18.706 HOMEWORK 6

DUE OCT.14, 2020

Theorems

In this part you will see theorems that we studied in the class. Please supply a detailed proof of each.

Theorem 1. *Let R be a left artinian ring and M be a finitely generated R-module. Then M admits a decomposition $M = \oplus_{i=1}^{n} M_i$ into indecomposable R-modules. For a fixed indecomposable R-module N, the numbers of summands isomorphic to N in any two such decompositions are the same.*

Theorem 2. *Let R be a left artinian ring and M be a finitely generated R-module. Then M has a projective cover P_M, and P_M is unique up to isomorphism.*

Exercises

Problem 1. For a finite-dimensional k-algebra R with $\{P_i\}_{i \in I}$ the set of indecomposable projective modules (up to isomorphism), we define its *Cartan matrix* to be $C = (C_{ij})_{i,j \in I}$ where $C_{ij} = \dim_k \mathrm{Hom}_R(P_i, P_j)$.

Let Q be a finite quiver without oriented cycles, and R_Q be its path algebra.

(1) Compute the Cartan matrix of R_Q in terms of Q.
(2) Show that any finite-dimensional R_Q-module M admits a two-step projective resolution, i.e., a short exact sequence of R_Q-modules

$$0 \to P_1 \to P_0 \to M \to 0$$

with P_1 and P_0 projective.

Problem 2. Let k be an algebraically closed field with $\mathrm{char}(k) = 2$ or 3. Describe indecomposable projective $k[S_3]$-modules.