18.706 HOMEWORK 3

DUE SEP.23, 2020

Theorems

In this part you will see theorems that we studied in the class. Please supply a detailed proof of each.

Theorem 1. *A ring R is semisimple if and only if it is a finite product $\prod_{i=1}^{s} M_{n_i}(D_i)$, where D_i are division rings.*

Theorem 2. *The following are equivalent for a ring R:*

(1) *R is simple and left artinian.*
(2) *R is left primitive (i.e., R has a faithful simple left module) and left artinian.*
(3) *$R \cong M_n(D)$ for some division ring D.*

Exercises

Problem 1. Let k be a field. Let R be a simple k-algebra with $Z(R) = k$ (here $Z(R)$ denotes the center of R).

(1) Define a ring structure on $R \otimes_k R^{op}$ in which R and R^{op} commute with each other.
(2) Show that R is a simple $R \otimes_k R^{op}$-module under left and right multiplication.
(3) Suppose $\dim_k R = n$, show that $R \otimes_k R^{op} \cong M_n(k)$ as k-algebras.

Problem 2. Let R be a k-algebra (it implicitly means k is in the center of R), and M be a semisimple module of R with $\dim_k M < \infty$. Show that

(1) $S = \mathrm{End}_R(M)$ is a semisimple ring.
(2) The action map $R \to \mathrm{End}_S(M)$ is surjective.