Lecture 12

10/14

- injective modules
- injective hulls

Defn/Prop: $R \circlearrowright I$, TFAE:

① $\mathrm{Hom}_R(-, I)$ is an exact functor.

② $\forall$

$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[d, "f"'] & N \arrow[dl, dotted] \\ I & \end{tikzcd}$$

$\exists$ extension $\tilde{f}$ of f to N.

③ $\forall$ $I \xhookrightarrow{i} I'$ = R-mod

$\exists$ retract $I' \overset{\pi}{\twoheadrightarrow} I$, $\pi \circ i = id_I$

($\Longleftrightarrow$ I is a direct summand of I').

④ For any left ideal $\mathfrak{a} \subset R$.

and $\underbrace{\text{any map}}_{R\text{-lin}}$ $\mathfrak{a} \xrightarrow[f]{} I$

$\exists$ an extension of f to $R \longrightarrow I$.

(I is call an injective R-mod if any of these holds).

<u>Pf</u> ① $0 \to \boxed{M'_\bullet \to M_\bullet} \to M''_\bullet \to 0$

$$0 \to \operatorname{Hom}_R(M'', I) \to \boxed{\operatorname{Hom}_R(M, I) \to \operatorname{Hom}_R(M', I)}$$

exact here   exact here

① $\Longleftrightarrow$ ② ✓

② $\Rightarrow$ ③ is a special case

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, equal] & I' \arrow[dl, dashed] \\ I & \end{tikzcd}$$

③ $\Rightarrow$ ②

$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[d, "f"'] & N \arrow[d] \\ I \arrow[r] & I \sqcup_M N \end{tikzcd} = (I \oplus N)\big/\underline{\Delta^-(M)}$$

$$\Delta^- : M \longrightarrow I \oplus N.$$
$$m \longmapsto \underline{(f(m), -i(m))}.$$

$$\begin{tikzcd} m \arrow[r, "i"] \arrow[d, "f"'] & i(m) \arrow[d] \\ f(m) \arrow[r] & \boxed{\begin{matrix}(0, i(m)) \\ (f(m), 0)\end{matrix}} \end{tikzcd}$$

make them equal in $I \oplus N / \ldots$

Universal property of pushout

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & N \arrow[d] \arrow[ddr, bend left] & \\ I \arrow[r] \arrow[drr, bend right] & I \sqcup_M N \arrow[dr, "\exists!"] & \\ & & J \end{tikzcd}$$

works without assuming i is inj.

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ I \arrow[r] & I \oplus N / \mathrm{Im}(\Delta^-) \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\tilde{i}"] & N \arrow[d] \\ I \arrow[r, hook] & J = I \sqcup_M N \end{tikzcd}$$

Ex: extending f to N.

$\Longleftrightarrow$ extending id_I to J.

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(4) $\Rightarrow$ (2)

extension of maps along ideal $\subset R$.

Start with any
$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[d, "f"] & N. \\ I & \end{tikzcd}$$

Zorn's lemma applied to:

$$\left\{ (M', f') \;\middle|\; \begin{matrix} M \subset M' \\ f' : M' \to I \text{ extends } f \end{matrix} \right\}$$

$$(M', f') \leq (M'', f'')$$

$\Rightarrow$ $\exists$ a max. elt in this poset.

call it (M', f')

$$\begin{tikzcd} M \arrow[dr, "f"'] & \subset M' \arrow[d, "f'"] & \subset N. \\ & I & \end{tikzcd}$$

f' cannot be extended further.

want to show $M' = N$.

If not, $x \in N \setminus M'$.

$$\underbrace{M' \underset{\neq}{\subset} M' + Rx}_{\text{quotient } R/\mathfrak{a}}$$

left ideal

$$\mathfrak{a} = \{ r \in R \mid r \cdot x \in M' \}.$$

$$\alpha : \mathfrak{a} \longrightarrow I$$

$$r \longmapsto f'(\underbrace{rx}_{\in M'}).$$

④ $\Rightarrow$ α can be ext'd to $R \xrightarrow{\tilde{\alpha}} I$

$$1 \longmapsto y \in I.$$

Define

$$\underline{M'} + \underline{Rx} \xrightarrow{f''} I$$

$$\boxed{m + rx} \longmapsto f'(m) + ry$$

check well-defn: guaranteed by $\tilde{\alpha}(r) = ry$

if $rx \in M'$, $r \in \mathfrak{a}$

$$0 + rx \qquad f''(0 + rx) = ry$$

$$(rx) + 0 \qquad f''(rx + 0) = f'(rx) = \alpha(r)$$

Analog of ④ for projectives ?

