

- injective modules

- injective hulls

Defn/Prop: $R \subseteq I$, TFAE:

①. $\text{Hom}_R(-, I)$ is an exact functor.

② $\forall \begin{array}{ccc} M & \xrightarrow{i} & N \\ f \downarrow & \swarrow \text{dashed} & \\ I & & \end{array} \exists \text{ extension } \tilde{f} \text{ of } f \text{ to } N.$

③ $\forall \begin{array}{ccc} I & \xrightarrow{i} & I' = R\text{-mod} \\ \exists \text{ retract } I' \xrightarrow{\pi} I, & \pi \circ i = \text{id}_I \end{array}$
 ($\Leftrightarrow I$ is a direct summand of I')

④ For any left ideal $\mathfrak{a} \subset R$.
 and any $\underbrace{\text{map } \mathfrak{a} \xrightarrow{f} I}_{R\text{-lin}}$
 \exists an extension of f to $R \rightarrow I$.

(I is call an injective R -mod if any of these holds)

Pf ① $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$

$0 \rightarrow \text{Hom}_R(M'', I) \rightarrow \text{Hom}_R(M, I) \rightarrow \text{Hom}_R(M', I) \rightarrow 0$

\uparrow exact here \uparrow exact here

① \iff ② \checkmark

② \implies ③ is a special case

$$\begin{array}{ccc} I & \hookrightarrow & I' \\ \parallel & & \swarrow \\ I & & \end{array}$$

③ \implies ②

$$\begin{array}{ccc} M & \xrightarrow{i} & N \\ f \downarrow & & \downarrow \\ I & \longrightarrow & I \amalg_M N = (I \oplus N) / \Delta^-(M) \end{array}$$

$\Delta^-: M \rightarrow I \oplus N$

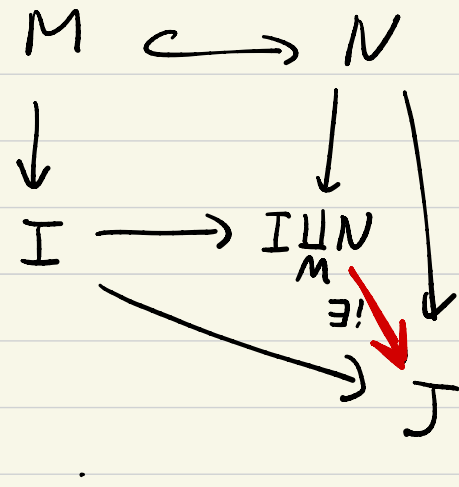
$m \mapsto (f(m), -i(m))$

$m \xrightarrow{i} i(m)$

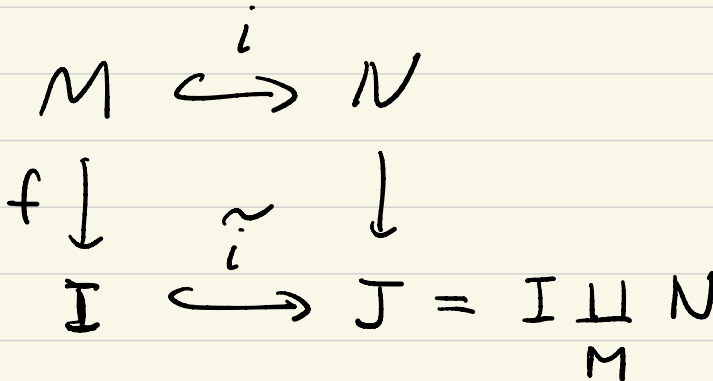
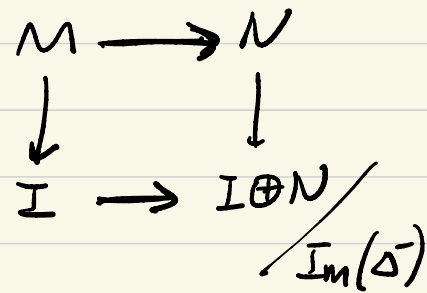
$$\begin{array}{ccc} & & \downarrow \\ & & (0, i(m)) \\ f \downarrow & & \downarrow \\ f(m) & \longrightarrow & (f(m), 0) \end{array}$$

make them equal in $I \oplus N / \dots$

Universal property of push out



works without assuming i is inj.



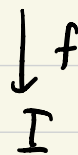
Ex. extending f to N .

\iff extending id_I to J .

(4) \implies (2)

extension of maps along ideal $I \subset R$.

Start with any $M \xrightarrow{i} N$.



