Lecture 11

10/9

- Projective covers

Projective resolution:

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

projective. R-mod.

Can we make sense of / construct the <u>minimal</u> projective obj $P \twoheadrightarrow M$.

<u>Ex.</u> Q quiver. R_Q

$$P_v = R_Q \cdot 1_v \longrightarrow\!\!\!\!\to k \cdot 1_v \quad (R_Q\text{-linear})$$

$\shortparallel$

Span{paths starting at v}

<u>Fact</u>: Any projective mod $P \twoheadrightarrow k \cdot 1_v$ has to contain P_v.

$$\begin{tikzcd} P_v \arrow[r] \arrow[rr, bend right] & P_\bullet \arrow[r, two heads] & 1_v \end{tikzcd}$$

Def: $M \overset{f}{\twoheadrightarrow} N$ (R-mod)
is called an <u>essential</u> surjection
if $\nexists$ proper submod $M' \subset M$
s.t. $f(M') = N$.

$$L \underset{f}{\twoheadrightarrow} M \underset{g}{\twoheadrightarrow} N$$

$g \circ f$ is ess $\Longleftrightarrow$ g, f are ess.

$\Longleftarrow$

$$\begin{tikzcd} L' \arrow[d] \arrow[r, phantom, "\subset"] & L \arrow[d, two heads, "f"] \\ M' = f(L') \arrow[d, two heads] \arrow[r, phantom, "\subset"] & M \arrow[d, two heads, "g"] \\ N \arrow[r, phantom, "="] & N \end{tikzcd}$$

g is ess
$\Downarrow$
$M' = f(L') = M$.

$f|_{L'}$ surj, f is ess $\Longrightarrow$ $L' = L$

Ex. M = f.g. R-mod
$J = J(R)$.

Claim: $M \twoheadrightarrow M/JM$ is essential.

Pf: $M' \subset M$. $M' \twoheadrightarrow M/JM$.

$$\Rightarrow \quad M' + JM = M.$$

Nakayama $\Rightarrow$ $M' = M$

<u>Def.</u> M = R-mod.
A projective cover of M is an essential surjection

$$\underset{\text{proj. } R\text{-mod.}}{\underset{\parallel}{P}} \twoheadrightarrow M$$

Non-existence in general:

$R = \mathbb{Z}$. $\quad M = \mathbb{Z}/p\mathbb{Z}$ $\quad$ ($\mathbb{Z}_p$ okay?)

$$P = \text{proj } \mathbb{Z}\text{-mod} \twoheadrightarrow M. \quad \underline{\text{essential}}.$$

$$\begin{array}{ccc} \Downarrow & & \Downarrow \\ \tilde{1} & \longmapsto & 1 \text{ mod } p. \end{array}$$

$$\begin{array}{ccc} \mathbb{Z}\cdot\tilde{1} & \subset & P \\ & \searrow & \downarrow \\ & & M \end{array}$$

equality (P ↠ M is ess)

$$\begin{array}{ccc} P = \mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/p = M \\ \cup\!\!\!\times & \nearrow & \\ n\mathbb{Z} & & \end{array}$$

$(n,p) = 1$ Not essential.

<u>Uniqueness</u>. (up to isom) true in general.

∃ isom, but not unique in general.

$$\begin{tikzcd} P \arrow[rr, dashed, "\sim"] \arrow[dr, "f"'] & & Q \arrow[dl, "g"] \\ & M & \end{tikzcd}$$

f, g ess.
P, Q proj

<u>Pf</u>.

$$\begin{tikzcd} & Q \arrow[dl, dashed, "\tilde{g}"'] \arrow[d, "g"] \\ P \arrow[r, two heads, "f"'] & M \end{tikzcd}$$

f is ess, $\quad \tilde{g}(Q) = P$.

$$\begin{tikzcd} Q \arrow[r, two heads, "\tilde{g}"'] & P \arrow[l, bend right] \end{tikzcd}$$

proj

$Q = P \oplus P'$.

$\tilde{g} =$ projection to P.

g is ess. $\qquad$ $\begin{tikzcd} P \arrow[r, phantom, "\subseteq_{\oplus}"] \arrow[rr, bend right, two heads, "f"'] & Q \arrow[r, two heads, "g"] & M \end{tikzcd}$.

$\Longrightarrow P' = 0$.

$\tilde{g} : Q \xrightarrow{\sim} P$. ∎

<u>Ex</u>. $R = \mathbb{Z}/p^2$, $M = \mathbb{Z}/p$

$$\begin{tikzcd} R \arrow[rr, "\alpha"] \arrow[dr, two heads] & & R \arrow[dl, two heads] \\ & M & \end{tikzcd}$$

any $\alpha \equiv 1 \ (p)$
non-unique.

