Lecture 13 10/19

- Injective hulls
- Indecomposable injectives.
- category theory.

<u>Def.</u> $0 \neq M \hookrightarrow N$ is <u>essential</u> (ess. extn).
if $\forall$ every nonzero $N' \subset N$, $N' \cap M \neq 0$.

<u>Def.</u> M: R-mod.
An injective hull of M is an <u>essential</u>

$$M \hookrightarrow I$$

where I is an injective R-mod.

<u>Existence</u> of I_M :

$$M \hookrightarrow I' = \text{injective } R\text{-mod.}$$

Consider all $\left\{ E \,\middle|\, M \underset{\text{ess.}}{\subset} E \subset I' \right\}$

Take ~~an~~ max. elt from this set, denoted E_{max}.

$$\left(\begin{array}{c} M \overset{\text{ess}}{\subset} E_1 \subset I' \\ M \underset{\text{ess}}{\subset} E_2 \subset I' \end{array}, \quad \text{then } M \underset{\text{ess.}}{\subset} E_1 + E_2 \subset I' \right)$$

We will show E_{max} is an injective R-mod.

Recall: An R-mod I is injective

$\iff$ I has no nontrivial essential extn.

$(I \subset_{ess} J \Rightarrow I = J.)$

Suffices to show E_{max} has no ess. extn.

Case 1. $\cancel{M \subset_{ess} E_{max} \subset_{ess} E' \subset I'}$ ess

Case 2

$$\begin{tikzcd} M \arrow[r, phantom, "\subset_{ess}"] \arrow[drr, "i"'] & E_{max} \arrow[r, phantom, "\subset_{ess}"] \arrow[dr] & E' \arrow[d, dashed, "\exists i'"] \\ & & I' \end{tikzcd}$$

(arbitrary)

$Ker(i') \subset E'$ submod.

$M \subset_{ess} E'$

if $ker(i') \neq 0, \Rightarrow ker(i') \cap M \neq 0$.

Contradiction: $i'|_M = i|_M =$ injective.

$Ker(i') = 0$.

Can view E' as a submod of I'.

back to Case 1.

$\Rightarrow$ E_{max} is injective.

Uniqueness (up to non-unique isom).

$$\begin{tikzcd}[column sep=small] & I_1 = \text{inj} \\ M \arrow[ur, hook, "\text{ess}", "i_1"'] \arrow[dr, hook, "\text{ess}\ i_2"'] & \\ & I_2 = \text{inj}. \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, hook, "i_1"] \arrow[dr, "i_2"'] & I_1 \arrow[d, dashed, "\exists f"] \\ & I_2 \end{tikzcd}$$

Claim: f is injective.

otherwise: $\operatorname{Ker}(f) \subset I_1$

$\operatorname{Ker}(f) \cap M \neq 0.$

contradict $f|_M = i_2$ inj.

$$I_1 \underset{\pi}{\overset{f}{\rightleftarrows}} I_2$$

$$\Rightarrow \quad I_1 \oplus I_3 = I_2.$$

If $I_3 \neq 0$, would have $M \oplus I_3 \subset I_2$.

contradict that $M \underset{\text{ess}}{\subset} I_2$.

$$\Rightarrow \quad I_3 = 0, \quad f \text{ is an isom.} \qquad \blacksquare$$

Ex. $\dim_k R < \infty$.

$R = k \,|\, k_0$

k^*

Recall: $R \twoheadrightarrow \overline{R} = R/J(R)$

is a projective cover.

$$(\overline{R})^* = \operatorname{Hom}_k(\overline{R}, k) \hookrightarrow R^* = \operatorname{Hom}_k(R, k).$$

$R \to \bar{R} =$ max. ss quotient of R. as ~~left~~ R-mod.

$\mathrm{soc}(R^*) = (\bar{R})^* \hookrightarrow R^*$ is an injective left R-mod.

this is essential.

General fact: R = left artinian

M = R-mod.

$\mathrm{soc}(M)$ = sum of simple submodules in M

$\mathrm{soc}(M) \subset M$ is essential.

Pf: $0 \neq M' \subset M$

may assume $M' = \boxed{R \cdot x} \subset M$.

has finite length (R = art.)