Zorn's lemma applied to:

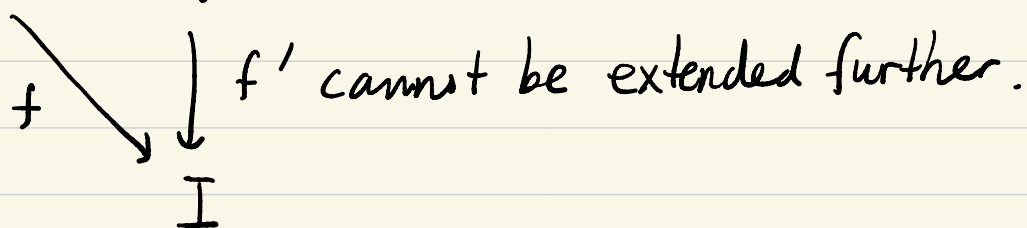
$$\left\{ (M', f') \mid \begin{array}{l} M \subset M' \\ f': M' \rightarrow I \text{ extends } f \end{array} \right\}$$

$$(M', f') \leq (M'', f'')$$

$\Rightarrow \exists$ a max. elt in this poset.

call it (M', f')

$$M \subset M' \subset N.$$



want to show $M' = N$.

If not, $x \in N \setminus M'$.

$$M' \subsetneq M' + Rx$$

⏟
quotient R/a ← left ideal

$$a = \{ r \in R \mid r \cdot x \in M' \}$$

$$\alpha: a \longrightarrow I$$

$$r \longmapsto \underbrace{f'(rx)}_{\in M'}$$

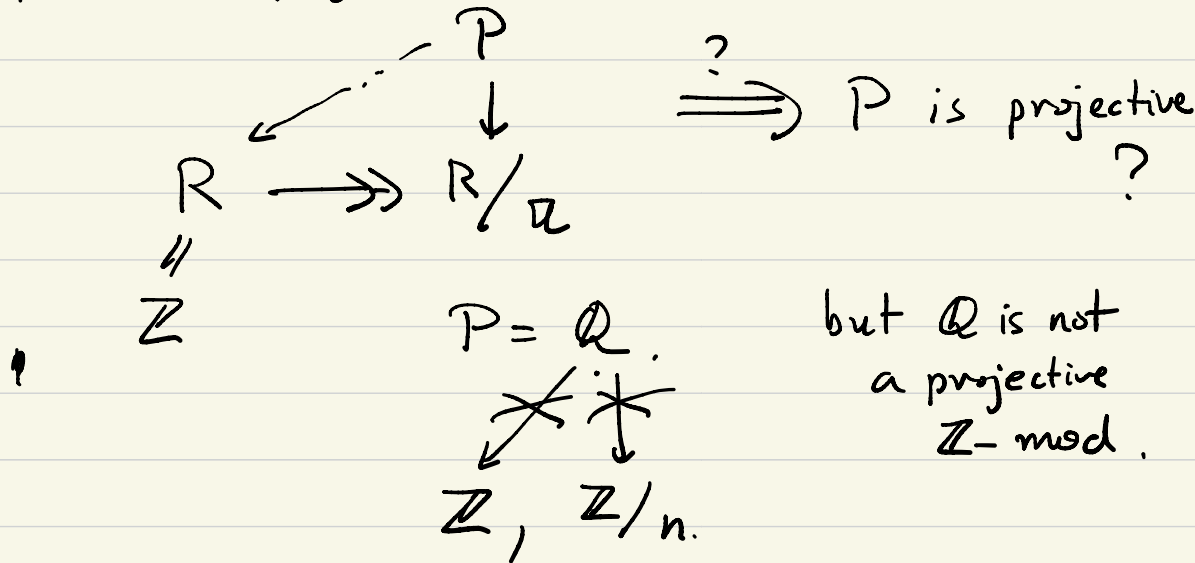
④ $\Rightarrow \alpha$ can be ext'd to $R \xrightarrow{\alpha} I$
 $1 \longmapsto y \in I$

Define

$$\begin{array}{ccc} M' + Rx & \xrightarrow{f''} & I \\ \hline \boxed{m + rx} & \longmapsto & f'(m) + ry \end{array}$$

check well-defn: guaranteed by $\tilde{\alpha}(r)$
 \parallel
 if $rx \in M'$, $f''(0+rx) = ry$
 $\quad \quad \quad r \in \mathfrak{a}$
 $\quad \quad \quad 0+rx$
 $\quad \quad \quad (rx)+0$ $f''(rx+0) = f'(rx)$
 $\quad \quad \quad \parallel$
 $\quad \quad \quad \alpha(r)$

Analog of $\textcircled{4}$ for projectives?



Construction of injective modules:

- R : k -algebra.
 For any R -mod M .

$\text{Hom}_k(R, M)$ is a left R -mod
 using right translation on R .

$$f: R \rightarrow M, \quad r \in R.$$

k -linear,

$$(r \cdot f)(x) = f(xr).$$

$$r_1(r_2 f) = (r_1 r_2) f$$

Claim: $\text{Hom}_k(R, M)$ is an injective R -mod.

