18.706 HOMEWORK 7

DUE OCT.21, 2020

Theorems

In this part you will see theorems that we studied in the class. Please supply a detailed proof of each.

Theorem 1. *Let R be a ring and M be an R-module. Then there exists an injective R-module I and an essential extension $M \hookrightarrow I$. Moreover, for another such essential extension $M \hookrightarrow I'$ with I' injective, there exists an isomorphism $I \xrightarrow{\sim} I'$ that is the identity on M.*

Theorem 2. *Let $\mathcal{C}$ be a category and $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ be the category of contravariant functors $\mathcal{C} \to \mathrm{Set}$. Then the functor $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ sending $X \in \mathcal{C}$ to the functor $h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ (for all $Y \in \mathcal{C}$) is a fully faithful embedding.*

Exercises

Problem 1. Let k be a field. A Frobenius k-algebra (always assuming k is in the center) is a finite-dimensional k-algebra R equipped with a k-linear function $\tau : R \to k$ such that the bilinear pairing $R \times R \to k$ given by $(x, y) \mapsto \tau(xy)$ is nondegenerate.

(1) Let G be a finite group. Show that $k[G]$ is a Frobenius k-algebra.
(2) Let V be a finite-dimensional k-vector space with a quadratic form q. Show that the Clifford algebra $R = \mathrm{Cl}(V, q)$ is a Frobenius k-algebra.
(3) If k'/k is a finite extension, and R is a Frobenius k'-algebra, show that R is also a Frobenius k-algebra.
(4) If R is a Frobenius k-algebra, show that R is injective as a left R-module (such a ring is called *self-injective*).

Problem 2. Let $R \to S$ be a ring homomorphism. Describe the endomorphism ring of the forgetful functor $\omega : (S - \mathrm{Mod}) \to (R - \mathrm{Mod})$.