18.706 HOMEWORK 4

DUE SEP.30, 2020

Theorems

In this part you will see theorems that we studied in the class. Please supply a detailed proof of each.

Theorem 1. *Let R be a left artinian ring and $J(R)$ be its Jacobson radical. Then $R/J(R)$ is semisimple.*

Theorem 2. *Let M be a finitely generated left module over a ring R. If $J(R)M = M$ then $M = 0$.*

Exercises

Problem 1. Let k be a field, V a k-vector space of infinite dimension. Let $R = \mathrm{End}_k(V)$.

(1) Show that $J(R) = 0$.

(2) Show that any nonzero two-sided ideal of R contains the ideal of finite-rank endomorphisms. In particular, $J(R)$ is not the intersection of all maximal two-sided ideals.

Problem 2. Let G be a finite group and k a field with $\mathrm{char}(k) = p > 0$. Let $R = k[G]$.

(1) If $|G|$ is prime to p, show that R is semisimple.

(2) Suppose G is a p-group, show that up to isomorphism there is only one simple R-module, the trivial module k. Conclude that $J(R)$ is the augmentation ideal of R, i.e., the kernel of $k[G] \to k$ sending g to 1 for all $g \in G$.

(3) (Optional) Again suppose G is a p-group. Can you prove directly that the augmentation ideal of R is a nilpotent ideal? If n is the smallest positive integer such that $J(R)^n = 0$, what can you say about n?

(4) Suppose the Sylow p-subgroup of G is normal in G, describe $J(R)$.