

- Projective covers

Projective resolution:

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

projective R -mod.

Can we make sense of / construct the minimal projective obj $P \twoheadrightarrow M$.

Ex. Q quiver, R_Q

$$P_v = R_Q \cdot 1_v \twoheadrightarrow k \cdot 1_v \quad (R_Q\text{-linear})$$

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Span {paths starting at v }

Fact. Any projective mod $P \twoheadrightarrow k \cdot 1_v$ has to contain P_v .

$$P_v \rightarrow P \twoheadrightarrow k \cdot 1_v$$

Def. $M \xrightarrow{f} N$ (R -mod) is called an essential surjection if \nexists proper submod $M' \subset M$ s.t. $f(M') = N$.

$$L \xrightarrow{f} M \xrightarrow{g} N$$

$g \circ f$ is ess \iff g, f are ess.

$$\begin{array}{ccc} \Leftarrow & L' \subset L & \\ & \downarrow & \downarrow f \\ & M' = f(L') \subset M & \\ & \downarrow & \downarrow g \\ g \text{ is ess} & & N = N \\ \downarrow & & \\ M' = f(L') = M & & \end{array}$$

$f|_{L'}$ surj, f is ess $\implies L' = L$

Ex. $M = f.g. R$ -mod
 $J = J(R)$.

Claim: $M \twoheadrightarrow M/JM$ is essential.

Pf: $M' \subset M$. $M' \twoheadrightarrow M/JM$.

$$\Rightarrow M' + JM = M.$$

$$\text{Nakayama} \Rightarrow M' = M$$

Def.

$$M = R\text{-mod.}$$

A projective cover of M is an essential surjection

$$\begin{array}{c} P \twoheadrightarrow M \\ \parallel \\ \text{proj. } R\text{-mod.} \end{array}$$

Non-existence in general:

$$R = \mathbb{Z}, \quad M = \mathbb{Z}/p\mathbb{Z}$$

(okay \mathbb{Z}_p ?)

$$P = \text{proj } \mathbb{Z}\text{-mod} \twoheadrightarrow M.$$

\Downarrow essential.

$$\begin{array}{c} \tilde{1} \longmapsto 1 \text{ mod } p. \\ \Downarrow \end{array}$$

$$\begin{array}{ccc} \mathbb{Z} \cdot \tilde{1} & \subset & P \\ & \searrow & \downarrow \\ & & M \end{array}$$

equality \checkmark
($P \twoheadrightarrow M$ is ess)

$$P = \mathbb{Z} \twoheadrightarrow \mathbb{Z}/p = M$$

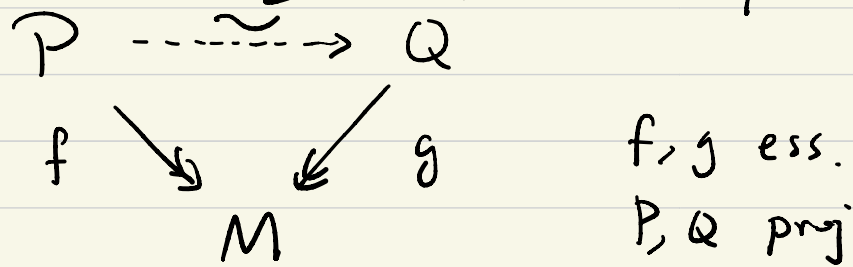
$$U \times$$

$$n \mathbb{Z}$$

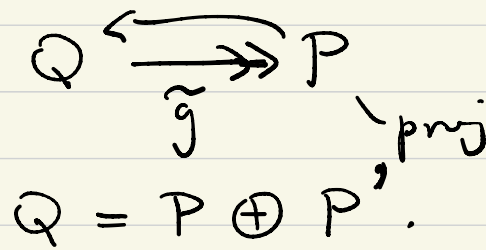
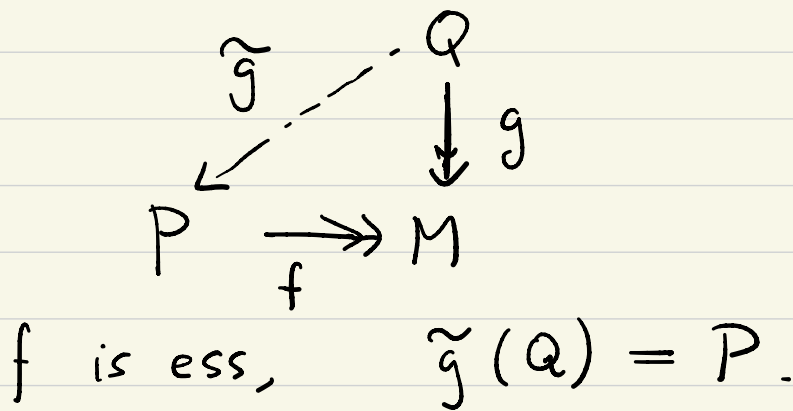
$$(n, p) = 1$$