$\Rightarrow$ has a simple submod. S

$\Rightarrow \quad M' \cap \mathrm{soc}(M) \supset S$. ∎

$\dim_k(R) < \infty$. $\quad \bar{R} = \bigoplus S^{\oplus m_S}$ right simple R-mod.

$\bar{R}^* = \bigoplus (S^*)^{\oplus m_S}$ simple left R-mod.

$S^* \hookrightarrow I_{S^*}$ = inj. hull of S^*.

$\bar{R}^* = \bigoplus (S^*)^{\oplus m_S} \hookrightarrow \bigoplus (I_{S^*})^{\oplus m_S} = I_{\bar{R}^*}$

$\oplus$ ess extn = ess extn.

$\bar{R}^* \hookrightarrow R^*$ is also an injective hull.

$\Rightarrow \quad R^* \cong I_{\bar{R}^*} \cong \bigoplus (I_{S^*})^{\oplus m_S}$.

Prop. R = left artinian.

$$S \longleftrightarrow I_S$$

$$\{\text{simple } R\text{-mods}\}/_{\cong} \longleftrightarrow \{\text{indecomp. injective } R\text{-mods}\}/_{\cong}$$

Pf. ① I_S is indecomp.

$$\underset{\text{ess}}{S \subset} I_S = \underset{\neq 0}{I_1} \oplus \underset{\neq 0}{I_2}$$

$I_i \cap S \neq 0 \Rightarrow I_i \supset S, \quad i = 1, 2$

$\Rightarrow I_1 \cap I_2 \supseteq S$. $\times$.

② I any indecomp injective.

R = artinian $\Rightarrow$ I contains a simple mod S.

$$\begin{tikzcd} S \arrow[r, hook, "\text{ess}"] \arrow[dr, hook] & I_S \arrow[d, dashed, "f"] \\ & I \end{tikzcd} \qquad \ker(f) = 0.$$

$f: I_S \hookrightarrow I$

$\underset{\text{indecomp.}}{I} = I_S \oplus \underset{\text{must be } 0}{(\cdots)}$. $\Rightarrow I \underset{f}{\cong} I_S$.

③. $I_S \overset{?}{\cong} I_{S'}$

If $S \not\cong S'$. $\qquad S \oplus S' \subset I_S \cong I_{S'}$

$\Longrightarrow$ $S \subset I_S$ is **not** essential.

R = artinian.

<u>Warning</u>: I_S may not be f.g.

(Lam, Lectures on rings and modules)
$\sim$ 3F, 3G.

Category theory:

$$\mathcal{C} = \Big(\underbrace{\mathrm{obj}\,\mathcal{C}}_{\text{vertices}},\ \big\{ \underbrace{\mathrm{Hom}_{\mathcal{C}}(X,Y)}_{\text{set}} \big\}_{X,Y \in \mathrm{Obj}\,\mathcal{C}} \Big)$$

arrows

axioms:

- composition of arrows

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

$$(g, f) \longmapsto g \circ f$$

+ associativity.
+ unit distinguished elt $id_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$.

$$\begin{tikzcd} X \arrow[dr, "f"'] \arrow[rr, "g \circ f"] & & Z \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

Functors (covariant) : $F: \mathcal{C} \longrightarrow \mathcal{D}$

$$\mathrm{Obj}\,\mathcal{C} \ni X \longmapsto F(X) \in \mathrm{Obj}\,\mathcal{D}$$

$$\left(X \xrightarrow{f} Y\right) \mapsto \left(F(X) \xrightarrow{F(f)} F(Y)\right)$$

s.t. $F(id_X) = id_{F(X)}$.

$F(g \circ f) = F(g) \circ F(f)$.

contravariant functor: $\mathcal{C} \to \mathcal{D}$

$\Longleftrightarrow$ functor $F: \underbrace{\mathcal{C}^{op}}_{\text{reverse all arrows.}} \longrightarrow \mathcal{D}$.