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dashed] \\ R \arrow[r, two heads] & R/\mathfrak{a} \end{tikzcd} \overset{?}{\Longrightarrow} P \text{ is projective?}$$

$R = \mathbb{Z}$

$$\begin{tikzcd} & P = \mathbb{Q} \arrow[dl, "\times"] \arrow[d, "\times"] \\ \mathbb{Z}, & \mathbb{Z}/n \end{tikzcd}$$

but $\mathbb{Q}$ is not a projective $\mathbb{Z}$-mod.

Construction of injective modules:

- R : k-algebra.

For any R-mod M.

$\mathrm{Hom}_k(R, M)$ is a left R-mod using right translation on R.

$f: R \xrightarrow[k\text{-linear}]{} M$, $r \in R$.

$$(r \cdot f)(x) = f(xr).$$

$$r_1(r_2 f) = (r_1 r_2) f$$

Claim: $\mathrm{Hom}_k(R, M)$ is an injective R-mod.

Pf. $\forall\ N.$ (R-mod)

$$\mathrm{Hom}_R(N, \underline{\mathrm{Hom}_k(R, M)}) \qquad S=k \circlearrowright \overset{B}{\overset{\parallel}{R}} \circlearrowleft R$$

$$\parallel$$

$$\mathrm{Hom}_k(R \otimes_R N, M) = \mathrm{Hom}_k(N, M).$$

$\uparrow$ exact in N.

General: $S \circlearrowright \underbrace{B \circlearrowleft R \circlearrowright N}$

$S \circlearrowright M.$

$$\mathrm{Hom}_S(B \otimes_R N, M) = \mathrm{Hom}_R(N, \boxed{\mathrm{Hom}_S(B, M)})$$

$$T_B = B \otimes_R (-) : R\text{-mod} \longrightarrow S\text{-mod}.$$

$$H_B = \mathrm{Hom}_S(B, -) : S\text{-mod} \longrightarrow R\text{-mod}.$$

$$\mathrm{Hom}_S(T_B(N), M) = \mathrm{Hom}_R(N, H_B(M)).$$

To prove, $\quad B \otimes_R N \longrightarrow M$, S-linear.

$$\Updownarrow$$

$B \times N \longrightarrow M$, $\underline{R\text{-bilinear}}$, S-linear.

$$\Updownarrow$$

$N \longrightarrow \mathrm{Hom}_S(B, M).$ R-linear.

$$\begin{array}{ccc} \mathrm{Hom}_R(R, M) & & \\ \| & & \\ M & \hookrightarrow & \mathrm{Hom}_k(R, M) = \text{injective.} \\ m & \longmapsto & (r \mapsto rm) \end{array}$$

Generalize this idea.. want to constr. injective R-mod.

$$S \circlearrowright B \circlearrowleft \underline{\underline{R}}$$

$S \circlearrowright M$ $\quad\quad \mathrm{Hom}_S(B, M)$. R-mod.

when is this injective?

$\left\{ \begin{array}{l} M \text{ is an injective } S\text{-mod.} \\ \\ B \text{ is a flat } R\text{-mod.} \end{array} \right.$

(i.e., $B \otimes_R (-)$ is an exact functor on R-mod)

Ex. $\quad R = \mathbb{Z}$, $\quad B = \mathbb{Q}$. flat $\mathbb{Z}$-mod.

$$S = \mathbb{Q} \circlearrowright B = \mathbb{Q} \circlearrowleft \mathbb{Z} = R.$$

$\Rightarrow \forall$ $\mathbb{Q}$-vector space V

$$\mathrm{Hom}_{\underset{S}{\overset{\|}{\mathbb{Q}}}}(\underset{B}{\overset{\|}{\mathbb{Q}}}, V) = V \text{ is an injective } \mathbb{Z}\text{-mod.}$$

(Fact: using ④ in the defn of injective modules,
$\mathbb{Z}$-mod M is injective $\Leftrightarrow$ M is divisible.
i.e., $\forall\, x \in M$, $n \in \mathbb{N}$, $\exists\, y \in M$
s.t. $ny = x$.)

A = comm. domain. $\quad K = \mathrm{Frac}(A)$.
(flat A-mod)

$\Rightarrow$ Any K-v.s. is an injective A-mod.

<u>Thm</u> $\forall R$, $\forall$ R-mod M.
$\exists$ $M \hookrightarrow I$ = injective R-mod.

<u>Pf</u>.

- $R = \mathbb{Z}$

- Apply general construction to

$$S = \mathbb{Z} \circlearrowright B = R \circlearrowleft R$$

$M \hookrightarrow M'$ = injective $\mathbb{Z}$-mod.

$$M = \mathrm{Hom}_{\dot{R}}(R, \dot{M}) \hookrightarrow \underbrace{\mathrm{Hom}_{\mathbb{Z}}(R, \dot{M'})}_{\text{injective } R\text{-mod}}.$$

M: $\mathbb{Z}$-mod

$M = F/N \qquad F$ = free $\mathbb{Z}$-mod.