Not essential.

Uniqueness. (up to isom) true in general.
 $\swarrow \exists$ isom, but not unique in general.



Pf.

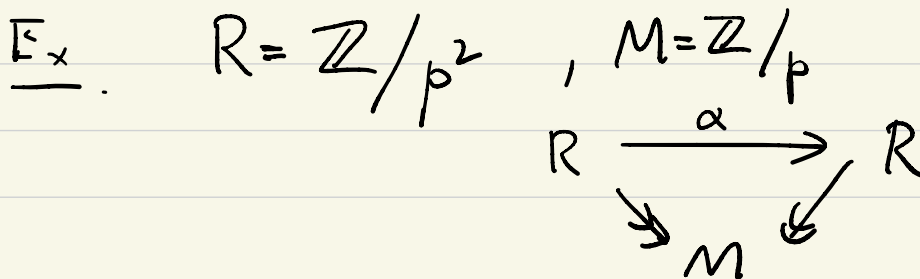


\tilde{g} = projection to P .

g is ess. $P \oplus Q \xrightarrow{g} M$

$\Rightarrow P' = 0$.

$\tilde{g}: Q \xrightarrow{\sim} P$.



any $\alpha \equiv 1 \pmod{p}$
 non-unique.

Thm $R = \text{left artinian}$, $M = f.g. R\text{-mod.}$
 then M has a projective cover.

Pf. Pick $P \xrightarrow{f} M$ s.t. $l(P)$ is minimal.
 $\begin{matrix} \text{proj.} \\ \uparrow \\ P \end{matrix}$ $\begin{matrix} f.g. \\ \downarrow \\ M \end{matrix}$

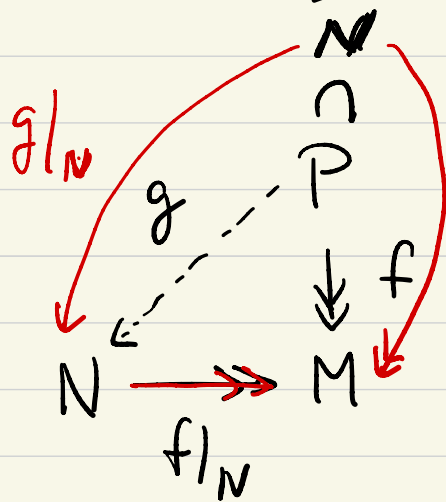
$$\left(R^n \twoheadrightarrow M \right)$$

↑ has length.

want to show P is a proj. cover of M .

Suppose $N \subset P$, $f(N) = M$.

Among all such N , choose one with minimal length.
 call it N .



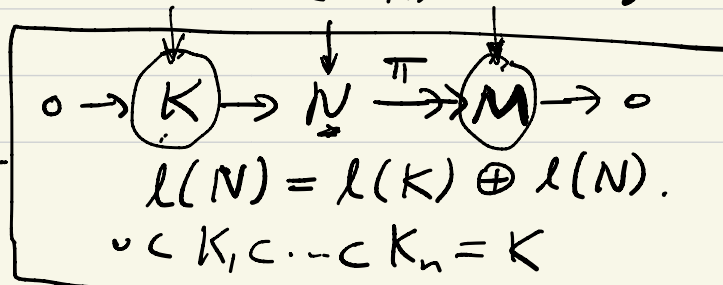
minimality of N .
 $\Rightarrow g(P) = N$.