Thm $R =$ left artinian. $M =$ f.g R-mod.
then M has a projective cover.

Pf. Pick $P \overset{f}{\twoheadrightarrow} M$ s.t. $\ell(P)$ is minimal.
P = proj. M f.g.

$$\left(R^n \twoheadrightarrow M \right)$$
R^n has length.

want to show P is a proj. cover of M.

Suppose $N \subset P$, $f(N) = M$.

Among all such N, choose one with minimal length. call it N.

$$\begin{tikzcd}
 & N \arrow[ddl, bend right, "g|_N"'] \arrow[dd, bend left] \arrow[d, phantom, "\cap"] \\
 & P \arrow[dl, dashed, "g"'] \arrow[d, "f"] \\
N \arrow[r, two heads, "f|_N"'] & M
\end{tikzcd}$$

minimality of N. $\Rightarrow g(P) = N$.
Indeed $g(N) = N$.

$g|_N : N \twoheadrightarrow N$.
finite length $\Rightarrow \ker(g|_N)$ has leng $= 0$

$\Rightarrow g|_N$ is an $\cong$.

$$0 \to K \to N \xrightarrow{\pi} M \to 0$$
$$\ell(N) = \ell(K) \oplus \ell(N).$$
$$0 \subset K_1 \subset \cdots \subset K_n = K$$

length is additive in s.e.s.

$$\begin{tikzcd} N \arrow[r, hook, "i"'] & P \arrow[l, bend right, "g"'] \end{tikzcd}$$

$0 \subset M_1 \subset \cdots \subset M_m = M.$

$0 \subset K_1 \subset \cdots \subset K_n \subset \pi^{-1}(M_1)$

$\subset \pi^{-1}(M_2)$

$\subset \cdots \subset \pi^{-1}(M_n)$

$\parallel$

N

$g \circ i = \text{autom. of } N.$

$\Longrightarrow \quad P = N \oplus \underbrace{N'}_{= \ker(g)}$

$\Longrightarrow \quad N$ is projective. $N \twoheadrightarrow M.$

$\Longrightarrow \quad P = N.$

R left artinian.

S : simple R-mod. $\rightsquigarrow P_S \twoheadrightarrow S$ proj. cover.

write ${}_R R$ into a direct sum of P_S.

(path alg R_Q. $\qquad P_v = R_Q \cdot 1_v$

$\bigoplus_{v \in \mathrm{vert}(Q)} P_v = R.$)

all paths.

<u>Prop 1</u> R = left artinian.

$$R \circlearrowright R \cong \bigoplus_{S \in \text{Simple } R\text{-mod}/\cong} P_S^{\oplus m_S}$$

$$m_S = \dim_{D_S} S \qquad D_S = \mathrm{End}_R(S).$$

Pf $\quad \bar{R} = R/J \cong \bigoplus S^{\oplus m_S}$

S simple R-mod $\longleftrightarrow$ simple $\bar{R}$-mod.

compute projective covers of both sides

$$\left(\begin{array}{l} \bar{R} = \prod M_{n_i}(D_i). \qquad n_i = m_{S_i} \\ S_i = D_i^{\oplus n_i} \\ M_{n_i}(D_i) = S_i^{\oplus n_i} \end{array} \right)$$

- $R \twoheadrightarrow \bar{R} = R/J(R)$ ess.

$\|$ This is a proj. cover of $\bar{R}$.

- $\bigoplus P_S^{\oplus m_S} \twoheadrightarrow \bigoplus S^{\oplus m_S}$ essential.

In general, if $\left.\begin{array}{l} M' \twoheadrightarrow M \\ N' \twoheadrightarrow N \end{array}\right\}$ ess

then $M' \oplus N' \twoheadrightarrow M \oplus N$ is ess.

Pf: $M' \oplus N' \twoheadrightarrow M' \oplus N \twoheadrightarrow M \oplus N$.

enough to show each one is essential.

e.g. $M' \oplus N \twoheadrightarrow M \oplus N$

$$\begin{array}{c} \cup \\ Q \end{array} \nearrow$$

since image of Q contains $(x,0)$ $\forall\ x \in M$.

$\Rightarrow \quad Q \cap M' \twoheadrightarrow M$

$M' \twoheadrightarrow M$ ess $\Rightarrow \quad Q \cap M' = M'$

i.e. $M' \oplus 0 \subset Q$

then use $Q \twoheadrightarrow N$ to show

$$Q = M' \oplus N.$$

Compare two calculations of proj covers

$$\Rightarrow \quad R \cong \bigoplus_S P_S^{\oplus m_S}.$$

$\square$

Def. $T(M)$ (top)

$$T(M) = M/JM \qquad J = J(R).$$

Claim: $R =$ left artinian.

