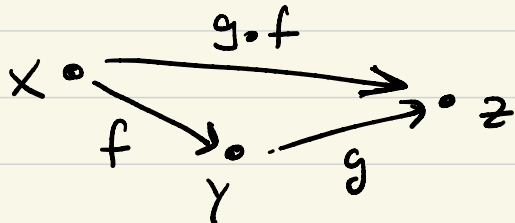
$$\text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{C}}(X, Z)$$
$$(g, f) \longmapsto g \circ f$$

+ associativity.

+ unit

distinguished elt

$$\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X).$$



Functors : $F: \mathcal{C} \rightarrow \mathcal{D}$
(covariant)

$$\text{Obj } \mathcal{C} \ni X \mapsto F(X) \in \text{Obj } \mathcal{D}$$

$$(X \xrightarrow{f} Y) \mapsto (F(X) \xrightarrow{F(f)} F(Y))$$

$$\text{s.t. } F(\text{id}_X) = \text{id}_{F(X)}.$$

$$F(g \circ f) = F(g) \circ F(f).$$

contravariant functor: $\mathcal{C} \rightarrow \mathcal{D}$

\Leftrightarrow functor $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$.

reverse all arrows.

$\text{Fun}(\mathcal{C}, \mathcal{D})$ is a category = $\left\{ \begin{array}{l} \text{Obj:} \\ \text{functors } F: \mathcal{C} \rightarrow \mathcal{D} \\ \\ \text{Hom}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(F_1, F_2) \\ = \text{natural transformations: } F_1 \xRightarrow{t} F_2 \\ \forall X \in \mathcal{C}. \quad t_X: F_1(X) \rightarrow F_2(X) \\ (t_X \in \text{Hom}_{\mathcal{D}}(F_1(X), F_2(X))) \end{array} \right.$

s.t. $\forall X \xrightarrow{f} Y$

$$\begin{array}{ccc} F_1(X) & \xrightarrow{t_X} & F_2(X) \\ \downarrow F_1(f) & & \downarrow F_2(f) \\ F_1(Y) & \xrightarrow{t_Y} & F_2(Y) \end{array}$$

is commutative.

Set category of sets (arbitrary maps).

$X \in \mathcal{C}$,

$$h_X : \mathcal{C}^{\text{op}} \longrightarrow \underline{\text{Set}}$$

$$Y \longmapsto \text{Hom}_{\mathcal{C}}(Y, X)$$

is the functor represented by X .

Thm (Yoneda Lemma)

$$h : \mathcal{C} \longrightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \underline{\text{Set}})$$
$$X \longmapsto h_X$$

is a fully faithful.
(bijection on Hom sets).

Pf.

$$\text{Hom}_{\mathcal{C}}(X_1, X_2) \ni f$$

$$\left\{ \begin{array}{l} \text{natural transf.} \\ h_{X_1} \Rightarrow h_{X_2} \end{array} \right\}$$

$$\begin{array}{ccc} \forall Y \in \mathcal{C} & & \\ h_{X_1}(Y) & \longrightarrow & h_{X_2}(Y) \\ \parallel & f \circ (-) & \parallel \\ \text{Hom}_{\mathcal{C}}(Y, X_1) & \longrightarrow & \text{Hom}_{\mathcal{C}}(Y, X_2) \end{array}$$

defines
 $t(f)$

$$h_{X_1} \Rightarrow h_{X_2}$$

Start with $h_{X_1} \xrightarrow{t} h_{X_2}$

$$t_{X_1}: h_{X_1}(X_1) \longrightarrow h_{X_2}(X_1)$$

$$\parallel \qquad \parallel$$

$$\text{Hom}_{\mathcal{C}}(X_1, X_1) \longrightarrow \text{Hom}_{\mathcal{C}}(X_1, X_2)$$

$$\text{id}_{X_1} \longmapsto t_{X_1}(\text{id}_{X_1}) = f(t)$$

$$f(t): X_1 \longrightarrow X_2$$

$$\left. \begin{array}{l} f \longmapsto t(f) \\ t \longmapsto f(t) \end{array} \right\} \text{inverse.}$$



$X \in \text{Obj } \mathcal{C}$ can be identified with $h_X: \mathcal{C}^{\text{op}} \rightarrow \underline{\text{Set}}$.
 \uparrow \longleftarrow \cup \nearrow
 (simpler cat than \mathcal{C})^{op}

Adjunction:

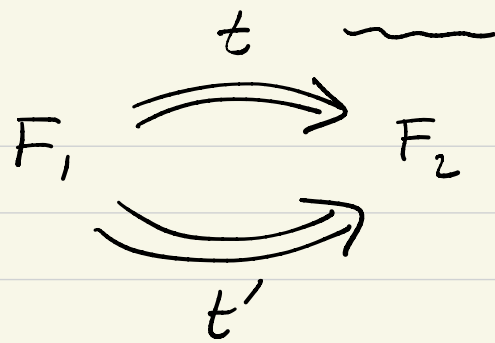
$$F: \mathcal{C} \rightleftarrows \mathcal{D}: G$$

$$F: \mathcal{C} \rightarrow \mathcal{D}, \quad G: \mathcal{D} \rightarrow \mathcal{C}.$$

$$\forall X \in \mathcal{C}. \quad Y \in \mathcal{D}.$$

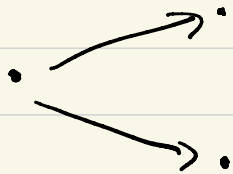
$$\alpha_{X,Y}: \text{Hom}_{\mathcal{C}}(X, G(Y)) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}}(F(X), Y)$$

$\{ \alpha_{X,Y} \}$ comp. with maps $X_1 \rightarrow X_2$ in \mathcal{C}
 $Y_1 \rightarrow Y_2$ in \mathcal{D} .



$$F_1(x) \xrightarrow{\textcircled{t_x}} F_2(x)$$

$$\xrightarrow{\quad} F_1(x) \xrightarrow{\textcircled{t'_x}} F_2(x)$$



$$t_x, t'_x \in \text{Hom}(F_1(x), F_2(x))$$

$\uparrow \mathcal{D}$