Indeed
 $g(N) = N$.

$$g|_N : N \twoheadrightarrow N.$$

finite length $\Rightarrow \ker(g|_N)$ has $\text{length} = 0$

$\Rightarrow g|_N$ is an \cong .



length is additive in s.e.s.

$$N \begin{array}{c} \xleftarrow{g} \\ \xrightarrow{i} \end{array} P \quad \left. \begin{array}{l} 0 \subset M_1 \subset \dots \subset M_m = M. \\ 0 \subset K_1 \subset \dots \subset K_n \subset \pi^{-1}(M_1) \\ \quad \subset \pi^{-1}(M_2) \\ \quad \subset \dots \subset \pi^{-1}(M_n) \\ \quad \quad \quad \parallel \\ \quad \quad \quad \text{---} \end{array} \right\}$$

$$g \circ i = \text{autom. of } N.$$

$$\Rightarrow P = N \oplus \underbrace{N'}_{\text{ker}(g)}$$

$$\Rightarrow N \text{ is projective. } N \rightarrow M.$$

$$\Rightarrow P = N. \quad \blacksquare$$

R left artinian.

$$S : \text{simple } R\text{-mod.} \rightsquigarrow P_S \twoheadrightarrow S$$

proj. cover.

write ${}^R R$ into a direct sum of P_S .

$$\left(\begin{array}{l} \text{paths alg } R_Q. \quad P_v = R_Q \cdot 1_v \\ \bigoplus_{v \in \text{vert}(Q)} P_v = R \\ \quad \quad \quad \uparrow \\ \quad \quad \quad \text{all paths.} \end{array} \right)$$

Prop 1 $R =$ left artinian.

$$R \cong R \cong \bigoplus_{S \in \text{Simple } R\text{-mod} / \sim} P_S^{\oplus m_S}$$

$$m_S = \dim_{D_S} S \quad D_S = \text{End}_R(S).$$

Pf.

$$\bar{R} = R/J \cong \bigoplus_{S \text{ simple } R\text{-mod}} S^{\oplus m_S}$$

\leftrightarrow simple \bar{R} -mod.

compute projective covers of both sides

$$\left(\begin{array}{l} \bar{R} = \prod M_{n_i}(D_i) \quad n_i = m_{S_i} \\ S_i = D_i^{\oplus n_i} \\ M_{n_i}(D_i) = S_i^{\oplus n_i} \end{array} \right)$$

• $R \xrightarrow{\text{ess.}} \bar{R} = R/J(R)$

\parallel This is a proj. cover of \bar{R} .

• $\bigoplus P_S^{\oplus m_S} \xrightarrow{\text{essential}} \bigoplus S^{\oplus m_S}$

In general, if $\left. \begin{array}{l} M' \twoheadrightarrow M \\ N' \twoheadrightarrow N \end{array} \right\} \text{ess}$
then $M' \oplus N' \twoheadrightarrow M \oplus N$ is ess.

Pf: $M' \oplus N' \twoheadrightarrow M' \oplus N \twoheadrightarrow M \oplus N$.
enough to show each one is essential.

e.g. $M' \oplus N \twoheadrightarrow M \oplus N$
 \cup
 $Q \twoheadrightarrow$

Since image of Q contains $(x, 0)$
 $\forall x \in M$.

$$\Rightarrow Q \cap M' \twoheadrightarrow M$$

$$M' \twoheadrightarrow M \text{ ess} \Rightarrow Q \cap M' = M'$$

$$\text{i.e. } M' \oplus 0 \subset Q$$

then use $Q \twoheadrightarrow N$ to show

$$Q = M' \oplus N$$

Compare two calculations of proj covers

$$\Rightarrow R \cong \bigoplus P_s^{\oplus m_s}$$



Def. $T(M)$ (top)

$$\begin{array}{c} \parallel \\ M/JM \end{array} \quad J = J(R).$$

Claim: $R = \text{left artinian.}$ the
 $T(M) = \text{max semisimple quotient of } M.$

$M \twoheadrightarrow \text{semisimple}$

Pf.

$$M \twoheadrightarrow \text{ss.} = \bigoplus S_i$$

$$J \subset_0 S_i \Rightarrow M \twoheadrightarrow \left(\begin{array}{c} M/JM \\ \uparrow \\ \bar{R} = \text{ss} \end{array} \right) \twoheadrightarrow \bigoplus S_i$$

\swarrow ss R -mod.