Pf. $\forall N. (R\text{-mod})$

$$\text{Hom}_R(N, \text{Hom}_k(R, M)) \quad S=k \hookrightarrow R \xrightarrow{B} R$$

\parallel

$$\text{Hom}_k(R \otimes_R N, M) = \text{Hom}_k(N, M).$$

\uparrow exact in N .

General: $S \hookrightarrow B \hookrightarrow R \hookrightarrow N$

$S \hookrightarrow M$.

$$\text{Hom}_S(B \otimes_R N, M) = \text{Hom}_R(N, \text{Hom}_S(B, M))$$

$$T_B = B \otimes_R (-) : R\text{-mod} \longrightarrow S\text{-mod}.$$

$$H_B = \text{Hom}_S(B, -) : S\text{-mod} \longrightarrow R\text{-mod}.$$

$$\text{Hom}_S(T_B(N), M) = \text{Hom}_R(N, H_B(M)).$$

To prove, $B \otimes_R N \rightarrow M, S\text{-linear}.$

\Downarrow

$$B \times N \rightarrow M, \text{R-bilinear, S-linear}.$$

\Downarrow

$$N \rightarrow \text{Hom}_S(B, M). \text{ R-linear}.$$

$$\text{Hom}_R(R, M)$$

$$\begin{array}{c} \text{Hom}_R(R, M) \\ \parallel \\ M \end{array} \hookrightarrow \text{Hom}_k(R, M) = \text{injective.}$$

$$m \longmapsto (r \mapsto rm)$$

Generalize this idea. want to constr. injective R -mod.

$$\begin{array}{c} S \supseteq B \supseteq R \\ \triangle \qquad \qquad \qquad = \end{array}$$

$$S \supseteq M \qquad \text{Hom}_S(B, M). \quad R\text{-mod.}$$

\uparrow when is this injective?

$$\left\{ \begin{array}{l} M \text{ is an injective } S\text{-mod.} \\ B \text{ is a flat } R\text{-mod.} \\ \left(\text{i.e., } B \otimes_R (-) \text{ is an exact functor} \right. \\ \left. \text{on } R\text{-mod} \right) \end{array} \right.$$

Ex. $R = \mathbb{Z}$, $B = \mathbb{Q}$. flat \mathbb{Z} -mod.

$$S = \mathbb{Q} \supseteq B = \mathbb{Q} \supseteq \mathbb{Z} = R.$$

$$\Rightarrow \forall \mathbb{Q}\text{-vector space } V$$

$$\text{Hom}_S \left(\begin{array}{c} \mathbb{Q} \\ \parallel \\ B \end{array}, V \right) = V \text{ is an injective } \mathbb{Z}\text{-mod.}$$

(Fact: using ④ in the defn of injective modules,
 \mathbb{Z} -mod M is injective \iff M is divisible.
 i.e., $\forall x \in M, n \in \mathbb{N}, \exists y \in M$
 s.t. $ny = x.$)

$$A = \overset{\text{Comm.}}{\text{domain}}. \quad K = \text{Frac}(A).$$

(flat A -mod)

\Rightarrow Any K -v.s. is an injective A -mod.

Thus $\forall R, \forall R\text{-mod } M.$

$$\exists M \hookrightarrow I = \text{injective } R\text{-mod}.$$

Pf.

—

- $R = \mathbb{Z}$

- Apply general construction to

$$S = \mathbb{Z} \hookrightarrow B = R \hookrightarrow R$$

$$M \hookrightarrow M' = \text{injective } \mathbb{Z}\text{-mod}.$$

$$M = \text{Hom}_R(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, M').$$

injective R -mod.

$$M: \mathbb{Z}\text{-mod}$$

$$M = F/N \quad F = \text{free } \mathbb{Z}\text{-mod}.$$

$$F \hookrightarrow F \otimes_{\mathbb{Z}} \mathbb{Q} = \text{inj. } \mathbb{Z}\text{-mod}$$

$$M = F/N \hookrightarrow \underbrace{(F \otimes_{\mathbb{Z}} \mathbb{Q})/N}_{\text{divisible}}.$$

(e.g. \mathbb{Q}/\mathbb{Z} divisible \Rightarrow injective.)

~~Q~~

Ex. Q (finite) quiver R_Q .

v : vertex.