$\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is a category $= \begin{cases} \text{Obj: functors } F: \mathcal{C} \to \mathcal{D} \\ \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F_1, F_2) \\ \quad = \text{natural transformations: } F_1 \overset{t}{\Rightarrow} F_2 \end{cases}$

$\forall X \in \mathcal{C}.\quad t_X: F_1(X) \longrightarrow F_2(X)$

$\left(t_X \in \mathrm{Hom}_{\mathcal{D}}(F_1(X), F_2(X))\right)$

s.t. $\forall\ X \xrightarrow{f} Y$

$$\begin{array}{ccc} F_1(X) & \xrightarrow{t_X} & F_2(X) \\ \downarrow F_1(f) & & \downarrow F_2(f) \\ F_1(Y) & \xrightarrow{t_Y} & F_2(Y) \end{array}$$

is commutative.

$\underline{\mathrm{Set}}$ category of sets (arbitrary maps).

$X \in \mathcal{C}$,

$$h_X : \mathcal{C}^{op} \longrightarrow \underline{\mathrm{Set}}$$
$$Y \longmapsto \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

is the functor <u>represented by</u> X.

<u>Thm</u> (Yoneda lemma)

$$h : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \underline{\mathrm{Set}})$$
$$X \longmapsto h_X$$

is a fully faithful.
(bijection on Hom sets).

<u>Pf</u>. $\mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \ni f$

$$\downarrow \qquad\qquad \downarrow$$

$\left\{ \begin{array}{c} \text{natural transf.} \\ h_{X_1} \Rightarrow h_{X_2} \end{array} \right\}$

$\forall Y \in \mathcal{C}$

$$\begin{array}{ccc} h_{X_1}(Y) & \longrightarrow & h_{X_2}(Y) \\ \| & & \| \\ \mathrm{Hom}_{\mathcal{C}}(Y, X_1) & \xrightarrow{f \circ (-)} & \mathrm{Hom}_{\mathcal{C}}(Y, X_2) \end{array}$$

defines

$$h_{X_1} \overset{t(f)}{\Longrightarrow} h_{X_2}$$

Start with $h_{X_1} \overset{t}{\Longrightarrow} h_{X_2}$

$$t_{X_1} : \; h_{X_1}(X_1) \longrightarrow h_{X_2}(X_1)$$

$$\begin{array}{ccc} \| & & \| \\ \mathrm{Hom}_{\mathcal{C}}(X_1, X_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \\ \mathrm{id}_{X_1} & \longmapsto & t_{X_1}(\mathrm{id}_{X_1}) = f(t) \end{array}$$

$$f(t) : X_1 \longrightarrow X_2$$

$$\left.\begin{array}{l} f \longmapsto t(f) \\ t \longmapsto f(t) \end{array}\right\} \text{ inverse.}$$

$\blacksquare$

$X \in \mathrm{obj}\,\mathcal{C}$ can be identified with $h_X : \mathcal{C}^{op} \to \underline{\mathrm{Set}}$.

$\cup$

$\left(\begin{array}{c}\text{simpler}\\ \text{cat than } \mathcal{C}\end{array}\right)^{op}$

Adjunction:

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

$$F : \mathcal{C} \to \mathcal{D}, \qquad G : \mathcal{D} \longrightarrow \mathcal{C}.$$

$\forall \; X \in \mathcal{C}, \; Y \in \mathcal{D}.$

$$\alpha_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$$

$\{\alpha_{X,Y}\}$ comp. with maps $X_1 \longrightarrow X_2$ in $\mathcal{C}$

$Y_1 \longrightarrow Y_2$ in $\mathcal{D}$.

$$\begin{tikzcd} F_1 \arrow[r, bend left, Rightarrow, "t"] \arrow[r, bend right, Rightarrow, "t'"'] & F_2 \end{tikzcd}$$

$t \rightsquigarrow F_1(x) \xrightarrow[t_x]{} F_2(x)$

$t' \rightsquigarrow F_1(x) \xrightarrow[t'_x]{} F_2(x)$

$$\begin{tikzcd} & \bullet \\ \bullet \arrow[ur] \arrow[dr] & \\ & \bullet \end{tikzcd}$$

$$t_x,\ t'_x \in \mathrm{Hom}_{\mathcal{D}}(F_1(x), F_2(x))$$