Socle of M is ^{the} max. ss submodule of M .

Lemma R . any ring.

M . f.g. R -mod.

then $P_M \cong P_{T(M)}$.

Pf. $P_M \twoheadrightarrow M \twoheadrightarrow M/JM = T(M)$
 $\begin{array}{ccc} \twoheadrightarrow & \twoheadrightarrow & \twoheadrightarrow \\ \text{ess} & \Downarrow \text{ess} & \text{ess} \\ & \text{ess} & \twoheadrightarrow \end{array}$ ▣

Prop. $R =$ left artinian. M . f.g. R -mod
 then $T(P_M) \cong T(M)$.

Pf. $P_M \twoheadrightarrow M \twoheadrightarrow T(M)$
 $T(M)$ is a ss quotient of P_M .

$P_M \twoheadrightarrow \boxed{T(P_M)} \twoheadrightarrow T(M)$
 $\text{ess} \quad \text{ess} \quad \twoheadrightarrow$
 ess
 These are \bar{R} -mod.
 \parallel
 ss.
 This map is essential
 \Rightarrow it is \cong . ▣

Another pf of Prop 1

want to show $R \cong \bigoplus P_s^{\oplus m_s}$.

? $P = \bigoplus P_s^{\oplus m_s} \xrightarrow{\text{ess.}} \bigoplus S^{\oplus m_s}$
 \parallel
 $T(P) = P/J_P$

$$T(P) = P/J_P = \bigoplus_S T(P_S)^{\oplus m_S}$$

$$\parallel$$

$$\bigoplus_S \oplus^{\oplus m_S}$$

$$M' \xrightarrow{\text{ess}} M$$

$$M' \oplus N \xrightarrow{\text{also ess.}} M \oplus N$$

\cup

Q

$$\boxed{Q \cap M'} \oplus 0 \longrightarrow M \oplus 0.$$

\parallel
 M'

$$\Rightarrow \underline{M' \oplus 0} \subset Q.$$

$$Q \twoheadrightarrow N. \Rightarrow M' \oplus N = Q.$$

General: $\left. \begin{array}{l} M' \longrightarrow M \\ N' \longrightarrow N \end{array} \right\} \text{ess}$

$$\Rightarrow M' \oplus N' \xrightarrow{\text{ess.}} M \oplus N.$$

Functors given by projective modules

R . ~~artinian~~ f.d. k -algebra ($k = \bar{k}$).

simple = $S \rightsquigarrow P_S$.

$$\text{End}_R(S) = k.$$

$F_S := \text{Hom}_R(P_S, \overset{\uparrow}{-})$ is an exact functor on R -mod.

$F_S: R\text{-mod} \rightarrow k\text{-v.s.}$

$$\boxed{S \mapsto 1d/k.}$$

$$P_S \xrightarrow[\text{f}]{\text{can}} S$$

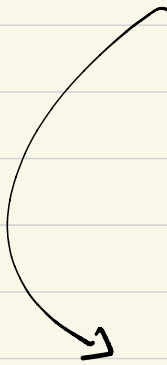
f induces a map

$$T(P_S) = S \xrightarrow{a \in k} S$$

$$f - a \cdot \text{can}: P_S \rightarrow S$$

induces 0 on top.

$$\Rightarrow f = a \cdot \text{can}.$$



$$P_S \xrightarrow{\text{any}} (S) = \text{simple}$$

$$P_S \twoheadrightarrow T(P_S) \rightarrow S$$

$$\quad \quad \quad \parallel$$

$$\quad \quad \quad S$$

$$S' \text{ simple} \neq S \quad \text{Hom}_R(P_S, S') = 0.$$

$$P_S \twoheadrightarrow T(P_S) \rightarrow S'$$

$$\quad \quad \quad \parallel$$

$$\quad \quad \quad S.$$

Using F_S is an exact functor
 \Rightarrow

$$\dim_k \text{Hom}(P_S, M) = \# \text{ of comp. factors of } M \text{ isom to } S.$$

||
fg. R-mod

comp. series of M

$S_1) S_2) \dots S_n).$

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$[M:S].$