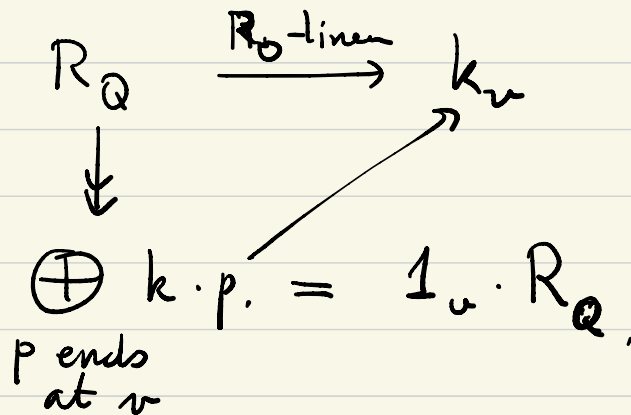
$$\prod_v k_v = R_0 \hookrightarrow R_Q$$

$\forall R_0$ -mod M , $\text{Hom}_{R_0}(R_Q, M)$ injective R_Q -mod.
 always injective. $M = k_v$ (1d mod at vertex v).

$$I_v = \text{Hom}_{R_0}(R_Q, k_v).$$

\parallel

$$\bigoplus k \cdot p.$$



$$I_v = \text{Hom}_k(1_v \cdot R_Q, k_v)$$

$$P_v = R_Q \cdot 1_v.$$

Ex. M : rep. of Q .

$$\text{Hom}_{R_Q}(M, I_v) = M_v^*$$

Next: $M \hookrightarrow$ minimal injective mod I ?

if \exists , I is called an injective hull of M .

Thm $\forall R, \forall R \supseteq M$. injective hull of M exists, and is unique up to isom.

Defn. $0 \neq M \hookrightarrow N$ is called essential, $\left(\begin{array}{l} N \text{ is an} \\ \text{essential} \\ \text{extn of } M \end{array} \right)$
if any nonzero submod $N' \subset N$,
 $N' \cap M \neq 0$.

$(\Leftrightarrow \text{cannot find } \underline{M \oplus N'} \subset N)$

Ex. $R = \mathbb{Z}$. $N = \mathbb{Z}/p^n\mathbb{Z}$

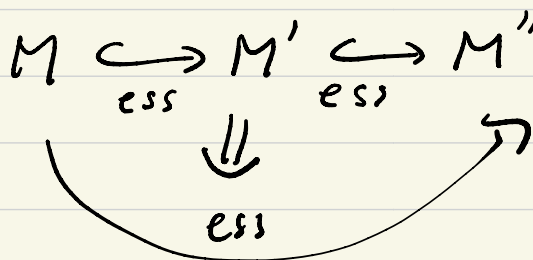
$0 \neq M \subset \boxed{N = \mathbb{Z}/p^n\mathbb{Z}}$
 \parallel
 $p^i\mathbb{Z}/p^n\mathbb{Z}$ \uparrow
 $(0 \leq i < n)$
 essential.

\therefore all submods contain $p^{n-1}\mathbb{Z}/p^n\mathbb{Z}$.

$(N$ is a uniform module.

any two nonzero submod intersect $\neq 0$).

Transitivity:

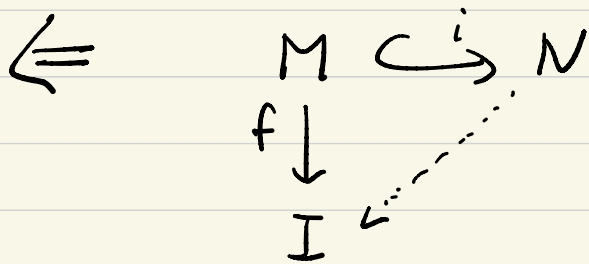


Criterion: $M \xrightarrow{i} N$.

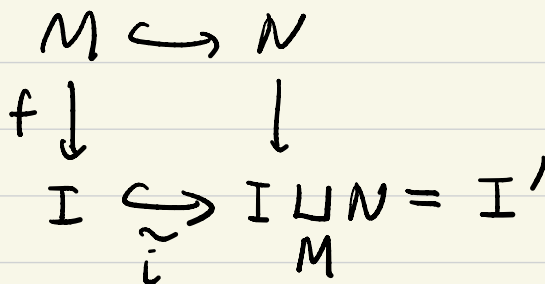
i is ess. $\iff \forall 0 \neq x \in N, \exists r \in R$
 s.t. $0 \neq rx \in M$.

Lemma: $R \ni I, I$ is injective $\iff I$ has no nontrivial ess extension.
 (i.e., $I \subsetneq I'$, then this ext is not ess).

Pf $\implies I$ inj.
 $I \subset I'$ $I' = I \oplus I''$
 if $I'' \neq 0$ this is not ess.



Zorn's Lemma. may assume (M, f) is max
 i.e., f cannot be further extended.



$\implies \text{id}_I$ cannot be further extended inside I' .
 Suppose $M \neq N$, then $I \neq I'$.
 $I \xrightarrow{i} I'$ is not essential.

$$I \oplus I'' \subset I'$$

$id \downarrow \swarrow 0$
 I

contracts that id_I cannot be
further ext'd.