$F \hookrightarrow F \otimes_{\mathbb{Z}} \mathbb{Q}$ = inj. $\mathbb{Z}$-mod

$$M = F/N \hookrightarrow \underbrace{(F \otimes_{\mathbb{Z}} \mathbb{Q})/N}_{\text{divisible.}}$$

(e.g. $\mathbb{Q}/\mathbb{Z}$ $\quad$ divisible $\Rightarrow$ injective.)

$\blacksquare$

Ex. $\overset{\text{(finite)}}{Q}$ quiver R_Q.

v : vertex.

$$\prod_v k_v = R_0 \subset R_Q$$

$\forall$ R_0-mod M, $\mathrm{Hom}_{R_0}(R_Q, M)$. injective R_Q-mod.

always injective. $M = k_v$ (1d mod at vertex v).

$$I_v = \mathrm{Hom}_{R_0}(R_Q, k_v).$$

$$\begin{array}{c} \| \\ \bigoplus k.p. \end{array}$$

$$\begin{tikzcd} R_Q \arrow[r, "R_0\text{-linear}"] \arrow[d] & k_v \\ \underset{\substack{p \text{ ends} \\ \text{at } v}}{\bigoplus} k\cdot p \arrow[ur] = 1_v \cdot R_Q, & \end{tikzcd}$$

$$I_v = \mathrm{Hom}_k(1_v \cdot R_Q, k_v)$$

$$P_v = R_Q \cdot 1_v.$$

Ex. M : rep. of Q.

$$\mathrm{Hom}_{R_Q}(M, I_v) = M_v^*$$

Next: $M \hookrightarrow$ <u>minimal</u> injective mod I ?

if $\exists$, I is called an <u>injective hull</u> of M.

<u>Thm</u> $\forall R$, $\forall R \curvearrowright M$. injective hull of M exists, and is unique up to isom.

<u>Defn.</u> $0 \neq M \hookrightarrow N$ is called <u>essential</u>, (N is an essential extn of M)
if any nonzero submod $N' \subset N$,

$$N' \cap M \neq 0.$$

($\iff$ cannot find $\underline{M \oplus N'} \subset N$).

<u>Ex.</u> $R = \mathbb{Z}$. $N = \mathbb{Z}/p^n\mathbb{Z}$

$$0 \neq \underset{\displaystyle p^i\mathbb{Z}/p^n\mathbb{Z}}{\overset{}{M}} \subset \boxed{N = \mathbb{Z}/p^n\mathbb{Z}} \qquad (0 \le i < n)$$

essential.

$\because$ all submods contain $p^{n-1}\mathbb{Z}/p^n\mathbb{Z}$.

(N is a <u>uniform</u> module.
any two nonzero submod intersect $\neq 0$).

Transitivity:

$$\begin{tikzcd} M \arrow[r, hook, "ess"'] \arrow[rr, hook, bend right=50, "ess"'] & M' \arrow[r, hook, "ess"'] \arrow[d, phantom, "\Downarrow"] & M'' \\ & {} & \end{tikzcd}$$

Criterion: $M \xhookrightarrow{i} N$.

i is ess. $\iff$ $\forall\, 0 \neq x \in N$, $\exists\ r \in R$. s.t. $0 \neq r x \in M$.

<u>Lemma</u>: $R \circlearrowright I$, I is injective $\iff$ I has no nontrivial ess extension.

(i.e., $I \subsetneq I'$, then this ext is not ess).

<u>Pf</u> $\Longrightarrow$ I inj.

$$I \subset I' \qquad I' = I \oplus I''$$

if $\neq 0$ this is not ess.

$\Longleftarrow$

$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[d, "f"'] & N \arrow[dl, dotted] \\ I & \end{tikzcd}$$

Zorn's lemma. may assume (M, f) is max

i.e. f cannot be further extended.

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "f"'] & N \arrow[d] \\ I \arrow[r, hook, "\tilde{i}"'] & I \sqcup_M N = I' \end{tikzcd}$$

$\Longrightarrow$ id_I cannot be further extended inside I'.

Suppose $\underline{M \neq N}$. then $I \neq I'$.

$I \xhookrightarrow[\tilde{i}]{} I'$ is not essential.

$$I \oplus I'' \subset I'.$$

$$\begin{tikzcd} {} \arrow[d, "\mathrm{id}"'] & {} \arrow[dl, "0"] \\ I & \end{tikzcd}$$

contracts that id_I cannot be further ext'd. $\blacksquare$