$T(M) =$ the max semisimple quotient of M.

$$M \twoheadrightarrow \text{semisimple}$$

Pf. $M \twoheadrightarrow$ ss. $= \bigoplus S_i$

$J \circlearrowright S_i$ is 0 $\Rightarrow$ $M \twoheadrightarrow (M/JM) \twoheadrightarrow \bigoplus S_i$ (ss R-mod.)

$\bar{R} = ss \circlearrowright M/JM$

$\square$

Socle of M is the max. ss submodule of M.

Lemma R. any ring.
M. f.g. R-mod.
then $P_M \cong P_{T(M)}$.

Pf.

$$\begin{tikzcd} P_M \arrow[r, two heads, "\text{ess}"'] \arrow[rr, two heads, bend right, "\text{ess}"'] & M \arrow[r, two heads, "\text{ess}"'] & M/JM = T(M) \end{tikzcd}$$

$\Downarrow$

∎

Prop. R = left artinian. M. f.g. R-mod
then $T(P_M) \cong T(M)$.

Pf.

$$\begin{tikzcd} P_M \arrow[r, two heads] \arrow[rr, two heads, bend right] & M \arrow[r, two heads] & T(M) \end{tikzcd}$$

$T(M)$ is a ss quotient of P_M.

$$\begin{tikzcd} P_M \arrow[r, two heads, "\text{ess}"'] \arrow[rr, two heads, bend right, "\text{ess}"'] & \boxed{T(P_M) \xrightarrow[\text{ess}]{} T(M)} \end{tikzcd}$$

These are $\bar{R}$-mod.
$\quad\quad\quad\quad\ \ \| $
$\quad\quad\quad\quad\ \ $ ss.

This map is essential
$\Rightarrow$ it is $\cong$.

∎

Another pf of Prop 1

want to show $R \cong \bigoplus P_s^{\oplus m_s}$.

$$? \ P = \bigoplus P_s^{\oplus m_s} \xrightarrow[\text{ess.}]{} \bigoplus S^{\oplus m_s}$$

$$\bigoplus S^{\oplus m_s} = T(P) = P/JP$$

$$T(P) = P/_{JP} = \bigoplus_S T(P_S)^{\oplus m_S}$$

$$\quad \| $$

$$\bigoplus_S S^{\oplus m_S}$$

$$M' \underset{ess}{\twoheadrightarrow} M$$

$$M' \oplus N \twoheadrightarrow M \oplus N$$

$$\cup$$

$$Q \xrightarrow{\text{also ess.}} M \oplus N$$

$$\boxed{Q \cap M'} \oplus 0 \twoheadrightarrow M \oplus 0 .$$

$$\| $$

$$M'$$

$$\Rightarrow \underline{M'} \oplus 0 \subset Q.$$

$$Q \twoheadrightarrow N. \Rightarrow M' \oplus N = Q.$$

General:

$$\left.\begin{array}{l} M' \twoheadrightarrow M \\ N' \twoheadrightarrow N. \end{array}\right\} \text{ess}$$

$$\Rightarrow M' \oplus N' \underset{ess.}{\twoheadrightarrow} M \oplus N.$$

Functors given by projective modules

R. ~~artinian~~ f.d. k-algebra $(k = \bar{k})$.

simple $= S \rightsquigarrow P_S$. $\qquad \mathrm{End}_R(S) = k$.

$F_S := \mathrm{Hom}_R(P_S, -)$ is an exact functor on R-mod.

$F_S : R\text{-mod} \longrightarrow k\text{-v.s.}$

$$\boxed{S \longmapsto 1d/k.}$$

$$P_S \underset{f}{\overset{\text{can}}{\rightrightarrows}} S$$

f induces a map

$T(P_S) = S \xrightarrow{a \in k} S$

$f - a \cdot \text{can} : P_S \longrightarrow S$

induces 0 on top.

$\Rightarrow f = a \cdot \text{can}.$

$P_S \xrightarrow{\text{any}} S = \text{simple}$

$P_S \twoheadrightarrow T(P_S) \longrightarrow S$
$\qquad\qquad \| $
$\qquad\qquad S$

S' : $\mathrm{Hom}_R(P_S, S') = 0$.
simple $\neq S$

$P_S \twoheadrightarrow T(P_S) \longrightarrow S'$.
$\qquad\qquad \|$
$\qquad\qquad S$.

Using F_S is an exact functor

$\Longrightarrow$

$$\dim_k \mathrm{Hom}(P_S, M) = \#\text{ of comp. factors of } M \text{ isom to } S.$$

M = f.g. R-mod

comp. series of M

$S_1) \; S_2) \; \cdots \; S_n)$.

$\#$ of comp. factors of M isom to S = $[M:S]$.